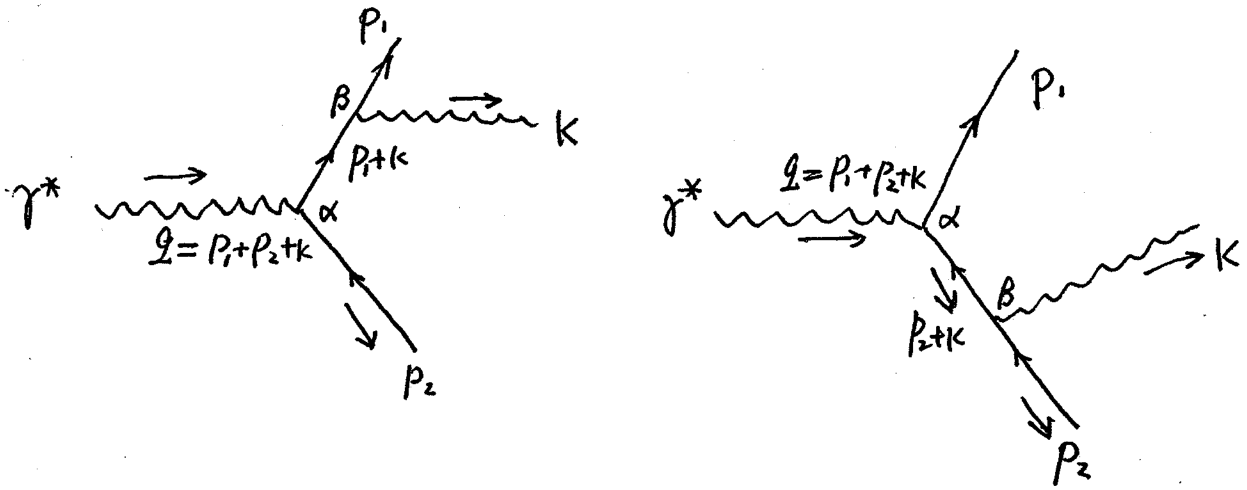
Consider QED radiation in Soft limit



$$i\mathcal{M} = \bar{u}(p_1,h_1)\Big((-ie\gamma^\beta)\,\epsilon^*_\beta(k)\cdot\frac{i(\not{p}_1+\not{k})}{(p_1+k)^2+i\epsilon}\cdot(-ie\gamma^\alpha)$$

$$+(-ie\gamma^\alpha)\cdot\frac{-i(\not{p}_2+\not{k})}{(p_2+k)^2+i\epsilon}\cdot(-ie\gamma^\beta)\,\epsilon^*_\beta(k)\Big)\, v(p_2,h_2)$$

$$=\bar{u}(p_1,h_1)\Big(\gamma^\beta\cdot\frac{\not{p}_1+\not{k}}{(p_1+k)^2+i\epsilon}\cdot(-ie\gamma^\alpha)$$

$$-(-ie\gamma^\alpha)\frac{\not{p}_2+\not{k}}{(p_2+k)^2+i\epsilon}\cdot\gamma^\beta\Big)\cdot e\,\epsilon^*_\beta\, v(p_2,h_2)$$

$$\overset{k\to 0}{=} \bar{u}(p_1,h_1)\left[\gamma^\beta\cdot\frac{\not{p}_1}{2p_1\cdot k}(-ie\gamma^\alpha) - (-ie\gamma^\alpha)\frac{\not{p}_2}{2p_2\cdot k}\cdot\gamma^\beta\right] e\,\epsilon^*_\beta\, v(p_2,h_2)$$

From the anticommutation relations of the Dirac matrices and the on-shell conditions, we obtain

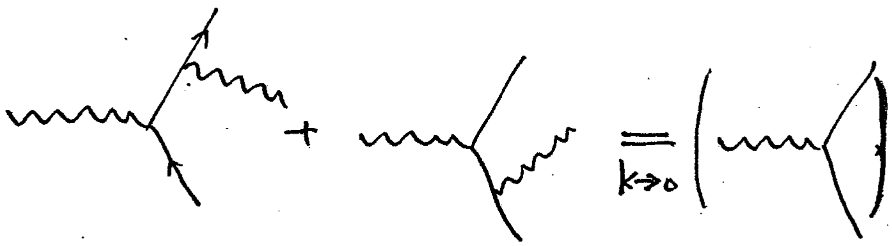
$$\bar{u}(p_1,h_1)\,\gamma^\beta\not{p}_1 = \bar{u}(p_1,h_1)\cdot 2p_1^\beta$$

$$\not{p}_2\,\gamma^\beta\, v(p_2,h_2) = 2p_2^\beta\cdot v(p_2,h_2)$$

Thus, when $k \to 0$

$$iM = \bar{u}(p_1, h_1)(-ie\gamma^\alpha)v(p_2, h_2)\cdot e\left[\frac{p_1^\beta\cdot\epsilon_\beta^*(k)}{p_1\cdot k} - \frac{p_2^\beta\cdot\epsilon_\beta^*(k)}{p_2\cdot k}\right]$$

$$= iM_0\cdot e\left[\left(\frac{p_1^\beta}{p_1\cdot k} - \frac{p_2^\beta}{p_2\cdot k}\right)\cdot\epsilon_\beta^*(k)\right]$$



$$\times e\left[\left(\frac{p_1^\beta}{p_1\cdot k} - \frac{p_2^\beta}{p_2\cdot k}\right)\cdot\epsilon_\beta^*(k)\right]$$

Let the virtual photon 4-momentum is

$$q = p_1 + p_2 + k$$

with

$$E_{cm}^{(3)} = Q = \sqrt{q^2}$$

The 3-body differential decay rate is

$$d\Gamma = \frac{1}{2E_{cm}}|M|^2 d\Phi_3$$

and the 3-body phase-space factor is given by

$$d\Phi_3 = \frac{d^3p_1}{(2\pi)^3(2E_1)}\cdot\frac{d^3p_2}{(2\pi)^3(2E_2)}\cdot\frac{d^3k}{(2\pi)^3(2E_k)}\cdot(2\pi)^4\delta^{(4)}(q-p_1-p_2-k)$$

When $k\to 0$, $\delta^{(4)}(q-p_1-p_2-k) \longrightarrow \delta^{(4)}(q-p_1-p_2)$

$$d\Phi_3 = \frac{d^3p_1}{(2\pi)^3(2E_1)}\cdot\frac{d^3p_2}{(2\pi)^3(2E_2)}\cdot(2\pi)^4\delta^{(4)}(q-p_1-p_2)\cdot\frac{d^3k}{(2\pi)^3(2E_k)}$$

$$= d\Phi_2\cdot\frac{d^3k}{(2\pi)^3(2E_k)}$$

Then,

$$d\Gamma(\gamma^* \to l^+ l^- \gamma) = \frac{1}{2E_{cm}} \overline{|M|^2} d\Phi_3$$

$$= \frac{1}{2E_{cm}} \overline{|M_0|^2} d\Phi_2 \cdot \frac{d^3k}{(2\pi)^3 (2E_k)} \cdot e^2 \sum_{\lambda=1,2} \left| \left(\frac{p_1{}^\beta}{p_1 \cdot k} - \frac{p_2{}^\beta}{p_2 \cdot k} \right) \cdot \epsilon_\beta^{*(\lambda)}(k) \right|^2$$

$$= d\Gamma(\gamma^* \to l^+ l^-) \cdot \frac{d^3k}{(2\pi)^3 (2E_k)} \cdot e^2 \cdot \sum_{\lambda=1,2} \left| \epsilon^{*(\lambda)} \cdot \tilde{j}(k) \right|^2$$

where

$$\tilde{j}(k) = \frac{p_1{}^\beta}{p_1 \cdot k} - \frac{p_2{}^\beta}{p_2 \cdot k}$$

2.

$$\sum_{\lambda=1,2} \left| \epsilon^{*(\lambda)} \cdot \tilde{j}(k) \right|^2 = \sum_\lambda \epsilon_\mu^* \epsilon_\nu \cdot \tilde{j}^\mu(k) \tilde{j}^{\nu*}(k),$$

For simplicity, we consider the case of a real massless photon propagating along the $\hat{z}$-axis with 4-momentum k given by

$$k^\mu = (k, 0, 0, k)$$

and the two transverse polarization 4-vector be chosen to be.

$$\epsilon_1{}^\mu = (0, 1, 0, 0)$$
$$\epsilon_2{}^\mu = (0, 0, 1, 0)$$

With these conventions, we have

$$\sum_\lambda \left| \epsilon^*(k) \cdot \tilde{j}(k) \right|^2 = \left| \tilde{j}^{(1)}(k) \right|^2 + \left| \tilde{j}^{(2)}(k) \right|^2$$

The current conservation requires

$$k \cdot \tilde{j}(k) = 0$$

In the frame chosen above,

$$k \cdot \tilde{j}(k) = k \cdot \tilde{j}^{(0)}(k) - k \cdot \tilde{j}^{(3)}(k) = 0$$

$$\Rightarrow \tilde{j}^{(0)}(k) = \tilde{j}^{(3)}(k)$$

So,

$$\sum_{\lambda} \left| \epsilon^{*}(k) \tilde{j}(k) \right|^2 = |\tilde{j}^{(1)}|^2 + |\tilde{j}^{(2)}|^2$$

$$= |\tilde{j}^{(1)}|^2 + |\tilde{j}^{(2)}|^2 + |\tilde{j}^{(3)}|^2 - |\tilde{j}^{(0)}|^2$$

$$= - g_{\mu\nu} \cdot \tilde{j}^{\mu}(k) \cdot \tilde{j}^{\nu *}(k)$$

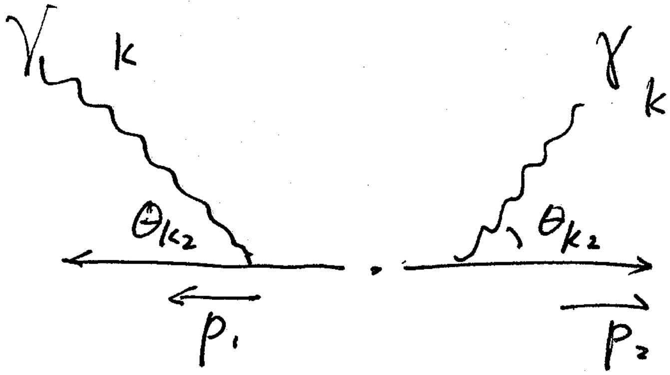
$$\Rightarrow \sum_{\lambda=1,2} \left| \frac{\varepsilon^{(\lambda)} \cdot p_1}{p_1 \cdot k} - \frac{\varepsilon^{(\lambda)} \cdot p_2}{p_2 \cdot k} \right|^2 = - \left(\frac{p_1}{p_1 \cdot k} - \frac{p_2}{p_2 \cdot k} \right)^2$$

$$= \frac{2\, p_1 \cdot p_2}{(p_1 \cdot k)(p_2 \cdot k)} - \frac{m_1^2}{(p_1 \cdot k)^2} - \frac{m_2^2}{(p_2 \cdot k)^2}$$

For $m_1 = m_2 = 0$, the soft photon factor is

$$\frac{d^3k}{(2\pi)^3 2k^0} \cdot (e^2) \cdot \frac{2 p_1 \cdot p_2}{(p_1 \cdot k)(p_2 \cdot k)}$$

P_1–P_2 c.m. Frame.



For $m_1 = m_2 = 0$,

$$P_1 = (|\vec{P}_1|, \vec{P}_1)$$

$$P_2 = (|\vec{P}_2|, \vec{P}_2)$$

$$k = (|k|, \vec{k})$$

$$\Rightarrow \quad \frac{2P_1 \cdot P_2}{(P_1 \cdot k)(P_2 \cdot k)} = \frac{2(1-\cos\theta_{12})}{|k|^2 \cdot (1-\cos\theta_{k1})(1-\cos\theta_{k2})}$$

The differential cross section diverges when the outgoing photon and lepton become parallel ($\cos\theta_{k1} \to 1$ or $\cos\theta_{k2} \to 1$).