Rob Neary

Notation:

1. $$\begin{cases} x_1 = r\cos\varphi_1 \\ x_2 = r\sin\varphi_1\cos\varphi_2 \\ \cdots\cdots \\ x_{n-1} = r\sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{n-2}\cos\varphi_{n-1} \\ x_n = r\sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{n-2}\sin\varphi_{n-1} \end{cases}$$

$$J = \underline{r^{n-1}\sin^{n-2}\varphi_1\cdot\sin^{n-3}\varphi_2\cdots\cdot\sin\varphi_{n-2}}$$

$$dx = \underline{r^{n-1}\sin^{n-2}\varphi_1\cdot\sin^{n-3}\varphi_2\cdots\cdot\sin\varphi_{n-2}\,dr\,d\varphi_1\cdots d\varphi_{n-2}\,d\varphi_{n-1}}$$

2. $$\int d^{N-1}p_2\,\delta^N(q-p_1-p_2) = \delta(Q-E_1-E_2)$$

$$\delta(p_0^2-|p|^2) = \frac{1}{2p_0}\delta(p_0-|p|)$$

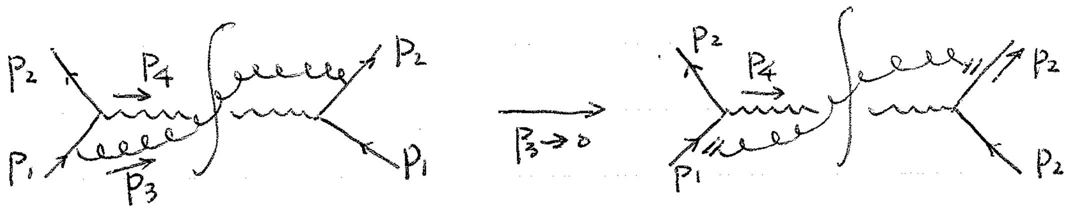
3. On-mass-shell propagator

$$\mu \overset{k}{\rightsquigarrow} \nu \qquad \frac{-ig^{\mu\nu}}{k^2+i\epsilon} \longrightarrow i(2\pi)(-ig^{\mu\nu})\cdot\delta^+(k^2)$$

where $\delta^+(k^2) = \delta(k^2)\,\Theta(k_0)$

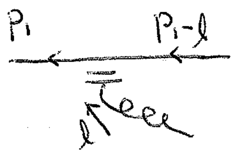
4. $$\int\frac{d^3p}{(2\pi)^3}\ [illegible] \int\frac{d^4p}{(2\pi)^4}(2\pi)\,\delta^+(p^2-m^2)\Big|_{p^0>0}$$

Eikonal Approximation of Soft Limit.

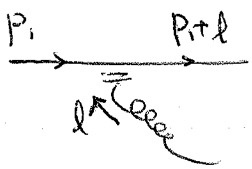


Feynman Rules :

(1)

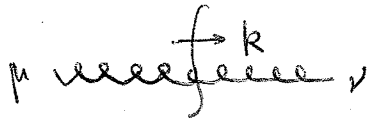


$$i(-ig_s)\,\frac{P_1^\alpha}{(-P_1\cdot l)+i\varepsilon} = g_s\cdot\frac{P_1^\alpha}{-P_1\cdot l+i\varepsilon}$$



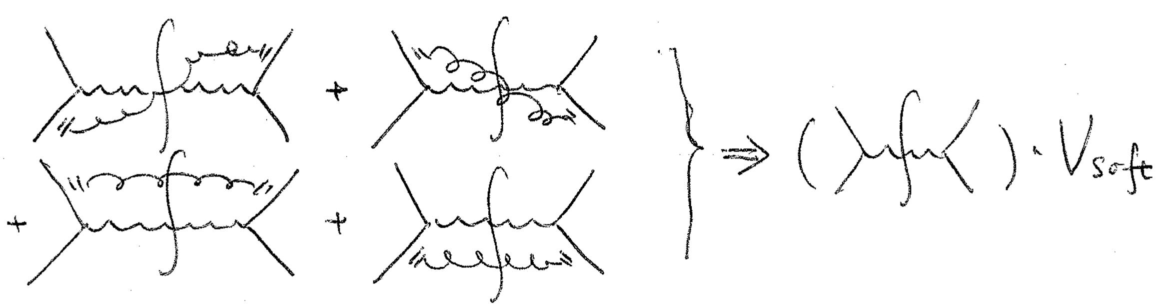
$$i(-ig_s)\,\frac{-P_1^\alpha}{P_1\cdot l+i\varepsilon} = g_s\cdot\frac{-P_1^\alpha}{P_1\cdot l+i\varepsilon}$$

(2) On-shell gluon propagator



$-ig_{\mu\nu}\left[P\left(\frac{1}{k^2}\right)\right.$ [redacted] $\left.i2\pi\delta^+(k^2)\right]$

After factorized, we obtain



Ignore the color factor and coupling constant, we obtain

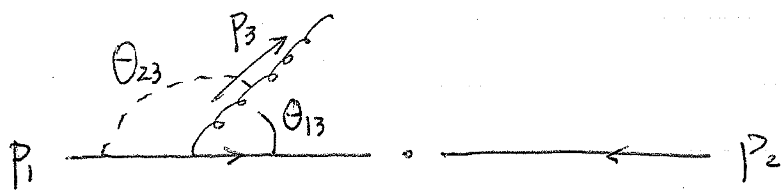
$$V_{soft} = \mu^{4-n}\int\frac{d^nP_3}{(2\pi)^n}\cdot\underbrace{\frac{2P_1\cdot P_2}{(P_1\cdot P_3)(P_2\cdot P_3)}}_{A}\cdot\underbrace{(2\pi)\delta^+(P_3^2)\cdot(2\pi)\delta^+(q^2-m^2)}_{\text{Momentum Conservation Relation.}}$$

Kinematics

In the C.M. Frame of $q\bar{q}$.

Let p_1 and p_2 propagate along z-axis, and gluon move along the $(N-1)^{th}$ direction, then we can write

$$P_3 = (|P_3|, \cdots\cdots, |P_3|\cos\theta)$$



The mandelstan Variables are

$$\hat{s} = (p_1+p_2)^2 = 2p_1\cdot p_2 = 4|p_1|\cdot|p_2| \quad\Rightarrow\quad |p_1| = \frac{\sqrt{\hat{s}}}{2}$$

$$\hat{t} = (p_1-p_3)^2 = -2p_1\cdot p_3 = -2|p_1|\cdot|p_3|(1-\cos\theta_{13}) = -\sqrt{\hat{s}}\cdot|p_3|(1-\cos\theta_{13}$$

$$\hat{u} = (p_2-p_3)^2 = -2p_2\cdot p_3 = -2|p_2|\cdot|p_3|(1-\cos\theta_{23}) = -\sqrt{\hat{s}}\cdot|p_3|(1-\cos\theta_{23}$$

From $\hat{s}+\hat{t}+\hat{u} = M^2$, we obtain

$$|p_3| = \frac{\hat{s}-M^2}{2\sqrt{\hat{s}}}.$$

Where the hat "^" denotes these variables are of the sub-process.

Then,

$$\hat{s} = 2p_1\cdot p_2$$

$$\hat{t} = -2p_1\cdot p_3 = -\hat{s}\left(1-\frac{M^2}{\hat{s}}\right)(1-y)$$

$$\hat{u} = -2p_2\cdot p_3 = -\hat{s}\left(1-\frac{M^2}{\hat{s}}\right)y. \qquad \Big|\; y = \frac{1}{2}(1+\cos\theta)$$

and factor A should be

$$A = \frac{2p_1\cdot p_2}{(p_1\cdot p_3)(p_2\cdot p_3)} = \frac{2\times\frac{\hat{s}}{2}}{\frac{\hat{t}}{2}\cdot\frac{\hat{u}}{2}} = 4\frac{\hat{s}}{\hat{t}\hat{u}}$$

$$= 4\cdot\frac{1}{\hat{s}\left(1-\frac{M^2}{\hat{s}}\right)^2}\cdot\frac{1}{y(1-y)}$$

From these variables and momentum conservation, it is easy to show that

$$p_4^2 - M^2 = (p_1+p_2)^2 - 2p_1\cdot p_3 - 2p_2\cdot p_3 + p_3^2 - M^2$$
$$= \hat{s} - M^2 - \sqrt{\hat{s}}\,|p_3|(1-\cos\theta_{13}) - \sqrt{\hat{s}}\,|p_3|(1-\cos\theta_{23})$$
$$= \hat{s} - M^2 - 2\sqrt{\hat{s}}\,|p_3|$$

thus,

$$\delta^+(p_4^2 - M^2) = \delta(\hat{s} - M^2 - 2\sqrt{\hat{s}}\,|p_3|) .$$

phase-space:

Now let's calculate the (N-2) angular integration for $\int d^N p_3$.

$$\int d^N p_3 = \int d|p_3|\cdot|p_3|^{N-1} \int \sin^{N-2}\theta_{N-1}\, d\theta_{N-1} \int \sin^{N-3}\theta_{N-2}\, d\theta_{N-2} \int\cdots\int d\theta_1$$

with $0 \le \theta_i \le \pi$, except $0 \le \theta_1 \le 2\pi$. For breviation, let's denote $p \equiv p_3$, then

$$\int d^N p\, \delta^+(p^2) = \int dp_0\, \delta^+(p_0^2 - |p|^2) \int d^{N-1}p$$
$$= \frac{1}{2p_0}\int_0^\infty d|p|\cdot|p|^{(N-1)-1} \int_0^\pi \sin^{(N-1)-2}\theta_{N-2}\, d\theta_{N-2} \underbrace{\int_0^\pi \sin^{(N-1)-3}\theta_{N-3}\, d\theta_{N-3} \cdots \int d\theta_1}_{\int d^{N-2}\Omega}$$

$\because p_0 = |p|$

$$= \frac{1}{2}\int_0^\infty d|p|\cdot|p|^{N-3} \int_0^\pi \sin^{N-3}\theta\, d\theta \int d^{N-2}\Omega$$

$$= \int_0^\infty d|p|\cdot|p|^{1-2\epsilon} \int_{-1}^{1} d(\cos\theta)\,(1-\cos^2\theta)^{-\epsilon} \left[\frac{1}{2}\int d^{N-2}\Omega\right]$$

For the angular integration $\int d^n\Omega$.

$$\int d^N\Omega = \int_0^\pi \sin^{N-2}\theta_{N-1}\, d\theta_{N-1} \int_0^\pi \cdots \int_0^\pi \sin\theta_2\, d\theta_2 \int_0^{2\pi} d\theta_1 ,$$

By using integration equation repeatly,

$$\int_0^\pi \sin^n\theta\, d\theta = \sqrt{\pi}\,\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+2}{2})}$$

We obtain

$$\int d^N\Omega = \sqrt{\pi}\,\frac{\Gamma(\frac{N-1}{2})}{\Gamma(\frac{N}{2})}\cdot\sqrt{\pi}\,\frac{\Gamma(\frac{N-2}{2})}{\Gamma(\frac{N-1}{2})}\cdots\sqrt{\pi}\cdot\frac{\Gamma(\frac{2}{2})}{\Gamma(\frac{3}{2})}\cdot(2\pi)$$

$$= \frac{2\pi^{N/2}}{\Gamma(N/2)}$$

Thus, $\int d^Np\,\delta^+(p^2) = \int_0^\infty d|p|\cdot|p|^{1-2\epsilon}\int_{-1}^1 d(\cos\theta)\,(1-\cos\theta)^{-\epsilon}\cdot\dfrac{\pi^{1-\epsilon}}{\Gamma(1-\epsilon)}$

$$= \frac{2^{1-2\epsilon}\cdot\pi^{1-\epsilon}}{\Gamma(1-\epsilon)}\cdot\int_0^\infty d|p|\cdot|p|^{1-2\epsilon}\int_0^1 dy\,[y(1-y)]^{-\epsilon}$$

where we let $N = 4-2\epsilon$, and $\cos\theta = 2y-1$

V-Factor

$$V_{soft} = \mu^{2\epsilon}\int\frac{d^Np}{(2\pi)^4}\,(2\pi)\,\delta^+(p^2)\,\frac{2P_1\cdot P_2}{(P_1\cdot P)(P_2\cdot P)}\cdot\delta(\hat{s}-M^2-2\sqrt{\hat{s}}|p|)$$

$$= \mu^{2\epsilon}\cdot\frac{1}{(2\pi)^{3-2\epsilon}}\cdot 2^{1-2\epsilon}\int_0^\infty d|p|\cdot|p|^{1-2\epsilon}\cdot\frac{4}{\hat{s}(1-\frac{M^2}{\hat{s}})^2}\cdot\frac{1}{2\sqrt{\hat{s}}}\,\delta(|P_3|-\frac{\hat{s}-M^2}{2\sqrt{\hat{s}}})$$

$$\times\frac{\pi^{1-\epsilon}}{\Gamma(1-\epsilon)}\cdot\int_0^1 dy\,[y(1-y)]^{-1-\epsilon}$$

$$= \frac{1}{4\pi^2}\cdot\frac{1}{\Gamma(1-\epsilon)}\cdot\left(\frac{4\pi\mu^2}{M^2}\right)^\epsilon\cdot\frac{1}{\hat{s}}\left(\frac{M^2}{\hat{s}}\right)^\epsilon\cdot\left(1-\frac{M^2}{\hat{s}}\right)^{-1-2\epsilon}\cdot\int_0^1 dy\,[y(1-y)]^{-1-\epsilon}$$

$$= \frac{1}{4\pi^2}\cdot\cancel{\frac{1}{\Gamma(1-\epsilon)}}\cdot\frac{1}{\hat{s}}\cdot\left(\frac{4\pi\mu^2}{M^2}\right)^\epsilon\cdot\tau^\epsilon\cdot(1-\tau)^{-1-2\epsilon}\cdot\frac{\Gamma^2(-\epsilon)}{\Gamma(1-\epsilon)\Gamma(-2\epsilon)}$$

Using the notation

$$z^\epsilon(1-z)^{-1-\epsilon} = -\frac{1}{\epsilon}\delta(1-z)+\frac{1}{(1-z)_+}-\epsilon\left(\frac{\ln(1-z)}{1-z}\right)_+ +\epsilon\frac{\ln z}{1-z}+O(\epsilon^2)$$

$$z^{-\epsilon} = 1-\epsilon\ln z+O(\epsilon^2)$$

with $\int_0^1 dz\,[g(z)]_+\,h(z) \equiv \int_0^1 dz\,g(z)\,[h(z)-h(1)]$, which is finite at $z=1$.

Then

$$\tau^{\epsilon}(1-\tau)^{-1-2\epsilon} = \tau^{2\epsilon}(1-\tau)^{-1-2\epsilon}\tau^{-\epsilon}$$
$$= \left[-\frac{1}{2\epsilon}\delta(1-\tau) + \frac{1}{(1-\tau)_+} - 2\epsilon\left(\frac{\ln(1-\tau)}{1-\tau}\right)_+ + 2\epsilon\frac{\ln\tau}{1-\tau}\right]\cdot(1-\epsilon\ln\tau)$$
$$= -\frac{1}{2\epsilon}\delta(1-\tau) + \frac{1}{(1-\tau)_+} - 2\epsilon\left(\frac{\ln(1-\tau)}{1-\tau}\right)_+ + 2\epsilon\frac{\ln\tau}{1-\tau} - \epsilon\frac{\ln\tau}{(1-\tau)_+} + O(\epsilon^2)$$

Since $\Gamma(1-\varepsilon) = -\varepsilon\Gamma(-\varepsilon)$ and $\Gamma(-\varepsilon) \approx -\frac{1}{\varepsilon} - \gamma + o(\varepsilon)$.

$\longrightarrow$ Euler constant

so easy to show that

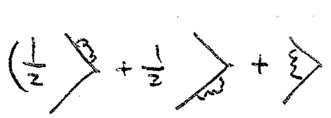
$$\frac{\Gamma^2(-\epsilon)}{\Gamma(1-\epsilon)\Gamma(-2\epsilon)} = -\frac{2}{\epsilon}.$$

Substitute these relations into the expression of V-Factor,

$$V_{soft} = \frac{1}{4\pi^2}\left(\frac{4\pi\mu^2}{M^2}\right)^{\epsilon}\cdot\frac{1}{s}\left[\frac{1}{\epsilon^2}\delta(1-\tau) - \frac{2}{\epsilon}\frac{1}{(1-\tau)_+} + 4\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+ - 4\frac{\ln\tau}{1-\tau} + 2\frac{\ln\tau}{(1-\tau)_+}\right]$$

Virtual - correction :

From Pavel's notes (PW216), We know that the infrared divergence of the virtual corrections would be



$$\Big)_{\substack{\text{Total}\\\text{DRED}}} = (\;\;)\cdot\frac{1}{16\pi^2}\left(\frac{4\pi\mu^2}{s}\right)^{\epsilon_{IR}}\cdot\frac{\Gamma(1-\epsilon_{IR})}{\Gamma(1-2\epsilon_{IR})} *$$
$$\left\{-\frac{2}{\epsilon_{IR}^2} - \frac{3}{\epsilon_{IR}} - 7 - \frac{\pi^2}{3} + o(\epsilon_{IR})\right\}$$

So,

$$V(Q) = \frac{1}{16\pi^2}\left(\frac{4\pi\mu^2}{s}\right)^{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\cdot\left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 7 - \frac{\pi^2}{3} + o(\epsilon)\right]$$

keep in mind ~~that the Facto~~ that $V(Q)$ is only in amptitude-level. After amptitude-squared, there should be another factor "2" such that $V(Q)$ should be double.

Thus.

$$V_{virtual} = \frac{1}{8\pi^2}\left(\frac{4\pi\mu^2}{s}\right)^{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left[-\frac{2}{\epsilon^2}-\frac{3}{\epsilon}-7-\frac{\pi^2}{3}+o(\epsilon)\right]$$

Now we could see that the divergences of the "$\frac{1}{\epsilon^2}$" items should vanish when we sum both virtual corrections and real corrections

$$\frac{1}{4\pi^2}\left(\frac{4\pi\mu^2}{M^2}\right)^{\epsilon}\frac{1}{\hat{s}}\frac{1}{\epsilon^2}\delta(1-\tau)+\frac{1}{16\pi^2}\left(\frac{4\pi\mu^2}{\hat{s}}\right)^{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(-\frac{2}{\epsilon^2}\right)\times 2$$

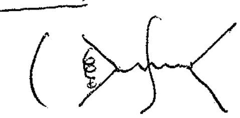
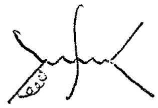
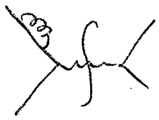
$$= \frac{1}{4\pi^2}\left(\frac{4\pi\mu^2}{M^2}\right)^{\epsilon}\cdot\frac{1}{\hat{s}}\cdot\frac{1}{\epsilon^2}\delta\left(1-\frac{M^2}{\hat{s}}\right)+\frac{1}{16\pi^2}\left(\frac{4\pi\mu^2}{\hat{s}}\right)^{\epsilon}\left(-\frac{2}{\epsilon^2}\right)\times 2$$

$$= \frac{1}{4\pi^2}\left(\frac{4\pi\mu^2}{M^2}\right)^{\epsilon}\cdot\frac{1}{\epsilon^2}\delta(\hat{s}-M^2)+\frac{1}{16\pi^2}\left(\frac{4\pi\mu^2}{\hat{s}}\right)^{\epsilon}\left(-\frac{2}{\epsilon^2}\right)\times 2$$

$$= \frac{1}{4\pi^2}\left(\frac{4\pi\mu^2}{\hat{s}}\right)^{\epsilon}\cdot\frac{1}{\epsilon^2}-\frac{1}{4\pi^2}\left(\frac{4\pi\mu^2}{\hat{s}}\right)^{\epsilon}\cdot\frac{1}{\epsilon^2}$$

$$\simeq 0$$

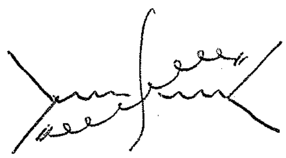
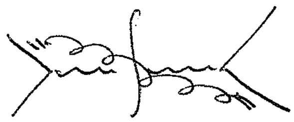
Virtual Corrections

$2\,\Big($  $+\frac{1}{2}$  $+\frac{1}{2}$  $\Big)$

$$C_F\; 2\tilde{\sigma}_0 \frac{1}{16\pi^2}\delta\left(1-\frac{M^2}{\hat{s}}\right)\times\left(\frac{4\pi\mu_R^2}{M^2}\right)^{\epsilon_{IR}}\frac{\Gamma(1-\epsilon_{IR})}{\Gamma(1-2\epsilon_{IR})}\left\{-\frac{2}{\epsilon_{IR}^2}-\frac{3}{\epsilon_{IR}}-7-\frac{\pi^2}{3}+\delta_{scheme}\right\}$$

where $\delta_{scheme}=\begin{cases}0\,, & \text{DRED}\\ -1\,, & \text{Naive } \gamma_5\,,\ \text{HVBM}\end{cases}$

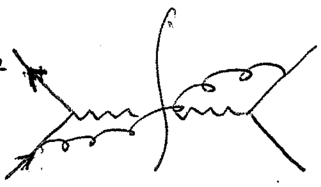
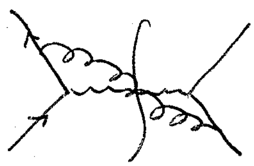
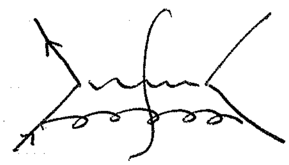
2. Soft-Limit

 $+$ 

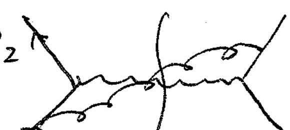
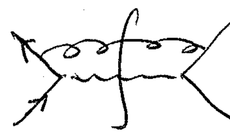
$$C_F\; 2\tilde{\sigma}_0\frac{1}{16\pi^2}\left(\frac{4\pi\mu_R^2}{\hat{s}}\right)^{\epsilon_{IR}}\frac{\Gamma(1-\epsilon_{IR})}{\Gamma(1-2\epsilon_{IR})}\left\{\frac{2}{\epsilon_{IR}^2}\delta(1-\hat{\tau})-\frac{4}{\epsilon_{IR}}\frac{1}{(1-\hat{\tau})_+}+8\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+\right\}$$

3. Collinear-Limit

(1) $P_3 /\!/ P_4$

P_2, P_1  $+$  $+$ 

(2) $P_3 /\!/ P_2$

P_2, P_1  $+$  $+$ 

P_1

mixed-diagrams' contributions:

$$2\tilde{\sigma}_0\left(\frac{4\pi\mu^2}{\hat{s}}\right)^{\epsilon}\cdot\frac{1}{16\pi^2}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\cdot\left\{\frac{4}{\epsilon^2}\delta(1-\hat{\tau})-\frac{4}{\epsilon}\frac{1+\hat{\tau}}{(1-\hat{\tau})_+}+8(1+\hat{\tau})\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+\right\}$$

unmixed-diagram's contributions:

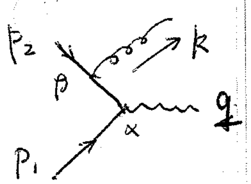
$$2\sigma_0\left(\frac{4\pi\mu^2}{\hat{s}}\right)^{\epsilon}\cdot\frac{1}{8\pi^2}\cdot\frac{1}{\Gamma(1-\epsilon)}\cdot(1-\hat{\tau})^{1-2\epsilon}\,B(-\epsilon,2-\epsilon)\,\underbrace{(1-\epsilon)}$$

The last item "$1-\epsilon$" comes from the Dirac Matrices Algebra in n-dimension

Use $(1-\hat{x})^{-1-2\epsilon}=-\frac{1}{2\epsilon}\delta(1-x)+\frac{1}{(1-x)_+}-2\epsilon\left(\frac{\ln(1-x)}{1-x}\right)_+$, then

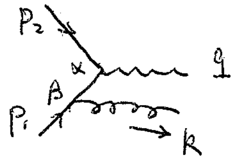
$$\Rightarrow 2\tilde{\sigma}_0\left(\frac{4\pi\mu^2}{\hat{s}}\right)^{\epsilon}\frac{1}{16\pi^2}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left\{-\frac{2}{\epsilon}(1-\hat{\tau})-4(1-\hat{\tau})^2\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+ +2(1-\hat{\tau})\right\}$$

4. $\boxed{\text{THE FULL REAL CORRECTIONS}}$



$$iM_1 = \bar{v}(p_2,h_2)\left[(-ig_s\gamma^\beta)\frac{-i(\not p_2-\not k)}{(p_2-k)^2+i\varepsilon}\, i\Gamma^\alpha\right]\bar{u}(p_1,h_1)\, T^a_{ij}\, \epsilon^*_\alpha(q)\,\epsilon^*_\beta(k)$$

$$M_1 = -\bar{v}(p_2,h_2)\left[\gamma^\beta(\not p_2-\not k)\Gamma^\alpha\right]\bar{u}(p_1,h_1)\cdot\frac{g_s T^a_{ij}}{(p_2-k)^2}\,\epsilon^*_\alpha(q)\,\epsilon^*_\beta(k)$$



$$iM_2 = \bar{v}(p_2,h_2)\left[i\Gamma^\alpha\frac{i(\not p_1-\not k)}{(p_1-k)^2+i\varepsilon}(-ig_sT^a_{ij}\gamma^\beta)\right]\bar{u}(p_1,h_1)\,\epsilon^*_\alpha(\vec{q})\,\epsilon^*_\beta(\vec{k})$$

$$M_2 = \bar{v}(p_2,h_2)\left[\Gamma^\alpha(\not p_1-\not k)\gamma^\beta\right]\bar{u}(p_1,h_1)\,\frac{g_sT^a_{ij}}{(p_1-k)^2}\,\epsilon^*_\alpha(\vec{q})\,\epsilon^*_\beta(\vec{k})$$

$$\Gamma^\mu = \frac{g_w}{2\sqrt{2}}\cdot\gamma^\mu(1-\gamma^5)\quad,\quad g_w = \frac{e}{s_w}$$

Thus, squaring and summarizing spins and colors, we obtain

$$\sum|M_R|^2 = M_{11} + M_{22} + M_{12}$$

$$M_{11} = |M_{1R}|^2_{\Sigma} = \mathrm{Tr}\left[\not p_2\gamma^\beta(\not p_2-\not k)\gamma^\alpha\not p_1\gamma_\alpha(\not p_2-\not k)\gamma_\beta\right]\times\frac{1}{(2p_2\cdot k)^2}$$

$$M_{22} = |M_{2R}|^2_{\Sigma} = \mathrm{Tr}\left[\not p_2\gamma^\alpha(\not p_1-\not k)\gamma^\beta\not p_1\gamma_\beta(\not p_1-\not k)\gamma_\alpha\right]\times\frac{1}{(2p_1\cdot k)^2}$$

$$M_{12} = |2M_1M^*_{2R}|^2_{\Sigma} = -2\,\mathrm{Tr}\left[\not p_2\gamma^\beta(\not p_2-\not k)\gamma^\alpha\not p_1\gamma_\beta(\not p_1-\not k)\gamma_\alpha\right]\times\frac{1}{2p_1\cdot k\; 2p_2\cdot k}$$

Here we omit the coupling const and color factor.

In the $q\bar{q}$ c.m. frame, we choose the particles moving along z-axis, and the Mandelstam Variables are

$$\hat{s} = (p_1+p_2)^2 = 2p_1\cdot p_2$$
$$\hat{t} = (p_1-p_3)^2 = -2p_1\cdot p_3$$
$$\hat{u} = (p_2-p_3)^2 = -2p_2\cdot p_3$$

We should do the γ-matrices in the $N=4-2\epsilon$ dimensions,

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_N$$

By using relations

$$\gamma^\mu\not a\gamma_\mu = -2(1-\epsilon)\not a$$
$$\gamma^\mu\not a\not b\gamma_\mu = 4\,a\cdot b - 2\epsilon\not a\not b$$
$$\gamma^\mu\not a\not b\not c\gamma_\mu = -2\not c\not b\not a + 2\epsilon\not a\not b\not c$$
$$\mathrm{Tr}(\not a\not b\not c\not d) = 4(a\cdot b\, c\cdot d + a\cdot d\, b\cdot c - a\cdot c\, b\cdot d)$$

We obtain the traces as following.

$$M_{11} = 16(1-\epsilon)^2 \frac{\hat{t}}{\hat{u}}$$

$$M_{22} = 16(1-\epsilon)^2 \cdot \frac{\hat{u}}{\hat{t}}$$

$$M_{12} = -32(1-\epsilon)\left[-\hat{s}M^2 + \epsilon \hat{t}\hat{u}\right]/\hat{t}\hat{u}$$

Thus.

$$\sum |M_k|^2 = 16(1-\epsilon)\left[(1-\epsilon)\left(\frac{\hat{t}}{\hat{u}} + \frac{\hat{u}}{\hat{t}}\right) + 2\cdot\frac{\hat{s}M^2}{\hat{t}\hat{u}} - 2\epsilon\right]$$

The coupling constants and color Factors are

$$\mu^{2\epsilon}\cdot\left(\frac{g_W}{2\sqrt{2}}\right)^2 \cdot g_s^2 \cdot Tr(T^A T^A) = \mu^{2\epsilon}\cdot\frac{g_W^2}{8}\cdot g_s^2\cdot Tr(T^A T^A)$$

$$Tr(T^A T^A) = C_A C_F = 4.$$

The spin and color average factor are

$$\left(\tfrac{1}{2}\cdot\tfrac{1}{2}\right) \text{ and } \left(\tfrac{1}{3}\cdot\tfrac{1}{3}\right).$$

Hence.

$$\overline{|M|^2} = \left(\tfrac{1}{2}\cdot\tfrac{1}{2}\right)\times\left(\tfrac{1}{3}\cdot\tfrac{1}{3}\, C_A C_F\right)\cdot\mu^{2\epsilon}\frac{g_W^2}{8}g_s^2\cdot 16(1-\epsilon)\left[(1-\epsilon)\left(\frac{\hat{t}}{\hat{u}}+\frac{\hat{u}}{\hat{t}}\right)+2\frac{\hat{s}M^2}{\hat{t}\hat{u}}-2\epsilon\right]$$

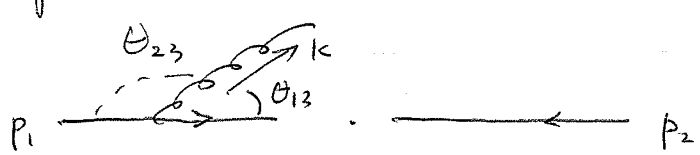
$$= \frac{1}{9}\cdot\mu^{2\epsilon}\, 2 g_W^2 g_s^2\cdot(1-\epsilon)\left[(1-\epsilon)\left(\frac{\hat{t}}{\hat{u}}+\frac{\hat{u}}{\hat{t}}\right)+2\frac{\hat{s}M^2}{\hat{t}\hat{u}}-2\epsilon\right]$$

And the phase-space integration

$$\int d^N p\, \delta^+(p^2) = \frac{2^{1-2\epsilon}\pi^{1-\epsilon}}{\Gamma(1-\epsilon)}\int_0^\infty d|p|\cdot|p|^{1-2\epsilon}\int_0^1 dy\,[y(1-y)]^{-\epsilon}$$

Kinematics.

In the CM Frame of $q\bar{q}$; Let P_1 and P_2 propagate along Z-axis. and gluon move along the $(N-1)^{th}$ direction. then $k = (|k|, \cdots, |k|\cos\theta)$



Thus.

$$\hat{s} = 2P_1\cdot P_2$$

$$\hat{t} = -2P_1\cdot P_3 = -\hat{s}\left(1-\frac{M^2}{\hat{s}}\right)(1-y)$$

$$\hat{u} = -2P_2\cdot P_3 = -\hat{s}\left(1-\frac{M^2}{\hat{s}}\right)y$$

where $y = \frac{1}{2}(1+\cos\theta)$

Thus,

$$(1-\epsilon)\left(\frac{\hat{t}}{\hat{u}}+\frac{\hat{u}}{\hat{t}}\right)+2\frac{\hat{s}M^2}{\hat{t}\hat{u}}-2\epsilon$$
$$=(1-\epsilon)\left(\frac{1-y}{y}+\frac{y}{1-y}\right)+\frac{2\tau}{(1-\tau)^2}\cdot\frac{1}{y(1-y)}-2\epsilon \qquad \left(\hat{\tau}=\frac{M^2}{\hat{s}}\right)$$

So

$$\sigma^{NLO}=\overline{|M|^2}\langle p.s\rangle$$
$$=\mu^{2\epsilon}\int\frac{d^nk}{(2\pi)^n}(2\pi)\delta^+(k^2)\cdot\overline{|M|^2}\,\delta(\hat{s}-M^2-2\sqrt{\hat{s}}\,|\mathbf{K}|)$$
$$=\mu^{2\epsilon}\frac{2^{1-2\epsilon}\pi^{1-\epsilon}}{(2\pi)^{n-1}\Gamma(1-\epsilon)}\int_0^\infty d|k|\cdot|k|^{1-2\epsilon}\,\delta(\hat{s}-M^2-2\sqrt{\hat{s}}|k|)\cdot\left[\frac{1}{9}\,2g_W^2g_s^2(1-\epsilon)\right]$$
$$\int_0^1 dy\,[y(1-y)]^{-\epsilon}\left[(1-\epsilon)\left(\frac{1-y}{y}+\frac{y}{1-y}\right)+\frac{2\hat{\tau}}{(1-\hat{\tau})^2}\cdot\frac{1}{y(1-y)}-2\epsilon\right]$$

$$=\frac{1}{16\pi^2}\left(\frac{4\pi\mu^2}{s}\right)^\epsilon\frac{1}{\Gamma(1-\epsilon)}(1-\hat{\tau})^{1-2\epsilon}\cdot\left[\frac{1}{9}\cdot 2g_W^2g_s^2(1-\epsilon)\right]$$
$$\cdot\left\{\int_0^1 dy\cdot y^{-\epsilon}(1-y)^{-\epsilon}\left[(1-\epsilon)\left(\frac{1-y}{y}+\frac{y}{1-y}\right)+\frac{2\hat{\tau}}{(1-\hat{\tau})^2}\frac{1}{y(1-y)}-2\epsilon\right]\right\}$$

Use $\int_0^1 dy\; y^\alpha(1-y)^\beta=\dfrac{\Gamma(1+\alpha)\Gamma(1+\beta)}{\Gamma(2+\alpha+\beta)}$,

We obtain the above y integral as

$$\{\ \}=-\frac{2}{\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\left[(1-\epsilon)^2+\frac{2\hat{\tau}}{(1-\hat{\tau})^2}\right]$$

Use $(1-x)^{-1-2\epsilon}=-\frac{1}{2\epsilon}\delta(1-x)+\frac{1}{(1-x)_+}-2\epsilon\left(\frac{\ln(1-x)}{1-x}\right)_++o(\epsilon^2)$.

$$(1-\hat{\tau})^{1-2\epsilon}=(1-\hat{\tau})^2(1-\hat{\tau})^{-1-2\epsilon}$$
$$=(1-\hat{\tau})^2\left[-\frac{1}{2\epsilon}\delta(1-\hat{\tau})+\frac{1}{(1-\hat{\tau})_+}-2\epsilon\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+\right]$$
$$=\frac{(1-\hat{\tau})^2}{(1-\hat{\tau})_+}-(1-\hat{\tau})^2\cdot 2\epsilon\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+$$

$$-\frac{2}{\epsilon}(1-\epsilon)^2(1-\hat{\tau})^{1-2\epsilon}=-\frac{2}{\epsilon}\frac{(1-\hat{\tau})^2}{(1-\hat{\tau})_+}+4\frac{(1-\hat{\tau})^2}{(1-\hat{\tau})_+}+4(1-\hat{\tau})^2\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+$$

and

$$-\frac{2}{\epsilon}\cdot 2\hat{\tau}(1-\hat{\tau})^{-1-2\epsilon}=\frac{2\hat{\tau}}{\epsilon^2}\delta(1-\hat{\tau})-\frac{4\hat{\tau}}{\epsilon}\frac{1}{(1-\hat{\tau})_+}+8\hat{\tau}\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+$$

Thus $\sigma^{NLO} = \frac{1}{16\pi^2}\left(\frac{4\pi\mu^2}{\hat{s}}\right)^{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left[\frac{1}{9}2g_w^2g_s^2(1-\epsilon)\right]\times$

$$\left\{\frac{2}{\epsilon^2}\delta(1-\hat{\tau}) - \frac{4\hat{\tau}}{\epsilon}\frac{1}{(1-\hat{\tau})_+} + 8\hat{\tau}\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+ - \frac{2}{\epsilon}\frac{(1-\hat{\tau})^2}{(1-\hat{\tau})_+} + 4\frac{(1-\hat{\tau})^2}{(1-\hat{\tau})_+} + 4(1-\hat{\tau})^2\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+ + O(\epsilon)\right\}$$

$$= \frac{1}{16\pi^2}\left(\frac{4\pi\mu^2}{\hat{s}}\right)^{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\cdot\left[\frac{2}{9}g_w^2g_s^2(1-\epsilon)\right]$$

$$\times\left\{\frac{2}{\epsilon^2}\delta(1-\hat{\tau}) - \frac{2}{\epsilon}\frac{1+\hat{\tau}^2}{(1-\hat{\tau})_+} + 4(1+\hat{\tau}^2)\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+ + 4\frac{(1-\hat{\tau})^2}{(1-\hat{\tau})_+}\right\}$$

$$= \left(\frac{1}{9}g_s^2\right)2\tilde{\sigma}_0\cdot\frac{1}{16\pi^2}\left(\frac{4\pi\mu^2}{\hat{s}}\right)^{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}$$

$$\times\left\{\frac{2}{\epsilon^2}\delta(1-\hat{\tau}) - \frac{2}{\epsilon}\frac{1+\hat{\tau}^2}{(1-\hat{\tau})_+} + 4(1+\hat{\tau}^2)\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+ + 4(1-\hat{\tau})\right\}$$

Eikonal Approximation at soft-Limit.

$$(\sigma^{NLO}_{soft}) = 2\tilde{\sigma}_0\frac{1}{16\pi^2}\left(\frac{4\pi\mu^2}{\hat{s}}\right)^{\epsilon_{IR}}\frac{\Gamma(1-\epsilon_{IR})}{\Gamma(1-2\epsilon_{IR})}\left\{\frac{2}{\epsilon_{IR}^2}\delta(1-\hat{\tau}) - \frac{4}{\epsilon_{IR}}\cdot\frac{1}{(1-\hat{\tau})_+} + 8\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+\right.$$

So we will see that when $\hat{\tau} = 1$,

$$\sigma^{NLO}(\hat{\tau}=1) = \sigma^{NLO}_{soft}.$$

$\left(\hat{\tau} = \frac{M^2}{\hat{s}} = 1 \Rightarrow \hat{s} = M^2 \Rightarrow t = u = 0 \Rightarrow \text{soft}\cdot\text{photon}\right)$

5. $\boxed{\text{Structure of the singularities at eikonal Approximation}}$

I omit the common Factors here:

$$(\sigma^{NLO}_{REAL}) = (\sigma^{NLO}_{coll})_{mixed} + (\sigma^{NLO}_{coll})_{unmixed} - (\sigma^{NLO}_{soft})$$

$$= \frac{2}{\epsilon^2}\delta(1-\hat{\tau}) - \frac{2}{\epsilon}\frac{1+\hat{\tau}^2}{(1-\hat{\tau})_+} + [8\hat{\tau} - 4(1-\hat{\tau})^2]\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+ + 2(1-\hat{\tau})$$

The singular items of eikonal approximation is ~~even~~ equal to the singular items of FULL real corrections, and only the finite items are different.

$$\sigma^{NLO}_{sing} = \sigma^{NLO}_{virt} + \sigma^{NLO}_{Real}$$

$$= -\frac{3}{\epsilon_{IR}}\delta(1-\hat{\tau}) - \frac{2}{\epsilon}\frac{1+\hat{\tau}^2}{(1-\hat{\tau})_+} + 2(1-\hat{\tau})$$

$$+ \left[8\hat{\tau} - 4(1-\hat{\tau})^2\right]\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+ - 7 - \frac{\pi^2}{3} + \delta_{scheme}$$

$$= -\frac{2}{\epsilon}\left(\frac{1+\hat{\tau}^2}{1-\hat{\tau}}\right)_+ + 2(1-\hat{\tau}) + \left[8\hat{\tau} - 4(1-\hat{\tau})^2\right]\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+ - 7 - \frac{\pi^2}{3} + \underbrace{\delta_{scheme}}$$