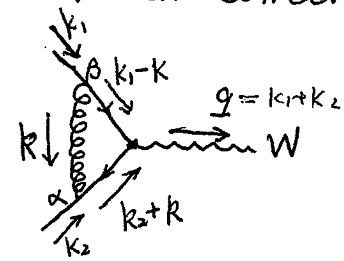
Vertex Correction.



$$G^\mu = \mu^{4-n}\int \frac{d^nk}{(2\pi)^n}\,(i\gamma_\alpha)\cdot\frac{i(-\not{k}_2-\not{k})}{(k_2+k)^2+i\epsilon}\cdot i\Gamma^\mu\cdot\frac{i(\not{k}_1-\not{k})}{(k_1-k)^2+i\epsilon}\cdot i\gamma_\beta\cdot\frac{-ig^{\alpha\beta}}{k^2+i\epsilon}$$

$$G^\mu = -\gamma_\alpha\gamma_{\delta_1}\Gamma^\mu\gamma_{\delta_2}\gamma_\beta\, g^{\alpha\beta}\,\mu^{4-n}\int\frac{d^nk}{(2\pi)^n}\,\frac{k_2^{\delta_1}+k^{\delta_1}}{(k_2+k)^2+i\epsilon}\times\frac{k_1^{\delta_2}-k^{\delta_2}}{(k_1-k)^2+i\epsilon}\times\frac{1}{k^2+i\epsilon}$$

$$= -\gamma_\alpha\gamma_{\delta_1}\Gamma^\mu\gamma_{\delta_2}\gamma^\alpha\cdot M_1$$

$$M_1 = \mu^{4-n}\int\frac{d^nk}{(2\pi)^n}\cdot\frac{k_2^{\delta_1}+k^{\delta_1}}{(k_2+k)^2}\cdot\frac{(k_1^{\delta_2}-k^{\delta_2})}{(k_1-k)^2}\times\frac{1}{k^2}$$

Feynman . . .

$$\frac{1}{ab} = \int_0^1 dx\,\frac{1}{[ax+b(1-x)]^2}$$

So.

$$\frac{1}{(k_2+k)^2\times(k_1-k)^2} = \int_0^1 dx\cdot\frac{1}{[x(k_2+k)^2+(1-x)(k_1-k)^2]^2}$$

$$= \int_0^1 dx\cdot\frac{1}{[x(k^2+2kk_2)+(1-x)(k^2-2kk_1)]^2}$$

$$= \int_0^1 dx\cdot\frac{1}{[k^2+2k(k_2x-(1-x)k_1)]^2}$$

Let $p^\mu = k_2^\mu x-(1-x)k_1^\mu$. $p^2 = -q^2x(1-x)$ —— (1)

$$= \int_0^1 dx\,\frac{1}{(k^2+2k\cdot p)^2}$$

$$\int_0^1 dx\,\frac{1}{k^2[k^2+2k\cdot p]^2} = \int_0^1 dx\int_0^1 dy\,\frac{2y}{[(k^2+2k\cdot p)y+k^2(1-y)]^3}$$

$$= \int_0^1 dx\int_0^1 dy\,\frac{2y}{[\cancel{k^2y}+2kpy+k^2-\cancel{k^2y}]^3}$$

$$= \int_0^1 dx \int_0^1 dy \frac{2y}{[k^2+2k\cdot py]^3}$$

~~Let~~ Substitute $k = l - yp$, $d^n k = d^n l$

$$= \int_0^1 dx \int_0^1 dy \frac{2y}{[l^2 - y^2p^2]^3}$$

Substitute $p^2 = -q^2 x(1-x)$,

$$= \int_0^1 dx \int_0^1 dy \cdot \frac{2y}{[l^2 + y^2 x(1-x) q^2]^3}$$

$$= \int_0^1 dx \int_0^1 dy \cdot \frac{2y}{[l^2+M^2]^3} \qquad M^2 = y^2 x(1-x) q^2$$

So, the denominator will be

$$\frac{1}{k^2(k_2+k)^2(k_1-k)^2} = \int_0^1 dx \int_0^1 dy \frac{2y}{[l^2+M^2]^3}$$

Numerator after substitution $k = l - yp$.

$$(k_2+k)^{\delta_1} \cdot (k_1-k)^{\delta_2} = (k_2+l-yp)^{\delta_1} (k_1-l+yp)^{\delta_2}$$
$$= -l^{\delta_1} l^{\delta_2} + l^{\delta_1}(k_1+yp)^{\delta_2} + (k_2-yp)^{\delta_1} l^{\delta_2}$$
$$+ (k_2-yp)^{\delta_1}(k_1+yp)^{\delta_2}$$

Drop the linear ~~terms~~ items,

$$= -l^{\delta_1} l^{\delta_2} + (k_2-yp)^{\delta_1}(k_1+yp)^{\delta_2}$$

Finally, $\int dx \int dy\, 2y$

$$M_1' = \mu^{4-n} \int \frac{d^n l}{(2\pi)^n} \cdot \frac{-l_1^{\delta_1} l_2^{\delta_2} + (k_2-yp)^{\delta_1}(k_1+yp)^{\delta_2}}{[l^2+M^2]^3}$$

~~When $l \to \infty$~~ $$= \underbrace{\mu^{4-n} \int \frac{d^n l}{(2\pi)^n} \frac{-l_1^{\delta_1} l_2^{\delta_2}}{[l^2+M^2]^3}}_{(\mathrm{I})} + \underbrace{\mu^{4-n} \int \frac{d^n l}{(2\pi)^n} \cdot \frac{(k_2-yp)^{\delta_1}(k_1+yp)^{\delta_2}}{[l^2+M^2]^3}}_{(\mathrm{II})}$$

When $l \to \infty$, item I $= \dfrac{\text{Momentum}^6}{\text{Momentum}^6} \to \ln(\ )$ divergent

item II $= \dfrac{(\text{Momentum})^2}{(\text{Momentum})^6} \to$ finite

When $l \to 0$, item I $= \dfrac{\text{Momentum}^6}{\text{Momentum}^6} \to \ln(\ )$ divergent

item II $= \dfrac{(\text{Momentum})^2}{(\text{Momentum})^6} \to \infty$ divergence

So, item I includes both UV and IR,
item II includes only IR.

In order to seperate the UV and IR in item I,
We introduce the item $\frac{M^2}{n} g_n^{\delta_1\delta_2}$ such that

$$\boxed{\begin{aligned} M_1' = {} & \mu^{4-n}\int\frac{d^n l}{(2\pi)^n}\left[-l^{\delta_1} l^{\delta_2} - \frac{M^2}{n} g_n^{\delta_1\delta_2}\right]\cdot\frac{1}{[l^2+M^2]^3} \qquad (\mathrm{I}) \\ & + \mu^{4-n}\int\frac{d^n l}{(2\pi)^n}\left[\frac{M^2}{n} g_n^{\delta_1\delta_2} + (k_2 - yP)^{\delta_1}(k_1 + yP)^{\delta_2}\right]\times\frac{1}{[l^2+M^2]^3} \qquad (\mathrm{II}) \end{aligned}}$$

$$\begin{aligned} M_1'(\mathrm{I}) &\overset{(-)}{=} \mu^{4-n}\int\frac{d^n l}{(2\pi)^n}\left[l^{\delta_1} l^{\delta_2} + \frac{M^2}{n} g_n^{\delta_1\delta_2}\right]\cdot\frac{1}{[l^2+M^2]^3} \\ &= -\mu^{4-n}\int\frac{d^n l}{(2\pi)^n}\cdot\frac{1}{n}[l^2+M^2]\, g_n^{\delta_1\delta_2}\cdot\frac{1}{[l^2+M^2]^3} \\ &= -\mu^{4-n}\int\frac{d^n l}{(2\pi)^n}\cdot\frac{1}{n}\,\frac{1}{(l^2+M^2)^2}\cdot g_n^{\delta_1\delta_2} \\ &= -\mu^{4-n}\cdot\frac{g_n^{\delta_1\delta_2}}{n}\cdot\int\frac{d^n l}{(2\pi)^n}\,\frac{1}{(l^2+M^2)^2} \end{aligned}$$

Feynman Parameterization:

$$\frac{1}{a_1 \cdot a_2 \cdots a_n} = \Gamma(n) \int_0^1 dx_1 \int_0^1 dx_2 \cdots \int_0^1 dx_n \cdot \frac{\delta(1-\sum_{i=1}^{n} x_i)}{|\sum a_i x_i|^n}$$

$$\frac{1}{a_1 a_2 a_3} = 2 \int_0^1 dx \int_0^1 dy \int_0^1 dz \cdot \frac{\delta(1-x-y-z)}{[ax+by+cz]^3}$$

$$\frac{1}{a^R b^M} = \frac{\Gamma(R+M)}{\Gamma(R)\Gamma(M)} \int_0^1 dx \int_0^1 dy \cdot \frac{x^{R-1} y^{M-1} \delta(1-x-y)}{[ax+by]^{R+M}}$$

$$= \frac{\Gamma(R+M)}{\Gamma(R)\Gamma(M)} \int_0^1 dx \cdot \frac{x^{R-1}(1-x)^{M-1}}{[ax+b(1-x)]^{R+M}}$$

Integral over d^4k ($N=4$ dimensions)

$$\int \frac{d^4k}{(2\pi)^4} \cdot \frac{(k^2)^R}{[k^2-c]^M} = \frac{i(-1)^{R-M}}{16\pi^2} c^{R-M+2} \cdot \frac{\Gamma(R+2)\Gamma(M-R-2)}{\Gamma(M)}$$

Integral over d^Nk (N dimensions)

$$\boxed{\int \frac{d^Nk}{(2\pi)^N} \cdot \frac{(k^2)^R}{[k^2-c]^M} = \frac{i(-1)^{R-M}}{(16\pi^2)^{N/4}} \cdot c^{R-M+N/2} \cdot \frac{\Gamma(R+\frac{N}{2})\Gamma(M-R-\frac{1}{2}N)}{\Gamma(\frac{1}{2}N)\Gamma(M)}}$$

So, $M_1(I) = -\mu^{4-n} \frac{g_n^{\delta_1\delta_2}}{n} \cdot \frac{i}{(4\pi)^{n/2}} \cdot \frac{\Gamma(2-\frac{n}{2})}{\Gamma(2)} \cdot (-M^2)^{\frac{n}{2}-2}$

minus

$$UV^{\delta_1\delta_2} = \int_0^1 dx \int_0^1 dy \cdot 2y \cdot \left[-\mu^{4-n} \cdot \frac{g_n^{\delta_1\delta_2}}{n} \cdot \frac{i}{(4\pi)^{n/2}} \cdot \frac{\Gamma(2-n/2)}{\Gamma(2)} [y^2 x(x-1) q^2]^{\frac{n}{2}-2}\right]$$

$$= -\mu^{4-n} \cdot \frac{g_n^{\delta_1\delta_2}}{n} \cdot \frac{i}{(4\pi)^{n/2}} \cdot \frac{\Gamma(2-n/2)}{\Gamma(2)} \cdot q^{n-4} \cdot \int_0^1 dx \int_0^1 dy \cdot 2y \cdot [y^2 x(x-1)]^{\frac{n}{2}-2}$$

$$= -\mu^{4-n} \cdot \frac{g_n^{\delta_1\delta_2}}{n} \cdot \frac{i}{(4\pi)^{n/2}} \cdot \frac{\Gamma(2-n/2)}{1} (q^2)^{\frac{n}{2}-2} \cdot 2 \cdot \int_0^1 dx\, [x(x-1)]^{\frac{n}{2}-2} \cdot \int_0^1 dy \cdot y^{n-3}$$

$$= -\mu^{4-n} \cdot \frac{g_n^{\delta_1\delta_2}}{n} \cdot \frac{i}{(4\pi)^{n/2}} \cdot \Gamma(2-n/2) \cdot (q^2)^{\frac{n}{2}-2} \cdot 2 \cdot \frac{1}{n-2} \cdot (-1)^{\frac{n}{2}-2} \cdot B(\tfrac{n}{2}-1, \tfrac{n}{2}-1)$$

$$= -\frac{i}{16\pi^2} \cdot \left(\frac{q^2}{4\pi\mu^2}\right)^{\frac{n}{2}-2} \cdot g_n^{\delta_1\delta_2} \cdot \Gamma(2-n/2) \cdot \frac{2}{n} \cdot \frac{1}{n-2} B(\tfrac{n}{2}-1, \tfrac{n}{2}-1)(-1)^{\frac{n}{2}-2}$$

Substitute $n_{UV} = 4 - 2\varepsilon_{UV}$, $\varepsilon_{UV} > 0$.

$$UV^{\delta_1\delta_2} = -\frac{i}{16\pi^2}\left(\frac{g^2}{4\pi\mu^2}\right)^{-\varepsilon_{UV}} g_n^{\delta_1\delta_2} \cdot \Gamma(\varepsilon_{UV}) \cdot \frac{2}{4-2\varepsilon_{UV}} \cdot \frac{1}{2-2\varepsilon_{UV}} B(1-\varepsilon_{UV}, 1-\varepsilon_{UV})(-1)^{\varepsilon_{u}}$$

$$= -\frac{i}{16\pi^2}\left(\frac{g^2}{4\pi\mu^2}\right)^{-\varepsilon_{UV}} g_n^{\delta_1\delta_2} \cdot \frac{1}{4-2\varepsilon_{UV}} \cdot \frac{1}{1-\varepsilon_{UV}} \cdot \Gamma(\varepsilon_{UV}) \cdot \frac{\Gamma(1-\varepsilon_{UV})\cdot\Gamma(1-\varepsilon_{UV})}{\Gamma(2-2\varepsilon_{UV})}$$

As $\Gamma(1+Z) = Z\Gamma(Z)$, $\Gamma(Z)\Gamma(1-Z) = \frac{\pi}{\sin\pi Z}$.

$$UV^{\delta_1\delta_2} = -\frac{i}{16\pi^2}\left(\frac{4\pi\mu^2}{g^2}\right)^{\varepsilon_{UV}} g_n^{\delta_1\delta_2} \cdot \underbrace{\frac{1}{4-2\varepsilon_{UV}} \cdot \frac{1}{1-\varepsilon_{UV}} \cdot \frac{\pi}{\sin\pi\varepsilon_{UV}} \cdot \frac{1}{(1-2\varepsilon_{UV})}}_{\text{Series}} \cdot \frac{\Gamma(1-\varepsilon_{UV})}{\Gamma(1-2\varepsilon_{UV})}$$

$$\text{Series} = \frac{1}{4}\left(1+\frac{\varepsilon_{UV}}{2}+\frac{\varepsilon_{UV}^2}{4}\right)\left(1+\varepsilon_{UV}+\varepsilon_{UV}^2\right)\left(1+2\varepsilon_{UV}+4\varepsilon_{UV}^2\right)\left[\frac{1}{\varepsilon_{UV}}+\pi^2\varepsilon_{UV}\right]$$

$$= \frac{1}{4}\left(1+\frac{\varepsilon_{UV}}{2}\right)(1+\varepsilon_{UV})(1+2\varepsilon_{UV})\left[\frac{1}{\varepsilon_{UV}}\right]$$

$$= \frac{1}{4}\left[1+\frac{\varepsilon_{UV}}{2}+\varepsilon_{UV}+2\varepsilon_{UV}+O(\varepsilon_{UV}^2)\right]\cdot\frac{1}{\varepsilon_{UV}}$$

$$= \frac{1}{4\varepsilon_{UV}} + \frac{7}{8} + O(\varepsilon_{UV})$$

$$UV^{\delta_1\delta_2} = -\frac{i}{16\pi^2}\left(\frac{4\pi\mu^2}{g^2}\right)^{\varepsilon_{UV}} g_n^{\delta_1\delta_2} \cdot \frac{\Gamma(1-\varepsilon_{UV})}{\Gamma(1-2\varepsilon_{UV})} \cdot \left[\frac{1}{4\varepsilon_{UV}} + \frac{7}{8} + O(\varepsilon_{UV})\right]$$

Next, Construct $UV^{\delta_1\delta_2}$ <u>with γ-matrix structure.</u>

(1) Naive γ^5 scheme:

$$\{\gamma^5, \gamma^\mu\} = 0$$

In the n-dimension,

$$- \gamma_\alpha \gamma_{\delta_1} \Gamma^\mu \gamma_{\delta_2} \gamma^\alpha -$$

$$= -\gamma_\alpha \gamma_{\delta_1} \gamma^\mu (g_V - g_A \gamma_5) \gamma_{\delta_2} \gamma^\alpha = -\gamma_\alpha \gamma_{\delta_1} \gamma^\mu \gamma_{\delta_2} \gamma^\alpha (g_V - g_A \gamma_5)$$

$$= \left[2 \gamma_{\delta_2} \gamma^\mu \gamma_{\delta_1} - 2\epsilon \gamma_{\delta_1} \gamma^\mu \gamma_{\delta_2} \right] \cdot (g_V - g_A \gamma_5)$$

$$UV^{\delta_1 \delta_2} \sim g^{\delta_1 \delta_2}.$$

So, γ matrix should be

$$[2 \gamma_{\delta_2} \gamma^\mu \gamma_{\delta_1} - 2\epsilon \gamma_{\delta_1} \gamma^\mu \gamma_{\delta_2}] \cdot (g_V - g_A \gamma_5) \, g^{\delta_1} g^{\delta_2}$$

$$= (2 - 2\epsilon) \gamma_{\delta_2} \gamma^\mu \gamma^{\delta_2} (g_V - g_A \gamma_5)$$

$$= 2(1-\epsilon) \cdot [-2(1-\epsilon)] \gamma^\mu (g_V - g_A \gamma_5)$$

$$= -4(1-\epsilon)^2 \gamma^\mu (g_V - g_A \gamma_5)$$

$$= -4(1-\epsilon)^2 \Gamma^\mu$$

Thus,

$$UV = (\gamma \text{ matrix}) \cdot UV^{\delta_1 \delta_2}$$

$$= -4(1-\epsilon)^2 \cdot \Gamma^\mu \cdot \left(-\frac{i}{16\pi^2}\right) \left(\frac{4\pi\mu^2}{q^2}\right)^{\epsilon_{UV}} \cdot \frac{\Gamma(1-\epsilon_{UV})}{\Gamma(1-2\epsilon_{UV})} \cdot \left[\frac{1}{4\epsilon_{UV}} + \frac{7}{8} + o(\epsilon)\right]$$

$$= \frac{i}{16\pi^2} \Gamma^\mu \left(\frac{4\pi\mu^2}{q^2}\right)^{\epsilon_{UV}} \cdot \frac{\Gamma(1-\epsilon_{UV})}{\Gamma(1-2\epsilon_{UV})} \cdot \left\{ 4(1-\epsilon)^2 \cdot \left[\frac{1}{4\epsilon_{UV}} + \frac{7}{8}\right] \right\}$$

$$= \frac{i}{16\pi^2} \Gamma^\mu \left(\frac{4\pi\mu^2}{q^2}\right)^{\epsilon_{UV}} \cdot \frac{\Gamma(1-\epsilon_{UV})}{\Gamma(1-2\epsilon_{UV})} \cdot \left\{ \frac{1}{\epsilon_{UV}} + \frac{3}{2} \right\}$$

$$\Rightarrow \quad G^\mu = i \Gamma^\mu \cdot UV$$

Where $UV = \frac{1}{16\pi^2} \left(\frac{4\pi\mu^2}{q^2}\right)^{\epsilon_{UV}} \cdot \frac{\Gamma(1-\epsilon_{UV})}{\Gamma(1-2\epsilon_{UV})} \left\{ \frac{1}{\epsilon_{UV}} + \frac{3}{2} \right\}$