Supplement to Chapter 1*

1.1 Planck's constant ($\hbar$) as the 'quantum of action'

The speed of light provides a 'benchmark' value against which the magnitudes of other velocities in a given problem can be compared. Familiar, non-relativistic Newtonian mechanics is valid in the limit when $v/c << 1$; we don't need relativity in 'everyday' situation because c is so large. In a similar way, the effects of wave mechanics on macroscopic systems often need not be considered because $\hbar$ is, in some sense, so small; but small compared to what? Eqn. (1.36) and **P1.12**(a) of the text suggests that $\hbar$ could be compared to the typical angular momentum in a problem, but this is clearly not a fundamental connection since quantum mechanics can be applied equally well to one-dimensional systems for which the notion of angular momentum does not arise.

The quantity in classical mechanics whose magnitude can most naturally be compared to $\hbar$ is the **classical action**†, S, which is used in the **Lagrangian formulation**‡ of classical mechanics.

The action takes as its argument any possible path, $x(t)$, connecting the initial (i) and final (f) points on a classical trajectory, i.e., which satisfies $x(t_i) = x_i$ and $x(t_f) = x_f$. It returns a numerical value given by the integral

$$S[x] = \int_{t_i}^{t_f} dt\ (T - V) = \int_{t_i}^{t_f} dt\ \left(\frac{1}{2}mv^2(t) - V(x(t))\right) . \tag{1.1}$$

Hamilton's principle states that

- The path, $x(t)$, which minimizes the classical action, $S[x]$, is the trajectory as given by Newton's equations of motion

†Our extremely abbreviated discussion does not do justice to the notion of action, in either classical or quantum mechanics. The most elegant and seamless connection between the two descriptions of nature is provided by the path integral approach to wave mechanics originated by Feynman. It is discussed in several undergraduate textbooks (see, e.g., Park (1992) and Sakurai (1994)), but the standard (and very readable) reference is Feynman and Hibbs (1965).

‡See, e.g., Marion and Thornton (1988).

and provides an alternative formulation of classical mechanics. We immediately note that the action integral has the dimensions of $energy \cdot time$, as does $\hbar$.

As an example of this connection, we calculate the classical action for an electron in the hydrogen atom in a circular orbit of radius r over a single period. It is easy to show that

$$S = \int_0^\tau dt\,(T - V) = \frac{3}{2}|E|\tau = 3\pi\sqrt{mKe^2 r}\,. \tag{1.2}$$

If we now make use the Bohr result (Eqn. (1.37) of the text) for the quantized radii,

$$r_n = a_0 n^2 = \left(\frac{\hbar^2}{m(Ke^2)}\right) n^2\,, \tag{1.3}$$

we find that that the action is given by

$$S = 3\pi n\hbar\,; \tag{1.4}$$

for this reason, $\hbar$ has sometimes been called the 'quantum of action'.

In classical mechanics, only the single path with the least action is realized in nature as the unique trajectory. Quantum mechanics, however, allows more 'leeway' in that other paths which have an action which is larger than the minimum, but by less than one unit of $\hbar$ (more or less), i.e.,

$$\mathcal{O}(\hbar) \gtrsim \Delta S = S[x_{path}^{alternative}] - S_{min}[x_{path}^{classical}] > 0 \tag{1.5}$$

are also possible. To better gauge the uncertainties this engenders in the notion of classical trajectory, we imagine an alternative, non-circular orbit as in Fig. 1.1. Using Eqn. (1.2) as a guide, we estimate the change in action corresponding to a small change in radius, Δr, is given by

$$\Delta S = (3\pi)\frac{1}{2}\sqrt{\frac{m(Ke^2)}{r}}\Delta r = \frac{1}{2}\frac{\Delta r}{r}S\,. \tag{1.6}$$

Only such changes for which $\Delta S \lesssim \hbar$ will contribute, so that we estimate that the uncertainty in a circular orbit due to quantum effects will scale roughly as

$$\frac{\Delta r}{r} \lesssim \frac{\Delta S}{S} \sim \frac{\hbar}{n\hbar} = \frac{1}{n}\,. \tag{1.7}$$

We note that:

- Low-lying and obviously quantum mechanical states (where n is small) have actions which are already close to $\hbar$ in magnitude, so that fluctuations of $\mathcal{O}(\hbar)$ around the classical path are dramatically different; many, quite different, paths are associated with such quantum states, and the notion of trajectory is not useful (or even well-defined).

- For highly excited, quasi-classical states (with $n >> 1$), an $\mathcal{O}(\hbar)$ change in the action makes little change in the physical dimensions of the path (**SP1.2**, **SP1.3**), and one approaches the classical limit of predictable trajectories.

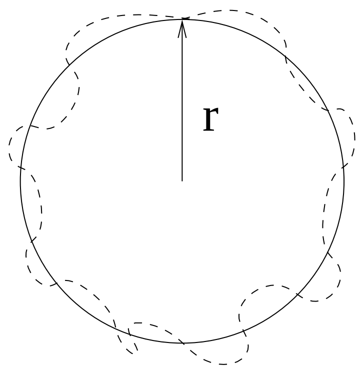

Figure 1.1: Classical circular orbit for hydrogen atom, plus path (not a classical orbit!) with larger classical action.

1.2 Questions and Problems

SP1.1. Repeat the argument (consulting a classical mechanics book if necessary) to show that the path, $x(t)$, which minimizes (or more generally gives an extremum of) the classical action in Eqn. (1.1) satisfies Newton's law of motion, namely

$$m\frac{d^2}{dt^2}x(t) = F(x) = -\frac{dV(x)}{dx} . \tag{1.8}$$

Hint: Consider small deviations around the minimum action path, characterized by $x(t)+\delta(t)$, where $\delta(t_i) = \delta(t_f) = 0$ so that the new path still goes through the specified initial and final points. Expand the argument of the action integral, assuming that $|\delta(t)| << |x(t)|$, keep only the first order change, use an integration by parts trick to write the *change* in action as an integral over $\delta(t)$, and argue that for arbitrary $\delta(t)$, the factor multiplying it must vanish.

SP1.2. Free particle action. In this problem, we gain some experience with the classical action and 'quantum trajectories'.
(a) A free particle of mass m moves in one dimension starting at the origin and arriving at position L in a time T. The path must then satisfy $x(0) = 0$ and $x(T) = L$. Such a free particle will, of course, travel in a straight line path at constant velocity so that $x(t) = Lt/T$ is the classical trajectory. Since the potential energy vanishes, the classical action is simply

$$S = \int_0^T dt\, \left(\frac{1}{2}mv^2\right) = \frac{1}{2}m\left(\frac{L}{T}\right)^2 T = \frac{mL^2}{2T} . \tag{1.9}$$

(b) Assume a more general path of the form $x(t) = a + bt + ct^2$, as in Fig. 1.2 and show that the boundary conditions imply that $a = 0$ and $c = (L - bT)/T^2$. Minimize the action, and show that

$$S(b) = S = \frac{m}{2}\left(\frac{4L^2}{3T} - \frac{2bL}{3} + \frac{b^2T}{3}\right) . \tag{1.10}$$

Show that this has a minimum when $b = L/T$ and $c = 0$ and therefore reproduces the classical trajectory as it should.
(c) Write $b = L/T \pm \delta$, and show that the action is bigger by an amount

$$\Delta S = S\left(b = \frac{L}{T} \pm \delta\right) - S\left(b = \frac{L}{T}\right) = \frac{mT\delta^2}{6} . \tag{1.11}$$

If paths with $\Delta S \lesssim \hbar$ are allowed, show that the 'quantum trajectories' differ at most from the classical straight line path by

$$|\Delta x|_{max} = \sqrt{\frac{3\hbar T}{8m}} \tag{1.12}$$

and evaluate this 'shift' for a typical macroscopic particle.
(d) Repeat as much of the problem as possible by using the even more general path $x(t) = a + bt + ct^2 + dt^3$ so that there are now two parameters in the minimization problem.

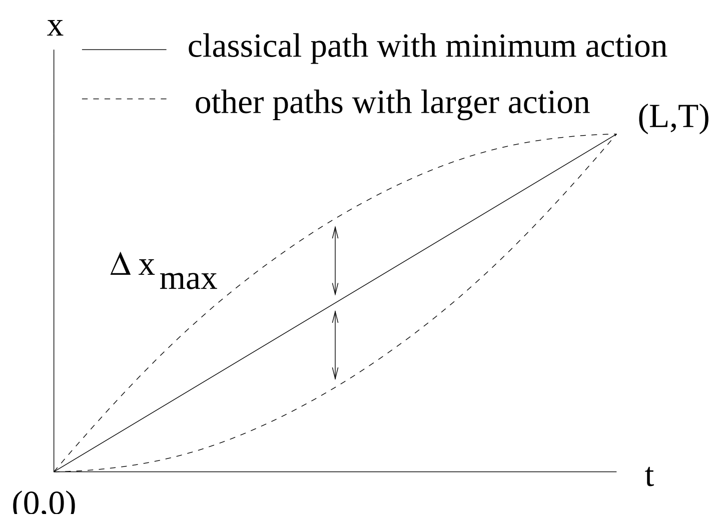

Figure 1.2: Classical straight-line path giving minimum action for a free particle (solid line) and other paths (dashed curves) with larger actions.

SP1.3. Repeat **SP1.2**, but for a particle undergoing a uniform acceleration, $+g$, for a time T starting from rest. In this case one has $x(t) = gt^2/2$ as the 'real' trajectory. Use the same trial path as in **SP1.2**(b), but remember that the potential energy function is now $V(x) = -mgx$.

SP1.4. Recall **P1.11** on **Simple-minded scaling laws** from the text, where simple de Broglie wavelength ideas were used for a general power law radial potential of the form

$$V(r) = Ar^s \qquad \text{giving} \qquad F(r) \propto Ar^{s-1}\,, \tag{1.13}$$

generalizing from the Coulomb result for the Bohr model where $s = -1$. You were asked to show that the classical period and energy scaled with quantum number (for large n) as

$$\tau \propto \left(\frac{m}{n\hbar}\right)\left(\frac{n^2\hbar^2}{Am}\right)^{2/(s+2)} \qquad \text{and} \qquad E \propto A\left(\frac{n^2\hbar^2}{Am}\right)^{s/(s+2)} . \tag{1.14}$$

Use these results to show that the action (when integrated over one period of a circular orbit) scales as

$$S = n\hbar \tag{1.15}$$

reinforcing the identification of $\hbar$ as the 'quantum of action'.

1.3 References

- Feynman, R. P. and A. R. Hibbs (1965). **Quantum Mechanics and Path Integrals**, McGraw-Hill, New York.
- Marion, J. B. and S. T. Thornton (1988). **Classical Dynamics of Particles and Systems**, 3rd Edition, Harcourt Brace Jovanovich, Fort Worth.
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