The uncertainty principle for energy and time. II

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The meaning and scope of a recent type of uncertainty relation of a very general character are elucidated using the notions of time-indicating dynamical variables (clock variables) and place-indicating dynamical variables (position variables). It is shown that if the total energy (momentum) of a system is certain, all time-indicating (place-indicating) dynamical variables are completely uncertain. The quantum clock is discussed as an illustration of the energy–time uncertainty relation. The relations can be successfully applied to the thought experiments that Einstein introduced into his debate with Bohr about the uncertainty principle(s) and, in particular, to the famous photon-box experiment. It is shown that due to this general relation the photon box can never serve its purpose, independent of the details of the experiment.

I. INTRODUCTION

In a previous paper[1] it was pointed out that an uncertainty relation for energy and time of the usual ''canonical'' type does not exist and some of the reasons why people have wanted to see such a relation were shown to be unfounded. It was also shown that there is a general uncertainty relation of a different type between energy and time which provides a satisfactory expression of the well-known relation between the lifetime and the energy spread of a quantum state. The existence of a similar relation between the momentum of a system and its position in space was mentioned briefly. In the present paper the meaning and consequences of these very general relations will be further discussed and elucidated.

As an introduction, we shall discuss in the rest of this section yet another source of confusion about the uncertainty principle of energy and time which was not mentioned in the previous paper, namely, the analogy with Fourier analysis.

Niels Bohr,[2,3] in particular, used the analogy with the Fourier analysis of a wave in space and of a time signal, respectively, to derive uncertainty relations by the following simple reasoning: A wave packet of limited extension in space and time can only be built up by a superposition of a number of elementary waves with a large range of wave numbers and frequencies. If Δx and Δt are the spatial and temporal extensions of the wave packet, and $\Delta\sigma$ and $\Delta\nu$ are the ranges of wave numbers and frequencies, then Fourier analysis tells us that $\Delta x\Delta\sigma=\Delta t\Delta\nu\approx 1$. Using the de Broglie relations $P=h\sigma$ and $E=h\nu$, one arrives at the relation $\Delta x\Delta P=\Delta t\Delta E\approx h$.[4]

Unfortunately, this simple ''derivation'' is a bit too simple. To see this, let us take a closer look at the relations between Fourier analysis and quantum mechanics.

Let $f(x)$ be a normalized function of the space coordinate and let $g(\sigma)$ be its Fourier transform:

$$f(x)=(2\pi)^{-1/2}\int g(\sigma)e^{i\sigma x}\,d\sigma,$$

then the widths of f and g are inversely related: If δx is the width of $|f(x)|^2$, the width $\delta\sigma$ of $|g(\sigma)|^2$ must at least be of the order of $1/\delta x$. This fact can be expressed mathematically in several ways by choosing suitable measures for the width of a function. If we think of $f(x)$ as representing a wave in space, the variable σ is the wave number. Thus, if a wave has extension δx in space, it must contain wave numbers in a range $\delta\sigma$ which is at least of the order of $1/\delta x$.

What is the counterpart of this in quantum mechanics? Since the observables of position and momentum of a system satisfy the commutation relation (operators are in bold type)

$$\mathbf{qp}-\mathbf{pq}=i\hbar, \tag{1}$$

they have continuous eigenvalues running from $-\infty$ to $+\infty$ on the real axis and, for any quantum state $|\Psi\rangle$, the probability amplitudes $\langle q|\Psi\rangle$ and $\langle p|\Psi\rangle$ are Fourier transforms of each other:

$$\langle q|\Psi\rangle=(2\pi\hbar)^{-1/2}\int\langle p|\Psi\rangle e^{(ipq)/\hbar}dp. \tag{2}$$

From this, there follow relations between the widths of the probability densities $|\langle q|\Psi\rangle|^2$ and $|\langle p|\Psi\rangle|^2$. In particular, taking the *standard deviation* as a measure of this width, the Heisenberg relation

$$\Delta q\Delta p\geqslant\tfrac{1}{2}\hbar \tag{3}$$

can be derived. Here,

$$(\Delta q)^2\equiv\int q^2|\langle q|\Psi\rangle|^2dq-\left(\int q|\langle q|\Psi\rangle|^2dq\right)^2,$$

and

$$(\Delta p)^2\equiv\int p^2|\langle p|\Psi\rangle|^2dp-\left(\int p|\langle p|\Psi\rangle|^2dp\right)^2.$$

We note that relation (3) may also be directly obtained from (1).

If one introduces the notations $\psi(q)\equiv\langle q|\Psi\rangle$ and $\phi(p)\equiv\langle p|\Psi\rangle$ for the wave functions in position and momentum space, respectively, the similarity of (2) with the Fourier formula becomes complete. However, there is an important shift in meaning. The function $f(x)$ is a function of the space coordinate, whereas the wave function $\psi(q)\equiv\langle q|\Psi\rangle$ is a function of the eigenvalues of the position operator $\mathbf{q}$ of a material system, e.g., a point particle. Likewise, $g(\sigma)$ is a function of the wave number, whereas the wave function $\phi(p)\equiv\langle p|\Psi\rangle$ is a function of the eigenvalues of the momentum operator $\mathbf{p}$. [This shift in meaning is obscured if one writes $\psi(x)$ instead of $\psi(q)$ for the wave function in q space, as is often done in the literature.]

The above similarity between x and σ, on the one hand, and q and p on the other, breaks down for energy and time because the energy operator $\mathbf{H}$ does not have a ''timelike'' companion operator in quantum mechanics. Since the energy eigenstates $|E\rangle$ form a complete set (we suppress degeneracies), one may consider the wave function $\langle E|\Psi\rangle$ of a state in energy space, but there is now no corresponding wave function of the state in ''time'' space. Hence, Bohr's argument cannot go through.

Let us instead consider the *time dependence* of the states. The eigenstates $|E\rangle$ have a very simple time dependence: $|E,t\rangle = |E\rangle e^{-iEt/\hbar}$. Expanding a state $|\Psi\rangle$ into energy eigenstates, we may write the time-dependent state as

$$|\Psi(t)\rangle = \int e^{-iEt/\hbar}|E\rangle\langle E|\Psi\rangle dE.$$

Thus the time-dependent wave function of the state in q space, say, is

$$\langle q|\Psi(t)\rangle = \int e^{-iEt/\hbar}\langle q|E\rangle\langle E|\Psi\rangle dE.$$

The left-hand side is the amplitude for finding the system at time t at position q in space. For a fixed value of q this amplitude is a function of time only and we may compare the above formula with the Fourier expansion of a time signal into its frequency components:

$$F(t) = (2\pi)^{-1/2}\int G(\omega)e^{-i\omega t}d\omega.$$

In this way, one still finds an analogy between energy and frequency. Again, an ''uncertainty'' relation holds[5] between the widths of $|\langle q|\Psi(t)\rangle|^2$ and $|\langle q|E\rangle\langle E|\Psi\rangle|^2$, but the first of these functions cannot be interpreted as a probability density in the variable t and the interpretation of the second function is not straightforward.[6] Moreover, unlike the functions $e^{-i\omega t}$ $(-\infty<\omega<\infty)$, the functions e^{-iEt} do not form a complete set of functions of t since E is bounded from below and the energy spectrum may be partly discrete.

Our discussion shows that the similarity between the uncertainty principles for position and momentum and for energy and time is by no means as close as Bohr's ''derivation'' suggests. Energy, momentum, and position are operators in quantum mechanics and the uncertainties in these quantities are represented as spreads of the probability distributions $|\langle E|\Psi\rangle|^2$, $|\langle p|\Psi\rangle|^2$, and $|\langle q|\Psi\rangle|^2$, respectively. On the other hand, an uncertainty in time cannot be construed this way, and a generally valid uncertainty relation of type (3) does not exist for energy and time. What, then, *is* the uncertainty principle for energy and time, if such a principle does indeed exist in quantum mechanics, and, in particular, what meaning can be given to an uncertainty in time? In the next section we shall address both problems and describe an approach which is satisfactory in several respects. (i) It is completely general. (ii) It covers the well-known physical applications and implies uncertainty relations of the usual type. (iii) It has a counterpart for ''space'' instead of ''time'' and allows a relativistically covariant formulation. (iv) It has interesting new applications.

In Sec. II the concept of an uncertainty in time in quantum mechanics is introduced using the notion of time-indicating variables or *clock* variables. A corresponding uncertainty relation of a very general character between the uncertainty in time and the total energy of the system is shown to exist and is discussed briefly. In Sec. III the results of the previous section are illustrated and further elucidated using the example of a *quantum clock*. In Sec. IV analogous results are derived for space-indicating variables or *position* variables, the total momentum of the system now taking the place of the energy. The relation with the usual type of uncertainty relations is discussed. The main result is summarized. In Sec. V the new uncertainty relations are applied to an episode from the Einstein–Bohr debate on quantum mechanics: the famous clock-in-the-box experiment.

II. UNCERTAINTY IN TIME IN QUANTUM MECHANICS AND THE UNCERTAINTY PRINCIPLE FOR ENERGY AND TIME

In classical as well as in quantum physics, excepting general relativity, space and time are assumed to be given as a fixed background for the description of physical systems. Although it is plausible that the notions of infinitely extended homogeneous and isotropic space and time are themselves abstractions somehow ''derived'' from the properties of material systems like rods and clocks, they appear in the usual formulation of physical theories as *given*. For example, the important conservation laws of energy, momentum, and angular momentum follow from the symmetry of this space–time background under translations and spatial rotations, respectively. Physical systems are situated in space and time; their dynamical state is characterized by dynamical variables like position, momentum, energy, and angular momentum, or the corresponding densities, which, in general, will depend on the space–time coordinates. In quantum mechanics the dynamical variables are turned into operators. On the other hand, the space–time coordinates are parameters in both quantum mechanics and classical mechanics; they do not correspond to operators. We shall denote the Cartesian coordinates of a point in space–time by x_1, x_2, x_3, t. By *time* we simply mean the time coordinate t.

A *clock* is a physical system possessing a dynamical variable, say a pointer position, the time dependence of which is particularly simple so that the value of t may be directly inferred from the value of the dynamical variable. We shall call such a variable a *time-indicating* variable or *clock* variable. Usually, the time dependence of such a clock variable is periodic. For example, let the direction of the hand of a clock be given by the angle ϕ in the interval $[0,2\pi]$. In classical physics this angle is a simple function of t: $\phi(t) = \omega t (\mathrm{mod}\, 2\pi)$. The value of t can be directly inferred $(\mathrm{mod}\, 2\pi/\omega)$ from the value of ϕ. In *quantum mechanics*, a clock variable, being a dynamical variable, is represented by an operator. In an arbitrary quantum state of the clock the outcomes of a measurement of the angle ϕ will now have a spread and, as a consequence, there will be an uncertainty in the time t. This comes about as follows. Let $|\phi\rangle$ denote the eigenstates of the angle operator $\mathbf{\Phi}$ (see the next section for a discussion of this operator and its eigenstates). The distribution $|\langle\phi|\Psi_t\rangle|^2$ of ϕ values in a quantum state $|\Psi_t\rangle$ at time t, and the distribution $|\langle\phi|\Psi_{t+\tau}\rangle|^2$ of ϕ values at a later time $t+\tau$ will, in general, show some overlap. This means that from the result of a measurement of $\mathbf{\Phi}$, the state, and therefore the value of the time coordinate, cannot be inferred with certainty. Here, then, we encounter an uncertainty in time of a purely quantum mechanical character: It is an uncertainty in the *parameter t* arising from the *quantum* character of the time-indicating dynamical variables. In the following we shall see that this kind of uncertainty is relevant in many physical situations.

Can we quantify this uncertainty? The uncertainty in t will clearly depend on the time τ it takes for the distributions of ϕ values at times t and $t+\tau$ to become distinguishable. A measure for this time will be the time it takes for the states $|\Psi_t\rangle$ and $|\Psi_{t+\tau}\rangle$ to become orthogonal.[7] However, it is not always practical to require total distinguishability. We shall, therefore, allow the two states to have some overlap. This rules out the possibility of distinguishing them with complete certainty, but it does allow one to make a distinction with a certain *degree of reliability*. Let us write, $|\Psi\rangle$ and $\mathbf{U}(\tau)|\Psi\rangle$ instead of $|\Psi_t\rangle$ and $|\Psi_{t+\tau}\rangle$, where $\mathbf{U}(t)=\exp(-i\mathbf{H}t)$ is the unitary operator of the evolution in time. The operator $\mathbf{H}$ is the Hamiltonian operator of the system and we have put $\hbar=1$. Let us then define τ_ρ as the smallest time at which the absolute value of the overlap integral between the two states has decreased to the value $1-\rho$:

$$|\langle\Psi|\mathbf{U}(\tau_\rho)|\Psi\rangle|=1-\rho, \quad 0\leqslant\rho\leqslant 1. \tag{4}$$

We shall call ρ the *reliability* with which the states $|\Psi\rangle$ and $\mathbf{U}(\tau_\rho)|\Psi\rangle$ can be distinguished. If the states coincide, this reliability is 0, whereas it attains its maximum value 1 when the states are orthogonal. Furthermore, for a system in the state $|\Psi\rangle$, we shall say that from a measurement of some given dynamical variable the time t may (at best) be inferred with an *uncertainty* τ_ρ with *reliability* ρ. Again, some dynamical variables are better suited to the task of distinguishing two states than others; τ_ρ refers to the *optimal* ones in this respect. Alternatively, $(\tau_\rho)^{-1}$ might be called the *time resolution* at reliability ρ of the measurement. Clearly, if one demands greater reliability from the time measurement, the resolution becomes poorer. These notions will be further illustrated by the example in the next section. We refer to Sec. IV for a discussion of the analogy of τ_ρ with the notion of resolving power in optics.

There now turns out to be a direct relationship between τ_ρ and the width of the *energy* distribution of the state. Inserting a complete set of energy eigenstates $|E\rangle$ into the matrix element in (4) we find that it is the Fourier transform of the probability density $|\langle E|\Psi\rangle|^2$:

$$\langle\Psi|\mathbf{U}(\tau)|\Psi\rangle=\int|\langle E|\Psi\rangle|^2e^{-i\tau E}dE. \tag{5}$$

(The integral may include a summation over discrete eigenvalues.) From (5) one can derive an uncertainty relation between τ_ρ and the width of the energy distribution. As a measure of this width one could take the standard deviation ΔE, but the standard deviation is generally not a suitable measure of the width of a quantum mechanical probability distribution: It diverges in many common cases (remember that a quantum mechanical probability distribution need not be Gaussian, its shape is quite arbitrary).[8] A suitable measure of width may be defined as follows. Let W_α^E be the size of the smallest energy interval W such that

$$\int_W|\langle E|\Psi\rangle|^2dE=\alpha.$$

Then, W_α^E is a reasonable measure for the uncertainty in energy if α is less than but close to 1. For example, if $\alpha=0.9$, then W_α^E is the smallest interval on which 90% of the energy distribution is situated. It can now be shown[9] that

$$\tau_\rho W_\alpha^E\geqslant 2\hbar\ \arccos\left(\frac{2-\alpha-\rho}{\alpha}\right), \quad \text{for } \rho\geqslant 2(1-\alpha). \tag{6}$$

This uncertainty relation holds for all states $|\Psi\rangle$. The only assumptions needed for its validity are the existence of the time-translation operator $\mathbf{U}(t)$ and the completeness of the energy eigenstates. We thus find that if the system is in a state which permits the time to be inferred with an uncertainty τ_ρ, the width W_α^E of the energy distribution of the state cannot be smaller than is allowed by the inequality (6). For sensible values of the parameters, say $\alpha=0.9$ or $\alpha=0.8$, and $0.5\leqslant\rho\leqslant 1$, the right-hand side of (6) is of the order of $\hbar$.

In fact, (6) comprises a whole set of relations. For example, taking $(1-\rho)=\sqrt{1/2}$ and $\alpha=0.9$ and writing $T_{1/2}$ for the corresponding τ_ρ, one finds

$$T_{1/2}W_{0.9}^E\geqslant 0.9\hbar. \tag{7}$$

From (4) we see that $T_{1/2}$ is the so-called *half-life* of the state, the (smallest) time at which the probability of finding the system in its original state has decreased to 50%. Thus (7) expresses the well-known relation between the lifetime of a quantum state and the width of its energy spectrum. This relation is usually obtained in the approximation in which the decay of the state is exponential, but we now see that the relation is completely general. As an extreme case, consider an eigenstate of the energy; then $W_\alpha^E=0$, and by (6) $\tau_\rho=\infty$, in agreement with the fact that, in this case, nothing changes.

III. THE QUANTUM CLOCK

Let us now return to our simple clock,[10] having a single hand, and let us see how the above results work out in this case. In quantum mechanics the angle-variable ϕ is represented by an operator $\mathbf{\Phi}$. The Hilbert space in the angle representation consists of the square-integrable functions f of ϕ on the interval $[0,2\pi]$. The operators of angle and angular momentum are represented by ($\hbar=1$)

$$\mathbf{\Phi}f(\phi)=\phi f(\phi),$$

$$\mathbf{L}f(\phi)=-i\frac{d}{d\phi}f(\phi).$$

The operator $\mathbf{\Phi}$ is self-adjoint on the whole Hilbert space, whereas $\mathbf{L}$ is self-adjoint on the subspace of the square-integrable, differentiable, functions satisfying $f(0)=f(2\pi)$. These operators have complete, orthonormal sets of generalized eigenstates $|\phi\rangle$ and $|m\rangle$:

$$\mathbf{\Phi}|\phi\rangle=\phi|\phi\rangle, \quad \langle\phi|\phi'\rangle=\delta(\phi-\phi'),$$

$$\mathbf{L}|m\rangle=m|m\rangle, \quad \langle m|m'\rangle=\delta_{m,m'},$$

where the eigenvalue ϕ runs through the interval $[0,2\pi]$ and $m=0,\pm1,\pm2,\ldots$.

In the ϕ representation the states $|\phi\rangle$ and $|m\rangle$ are represented by the wave functions $\langle\phi|\phi'\rangle=\delta(\phi-\phi')$ and $\langle\phi|m\rangle=(2\pi)^{-1/2}e^{-im\phi}$, respectively. The situation is very similar to that of position and momentum, except for the fact that the interval on which the functions $f(\phi)$ are defined is finite. In particular, we have

$$|\phi\rangle=(2\pi)^{-1/2}\sum_{-\infty}^{+\infty}e^{im\phi}|m\rangle, \tag{8}$$

where the sum (not an integral now) runs over all values of m.

The dynamics of the system is introduced by specifying the Hamiltonian; we put $\mathbf{H}=\omega\mathbf{L}$, where ω is a constant frequency. With the help of (8) we find

$$\mathbf{U}(t)|\phi\rangle=e^{-i\mathbf{H}t}|\phi\rangle$$
$$=(2\pi)^{-1/2}\sum e^{im\phi-im\omega t}|m\rangle$$
$$=|\phi-\omega t\rangle. \quad (9)$$

This is precisely the behavior to be expected of the hand on an ideal clock: It rotates at constant angular velocity and after an arbitrarily short time an eigenstate $|\phi\rangle$ of the hand position goes over into an orthogonal state, that is, the clock, in this state, has zero uncertainty (i.e., infinite resolution) at maximum reliability: $\tau_1=0$. Accordingly, by (6), the width of the energy distribution of the states $|\phi\rangle$ must be infinite: $W_\alpha^E=\infty$. This is indeed the case: The energy eigenvalues are ωm, with m running over all integers; hence, the spectrum extends from $-\infty$ to $+\infty$, and all eigenstates $|m\rangle$ of $\mathbf{H}$ appear in $|\phi\rangle$ with equal amplitude.

A more realistic case is obtained if we require the energy to be bounded. We then restrict the sum in (8) to values of m which satisfy the condition $-l\leqslant m\leqslant l$, where l is a positive integer, and consider the states

$$|\theta\rangle=(2l+1)^{-1/2}\sum_{m=-l}^{l} e^{im\theta}|m\rangle, \quad \theta\in[0,2\pi].$$

The wave functions in the ϕ representation are[11]

$$\langle\phi|\theta\rangle=(2\pi)^{-1/2}(2l+1)^{-1/2}\sum_{-l}^{l} e^{im(\theta-\phi)}$$
$$=[2\pi(2l+1)]^{-1/2}\frac{\sin[(2l+1)(\theta-\phi)/2]}{\sin[(\theta-\phi)/2]}. \quad (10)$$

Here, use is made of the identity

$$\sum_{m=-l}^{l} x^m\equiv\frac{x^{l+1/2}-x^{-l-1/2}}{x^{1/2}-x^{-1/2}}.$$

The functions (10) are no longer δ functions of ϕ but peak around the value $\phi=\theta$ and have width $\approx 2\pi/(2l+1)$. For the time evolution we again find

$$\mathbf{U}(t)|\theta\rangle=|\theta-\omega t\rangle.$$

Unlike the states (8), the states $|\theta\rangle$ and $|\theta-\omega t\rangle$ overlap for short times, but they become orthogonal after time intervals which are multiples of $\tau=2\pi[(2l+1)\omega]^{-1}$ $(l\neq 0)$. Thus the $(2l+1)$ states $|\theta+2\pi k/(2l+1)\rangle$, with $k=-l,\ldots,l$, form a complete set of orthonormal states in the space spanned by the states $|m\rangle$ with $-l\leqslant m\leqslant l$. Our clock, in a state $|\theta\rangle$, is no longer ideal; its time resolution at maximum reliability is $\tau_1=2\pi[(2l+1)\omega]^{-1}$. On the other hand, the width of the energy distribution of the states $|\theta\rangle$ is no longer infinite: $W_\alpha^E=(2l+1)\alpha\omega$. Of course, the product $\tau_1\, W_\alpha^E$ satisfies the uncertainty relation (6).

The upshot of the above discussion is that there is an uncertainty relation between the accuracy with which a physical system can indicate time and the width of the energy distribution of its quantum state. This is expressed quantitatively by relation (6). This uncertainty relation does not, in itself, rule out the existence of an ideal clock; it says only that the energy spread of an ideal clock must be infinite.

Note. Relations of type (6) were first derived by Uffink,[12,13] and their meaning has been studied by Hilgevoord and Uffink[8,14,15] (cf. also the next section). A discussion of the quantum clock in much the same spirit as ours has been given by Busch.[16] We may also refer to his article for an excellent survey of the whole subject of the uncertainty principle for energy and time.

IV. THE ANALOGY BETWEEN SPACE AND TIME

The above discussion of the uncertainty in time can be repeated, almost word for word, with respect to the space coordinates. For simplicity, let us consider just one space coordinate x. Like the time coordinate, the space coordinate must be sharply distinguished from dynamical variables attached to a physical system and indicating the position of the system in space.[17] As the simplest example, consider the position-variable q of a point particle. The numerical relation between this dynamical variable and the space coordinate is very simple: $q=x$. That is, from the value of q the value of x can be inferred directly. This simple relation is to be compared with the simple relation $\phi=\omega t$ between a classical clock variable and the time coordinate in Sec. II. Again, in quantum mechanics, q is represented by an operator whereas x remains a c number. Let $\mathbf{U}(x)=\exp(i\mathbf{P}x)$ be the unitary operator of translations in space. Here, $\mathbf{P}$ is the operator of the total momentum of the system. The matrix element $\langle\Psi|\mathbf{U}(x)|\Psi\rangle$ is the overlap integral between the state $|\Psi\rangle$ and the space-translated state $\mathbf{U}(x)|\Psi\rangle$. Inserting a complete set of momentum eigenstates $|P\rangle$ we get the analogue of Eq. (5):

$$\langle\Psi|\mathbf{U}(\xi)|\Psi\rangle=\int|\langle P|\Psi\rangle|^2 e^{i\xi P}dP. \quad (5')$$

One can now define an uncertainty ξ_ρ in x in the same way as we defined an uncertainty in t:

$$|\langle\Psi|\mathbf{U}(\xi_\rho)|\Psi\rangle|=1-\rho, \quad 0\leqslant\rho\leqslant 1. \quad (4')$$

Let us call ξ_ρ the (spatial) *translation width* of the state. The translation width is a measure of the place-indicating capacity of the state just as τ_ρ is a measure of the time-indicating capacity of the state. The best place-indicating states have the smallest translation widths. In particular, the eigenstates of position variables like $\mathbf{q}$ are optimal in this respect: By an arbitrarily small translation an eigenstate of $\mathbf{q}$ goes over into an orthogonal eigenstate: $\mathbf{U}(\xi)|q\rangle=|q-\xi\rangle$. [Compare (9)]. Hence, for these states, $\xi_1=0$. The position variables, therefore, are ideal place-indicating variables of the system, just as the clock variables of an ideal clock are ideal time-indicating variables. We have seen that the existence of ideal clocks is impossible if we require the energy to be bounded. The same is true of the place-indicating variables in systems where the total momentum is bounded. On the other hand, if the spectrum of the total momentum is the whole real axis, all position variables are ideal place-indicating variables.

From (5′) and (4′), there follows the uncertainty relation

$$\xi_\rho W_\alpha^P\geqslant 2\hbar\ \arccos\left(\frac{2-\alpha-\rho}{\alpha}\right) \quad \text{for } \rho\geqslant 2(1-\alpha), \quad (6')$$

where the width W_α^P of the momentum distribution is defined as the smallest momentum interval W such that

$$\int_W |\langle P|\Psi\rangle|^2 dP = \alpha.$$

The uncertainty relation (6′) holds for all states under the sole condition that the generator **P** of translations in space exists and has a complete set of eigenstates.

We now consider some consequences of (6′). Suppose that the total momentum of the system is reasonably well determined, i.e., W_α^P is small. Then, the translation width ξ_ρ of the state must be large. This implies that the spread in *all* position variables must be large. Hence, it is the *total* momentum which determines whether or not the position variables of the system can be sharply determined. To illustrate this, let us consider a system with two position variables **q** and **Q**, e.g., a two-particle system. The eigenstates of the total momentum can be written in the form

$$\psi(q,Q) = \chi(q-Q)e^{iP(q+Q)/2}, \tag{11}$$

where χ is an arbitrary wave function and the eigenvalue is P. Note that (11) need not be an eigenstate of the momenta of the separate particles. The probability density in q is $\int|\psi(q,Q)|^2\, dQ = \int|\chi(q-Q)|^2\, dQ$, which is uniform in q. Evidently, the same is true for Q. Hence, if the total momentum is certain, both q and Q are completely uncertain. On the other hand, if the state is an eigenstate of **q**, say, then it has the form

$$\psi(q,Q) = \delta(q-a)\chi(q,Q),$$

where χ is an arbitrary wave function. Every finite translation of the whole system transforms this state into an orthogonal one; hence, the translation width of this state is zero. From (6′), it follows that the width W_α^P of the momentum distribution of this state must be infinite. To verify this, we consider the wave function of the above state in momentum space:

$$\varphi(k,K) = (2\pi)^{-1}\int\int \delta(q-a)\chi(q,Q)e^{-ikq-iKQ}dq\,dQ$$

$$= (2\pi)^{-1}e^{ik\alpha}\int \chi(a,Q)e^{-iKQ}dQ.$$

One sees that $|\varphi(k,K)|^2$ depends only on K: $|\varphi(k,K)|^2 = F(K)$. The probability of finding total momentum P is $\int\int\delta(k+K-P)|\varphi(k,K)|^2\, dk\, dK = \int F(K)\, dK$, which is independent of P. Thus the probability distribution of the total momentum is uniform and, hence, W_α^P is infinite.

We conclude: If the total momentum is certain, *all* position variables are completely uncertain; if at least one position variable is certain, the total momentum is completely uncertain. All this is similar to the relation between clock variables and the total energy.

Since **q** is an ideal place-indicating variable it can be used to reexpress the translation width of a state. For simplicity, consider the one-dimensional case. Inserting a complete set of eigenstates $|q\rangle$ of **q** into the matrix element $\langle\Psi|\mathbf{U}(\xi)|\Psi\rangle$ gives

$$\langle\Psi|\mathbf{U}(\xi)|\Psi\rangle = \int \psi^*(q)\psi(q-\xi)dq, \tag{12}$$

where $\psi(q) \equiv \langle q|\Psi\rangle$ is the wave function of the state in q space. Hence, alternatively, we could have introduced ξ_ρ as the smallest displacement such that $|\int\psi^*(q)\psi(q-\xi_p)dq| = 1-\rho$. However, the definition (4′) is more general because position variables of a system do not always exist. An example is provided by the photon.[18] In such a case, definition (4′) can still be used!

One may wonder what the relationship is between the uncertainty relation (6′) and the usual uncertainty relation (3). Suppose that $\psi(q)$ is a simple wave packet having a single pronounced peak of width a. Then, as is clear from (12), $\xi_\rho \approx a$ for values of ρ close to 1. That is, for such wave packets, the translation width is of the same order of magnitude as the standard deviation and other measures of total width like W_α. For such wave functions the relations (3) and (6′) are roughly equivalent. However, if the wave function consists of several equidistant peaks of width b, as in an interference pattern, then, clearly, $\xi_\rho \approx b$, whereas the standard deviation and measures like W_α, which are not sensitive to the fine structure of the wave function, are still of the, much bigger, order of the total width a. It follows that the uncertainty relation (6′) is essentially *stronger* than relations of type (3).

There is a close similarity between the notion of the translation width and the notion of the *resolving power* of an optical system. The resolving power of an optical instrument is defined as the minimum distance between two point sources such that the images of the point sources formed by the instrument can still be distinguished. These images are diffraction patterns which are displaced relative to each other in the image plane. The usual criterion for their distinguishability, viz. Rayleigh's criterion, rests on the assumption that the patterns have a single pronounced peak. The condition (4′) may be considered as a generalization of Rayleigh's criterion for the distinguishability of two states. If an optical instrument is to have a great resolving power, the translation width of the images it produces must be small!

Likewise, condition (4) can be seen as a generalization of the time analogue of Rayleigh's criterion. Suppose that, for a *fixed* value of q, the wave function $\langle q|\Psi(t)\rangle$, as a function of t, has a simple peak of duration T. This would happen if, for instance, $\langle q|\Psi(t)\rangle$ is a Gaussian wave packet moving through space with negligible dispersion and passing through q. Then, for values of ρ close to 1, τ_ρ is of the order of T, and from (6) it follows that the width of the energy distribution of the state must be at least of the order of $\hbar/T$. Thus one recovers the well-known relation, mentioned in the Introduction, between the duration of a signal and the width of its energy or frequency spectrum. Again, if the signal has a fine structure, condition (4) will ''see'' this fine structure and τ_ρ will be smaller than the total width of the signal. Hence, relation (6) is stronger than the usual relationship between the duration of a signal and its frequency spread.

We may summarize the results of this section as follows.

The uncertainty relation (6) expresses, for a given quantum state, the reciprocal relationship between the spread in the total energy of the system and the accuracy with which the time-indicating dynamical variables (''clock variables'') of the system can indicate the time.

The uncertainty relation (6′) expresses, for a given quantum state, the reciprocal relationship between the spread in the total momentum of the system and the accuracy with which the place-indicating dynamical variables (''position variables'') of the system can indicate the place.

Here, ''time'' and ''place'' refer to the space–time reference frame with respect to which the system is described.

In particular, it follows that if the total energy (momentum) of a system is known precisely, all clock variables (position variables) are completely uncertain.

Finally, we note that in a *relativistic* quantum theory, space and time are united into space–time, and the coordinates x_1, x_2, x_3, t are the components of a 4-vector. Likewise, the operators $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$ of translations in three-dimensional space and the operator $\mathbf{H}$ of translations in time form a 4-vector operator. In this way the uncertainty relations (6) and (6′) obtain a joint covariant basis.

V. THE PHOTON BOX: AN EPISODE FROM THE EINSTEIN–BOHR DEBATE

In his contribution to the Einstein Seventieth Birthday Volume: ‘‘Albert Einstein: Philosopher-Scientist’’ [3] entitled ‘‘Discussion with Einstein on Epistemological Problems in Atomic Physics,’’ Bohr gave a careful exposition of his views on quantum mechanics. He dealt extensively with both the uncertainty principles for momentum and position, and for energy and time, and in particular with arguments, raised by Einstein, which question the consistency of his (Bohr's) reading of these principles.[19] Einstein's stand on the issue is not completely clear. Presumably, Einstein would not have objected to a statistical interpretation of the principle in which the uncertainties are interpreted as spreads in the outcomes of a long series of independent measurements. But for Bohr the principle also applies to individual quantum systems. For Bohr the uncertainties symbolize the bounds within which quantities like momentum and position or energy and time can be considered to be simultaneously determined in an individual system. Against this view Einstein advanced several counterexamples. Undoubtedly, the most famous of these is the photon-box or clock-in-the-box experiment (1930) pertaining to the uncertainty principle of energy and time. There is still some discussion[20] about the question of whether Einstein directed this thought experiment, in the first instance, against Bohr's interpretation of the uncertainty principle, as Bohr assumes he did, or that he had something different in mind, namely, a ‘‘locality’’ type argument of the kind put forward, five years later, in the famous Einstein–Podolsky–Rosen paper. We will not enter this discussion but follow Bohr and take the photon-box experiment as being directed against the uncertainty principle of energy and time.

Let us recall the idea of the photon-box experiment. A box containing some radiation has a hole in one of its walls which can be opened and closed by a shutter controlled by a clock inside the box. At a preset time the clock opens the shutter for a short interval of time during which a single photon escapes. In this way the time of escape of the photon can be determined with arbitrary precision. Since energy is equivalent to inertial mass, and mass has weight, the energy of the box can be determined by weighing it in a gravitational field. Hence, by weighing the box before and after the escape of the photon, the energy of the photon can also be found with arbitrary precision. But this violates the uncertainty principle of energy and time. Here, Einstein used relativity theory to show the inconsistency of Bohr's interpretation of the quantum mechanical uncertainty principle! Bohr refuted Einstein by a famous argument in which, for his part, he used the relativistic effect of a gravitational field on the rate of the clock in the box. I reproduce his argument[21] (cf. Fig. 1):[22]

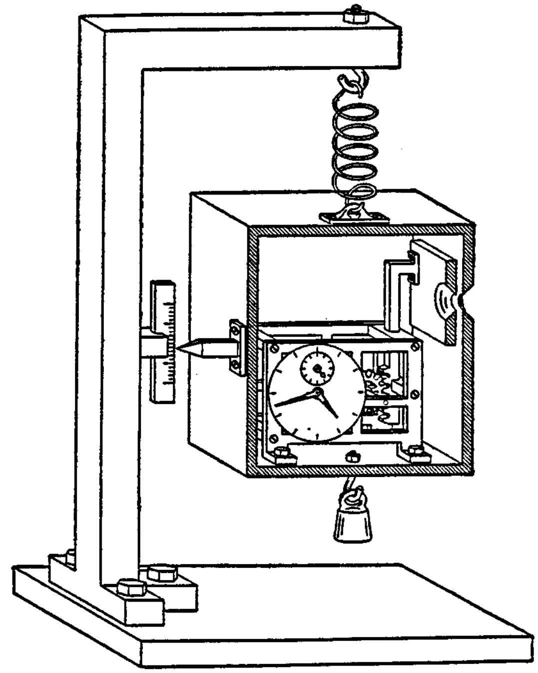

Fig. 1. The photon box experiment.

‘‘The box,..., is suspended in a spring-balance and is furnished with a pointer to read its position on a scale fixed to the balance support. The weighing of the box may thus be performed with any given accuracy Δm by adjusting the balance to its zero position by means of suitable loads. The essential point is now that any determination of this position with a given accuracy Δq will involve a minimum latitude Δp in the control of the momentum of the box connected with Δq by the relation $\Delta q \Delta p \approx h$. This latitude must obviously again be smaller than the total impulse which, during the whole interval T of the balancing procedure, can be given by the gravitational field to a body with mass Δm, or

$$\Delta p \approx h/\Delta q < Tg\Delta m \qquad \text{(a)}$$

where g is the gravity constant. The greater the accuracy of the reading q of the pointer position, the longer must, consequently, be the balancing interval T, if a given accuracy Δm of the weighing of the box with its content shall be obtained.

Now, according to general relativity theory, a clock, when displaced in the direction of the gravitational force by an amount Δq, will change its rate in such a way that its reading in the course of a time interval T will differ by an amount ΔT given by the relation

$$\Delta T/T = c^{-2} g \Delta q. \qquad \text{(b)}$$

By comparing (a) and (b) we see, therefore, that after the weighing procedure there will in our knowledge of the adjustment of the clock be a latitude

$$\Delta T > h/c^2 \Delta m.$$

Together with the formula $E = mc^2$, this relation again leads to

$$\Delta T \Delta E > h,$$

in accordance with the indeterminacy principle. Consequently, a use of the apparatus as a means of accurately measuring the energy of the photon will prevent us from controlling the moment of its escape.''

This ingenious argument has been applauded by many,[23,24] but it has failed to convince many others.[25,26] It is interesting to note that, whereas the original discussion with Einstein took place in 1930, Bohr did not publish his rebuttal until 1948 and, apparently, went on thinking about the problem, for, on the last day of his life (18 November 1962), he sketched the photon box on the blackboard in his home.[27] It is not my purpose, here, to analyze the argument which would surely be no easy task. I only remark that Bohr fixed his attention on the procedure by which the energy of the box is *measured* rather than on the internal dynamics of the system. In particular, he did not apply quantum mechanics to the clock itself but only to the vertical motion of the box's center of mass, and the uncertainty principle of position and momentum is called on to save the uncertainty principle of energy and time. Under close scrutiny Bohr's argument raises many questions which, in my view, have never been satisfactorily answered.

If, however, the results of the previous sections are applied directly to the clock-shutter system, the problem gets an easy solution. The shutter may be considered as the hand of a clock passing over the hole. Now, if one wants to determine the energy of the escaping photon by a subsequent measurement of the energy of the box, the total initial energy of the system must be accurately known. But from the general result stated at the end of the previous section it follows that all time-indicating dynamical variables of the system must then be unsharp, and, in particular, the time when the shutter opens the hole is uncertain. Therefore, under the circumstance that the energy of the photon can be determined by weighing the box afterwards, the time of escape is uncertain, the uncertainties in energy and time being related by relation (6). Hence, quite independent of the way the energy of the box is measured, Einstein's idea of violating the uncertainty principle by appealing to the conservation law of energy cannot succeed.

We note that Bohr's relation (b) depends on the classical redshift formula relating the time shown by a clock in a gravitational field and coordinate time.[28] It is precisely the relation between the clock time and the coordinate time which, in the previous sections, was found to become uncertain if the energy of the system is accurately known. From this point of view the uncertainty ΔT may be seen as an extra uncertainty, arising from the way the energy is measured, in addition to the uncertainty τ_ρ which, on account of relation (6), is already present in the initial state.

In the same article Bohr discussed a proposal of Einstein's to violate the uncertainty principle of position and momentum by using the conservation law of momentum.[29] It will not come as a surprise that this proposal also fails on account of our general uncertainty relation between the total momentum and all place-indicating dynamical variables of the system.

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[7]It should be noted that, even when the states are orthogonal, a measurement of ϕ may not be capable of distinguishing with certainty between them, i.e., the distributions of ϕ values in the two states may still overlap. For example, a measurement of the spin component of a spin one-half particle in any direction other than the z direction does not allow one to distinguish with certainty between the two orthogonal eigenstates of the spin in the z direction. However, a measurement of the spin component in the z direction does. Generally, if two states are orthogonal, there are variables a measurement result of which enables one to distinguish with certainty between the states.
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