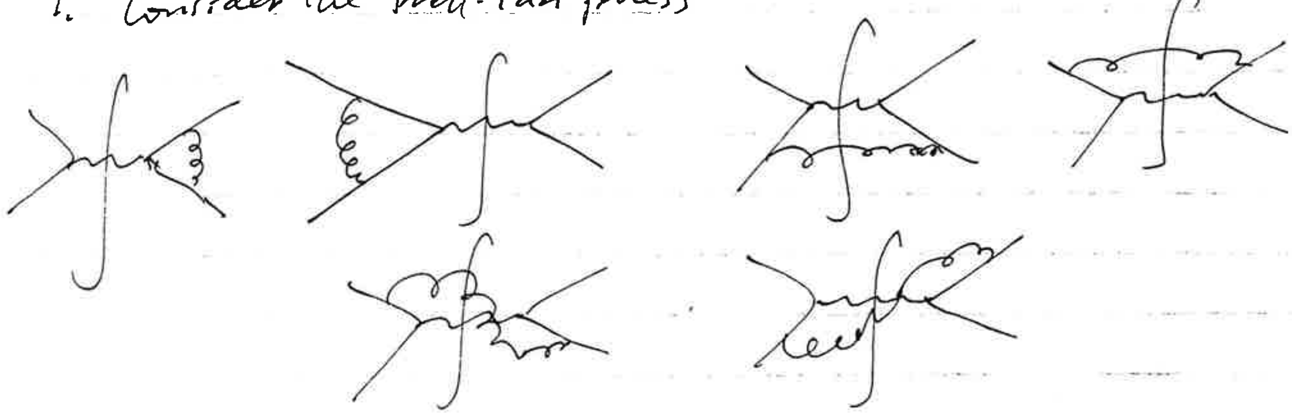
E-1

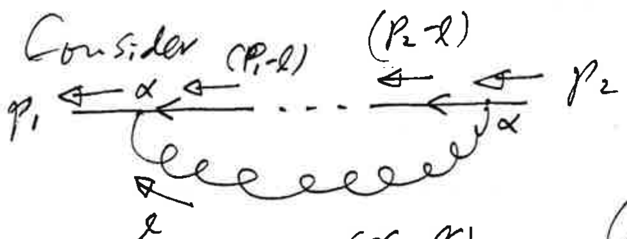
Eikonal Approximation

1. Consider the Drell-Yan Process



2. Eikonalization in <u>Collinear</u> limit

1) Consider



light-cone coordinate

$$\mathcal{M} = \bar{v}(p_1)\gamma_\alpha \frac{(\not p_1 - \not l)}{(p_1-l)^2}\cdots\frac{(\not p_2-\not l)}{(p_2-l)^2}\gamma^\alpha u(p_2)$$

if $l \,/\!/\, p_2$ (<u>Collinear gluon</u>), then for

$$p_{2\mu} = (p_2^+, p_2^-, \underline{p}_2^T) \equiv p_2^+ \bar{n}_\mu + p_2^- n_\mu + \underline{p}_2^T \cdot \underline{n}_T$$

with

$$\begin{cases} \bar{n}_\mu = (1, 0, \underline{0}) \\ n_\mu = (0, 1, \underline{0}) \\ \underline{n}_T = (0, 0, \underline{1}) \end{cases}$$

$\left(p^2 = p_\mu p^\mu = 2p^+p^- - \underline{p}_T^2 \right)$, and $p_2^- = p_2^T = 0$,

we have $(\not p_2 - \not l)\gamma^\alpha u(p_2) = [(p_2-l)^+\gamma^-\gamma^\alpha u(p_2)$

2) Consider $\gamma^- \gamma^\alpha u(p_2)$

① If $\alpha = -$, then $\gamma^- u(p_2) = 0$ (on-shell condition)

$\gamma^- \gamma^- = 0$

$\not{p}_2 u(p_2) = 0$

$p_2^+ \gamma_+ u(p_2) = 0$

$= p_2^+ \gamma^- u(p_2)$

② If $\alpha = \perp$, then $\gamma^- \gamma^\perp = -\gamma^\perp \gamma^-$ (from $\{\gamma_\mu, \gamma_\nu\} = 2 g_{\mu\nu}$)

Hence, on-shell condition gives $\gamma^- \gamma^\perp u(p_2) = 0$

$$\Rightarrow \quad \gamma^- \gamma^\alpha u(p_2) = \gamma^- \gamma^+ U(p_2) \quad \text{for} \quad p_2 = (p_2^+, 0, \underline{0})$$

Since $\gamma_\alpha (\cdots) \gamma^\alpha \longrightarrow \boxed{\gamma_\alpha (\cdots) \gamma^+ = \gamma_+ (\cdots) \gamma^+ = \gamma^- (\cdots) \gamma^+}$

So the only non-vanishing contribution of $\mathcal{M}$ in the collinear limit ($\ell \parallel p_2$) comes from

$$\mathcal{M} = \bar{v}(p_1) \frac{\gamma^- (\not{p}_1 - \not{\ell})}{(p_1 - \ell)^2} \cdots \frac{(p_2 - \ell)^+ \gamma^- \gamma^+}{(p_2 - \ell)^2} u(p_2)$$

Since on-shell condition gives

$\bar{v}(p_1) \not{p}_1 = 0$, we can use the commutation relation

$$\gamma_\mu \not{p}_1 = 2 p_{1\mu} - \not{p}_1 \gamma_\mu$$

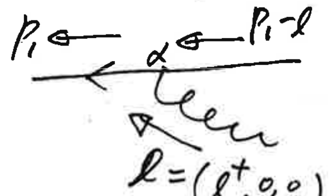
to obtain $\bar{v}(p_1) \frac{\gamma_\mu (\not{p}_1 - \not{\ell})}{(p_1 - \ell)^2} (\cdots) = \bar{v}(p_1) \left[\frac{2 p_{1\mu}}{(p_1 - \ell)^2} + \frac{-\gamma_\mu \not{\ell}}{(p_1 - \ell)^2} \right] (\cdots)$

Note that $\ell \parallel p_2$, so $\not{\ell} = \ell^+ \gamma^-$.

Since $\gamma^- \gamma^- = 0$, therefore $\bar{v}(p_1) \frac{\gamma^- (\not{p}_1 - \not{\ell})}{(p_1 - \ell)^2} = \bar{v}(p_1) \frac{2 p_1^-}{(p_1 - \ell)^2}$

E-3

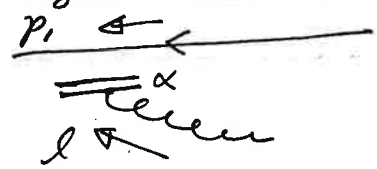
3) Thus, we conclude that for $l \parallel p_2$ with $p_2 = (p_2^+, 0, \underline{0})$



$l \parallel n^+$

$$= \bar{v}(p_1) \frac{2p_1^-}{(p_1 - l)^2} = \underline{\underline{\bar{v}(p_1) \frac{p_1^-}{(-p_1 \cdot l)}}}$$

$\equiv$ Eikonized line

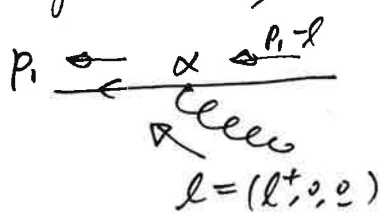


~~If $l \parallel p_1$ then~~ Since $p_{1\alpha} = p_1^+ \bar{n}_\alpha + p_1^- n_\alpha + p_1^T \underline{n}^T$,

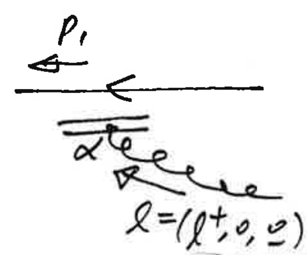
$$\underline{\frac{p_1^-}{(-p_1 \cdot l)} = \frac{n_\alpha}{(-n \cdot l) + i\varepsilon}}$$

(for propagator $\longrightarrow \frac{1}{p^2 - m^2 + i\varepsilon}$)

Diagramatically,



$$\underline{\underline{}}_{(l_\mu \parallel \bar{n}_\mu)}$$



$$\bar{v}(p_1) \frac{\gamma_\alpha (\not{p}_1 - \not{l})}{(p_1 - l)^2 + i\varepsilon} \qquad\qquad \bar{v}(p_1) \boxed{\frac{n_\alpha}{(-n \cdot l) + i\varepsilon}}$$

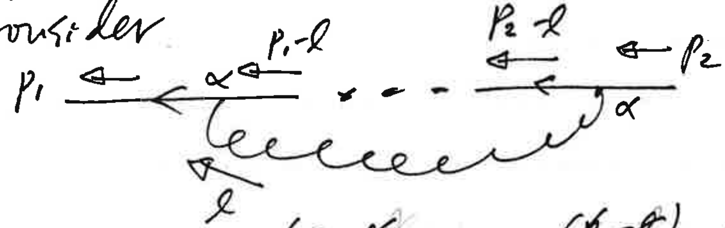
with $n_\alpha = (0, 1, \underline{0})$

$\Longrightarrow$ Factorized

E-4

3. Eikonalization in Soft limit

Consider



$$\mathcal{M} = \bar{v}(p_1)\,\gamma_\alpha \frac{(\not p_1 - \not l)}{(p_1-l)^2} \cdots \frac{(\not p_2 - \not l)}{(p_2-l)^2}\,\gamma^\alpha u(p_2)$$

In the soft limit, $l \to 0$,

$$\frac{(\not p_2-\not l)}{(p_2-l)^2}\gamma^\alpha u(p_2) \xrightarrow[l\to 0]{} \frac{\not p_2\, \gamma^\alpha}{(p_2-l)^2}\, u(p_2)$$

$$= \left[\frac{2p_2^\alpha - \gamma^\alpha \not p_2}{(p_2-l)^2}\right] u(p_2)$$

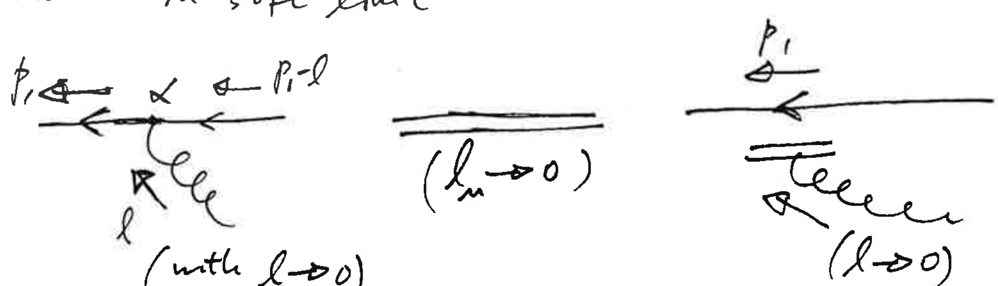
always plus

In real calculate, the sign of ϵ is very important. It is always "plus"

$$= \frac{p_2^\alpha}{(-p_2\cdot l)+i\varepsilon}\, u(p_2)$$

(On-shell condition $\not p_2\, u(p_2)=0$)

Hence in soft limit



$$\bar{v}(p_1)\,\frac{\gamma_\alpha(\not p_1-\not l)}{(p_1-l)^2+i\varepsilon}$$

$$\bar{v}(p_1)\,\frac{p_1^\alpha}{(-p_1\cdot l)+i\varepsilon}$$

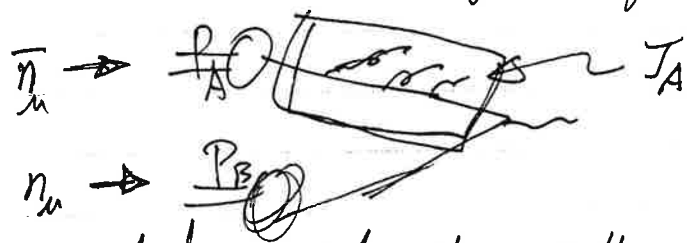
The sign is very important!

$\Longrightarrow$ Factorized

4. Classify the Feynman diagrams in Drell-Yan process

(E-5)

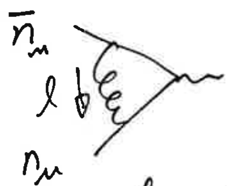
1) To define the incoming jet of J_A in



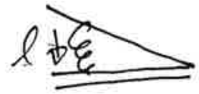
, (with $\bar{n}_\mu = (1, 0, \underline{0})$, $n_\mu = (0, 1, \underline{0})$)

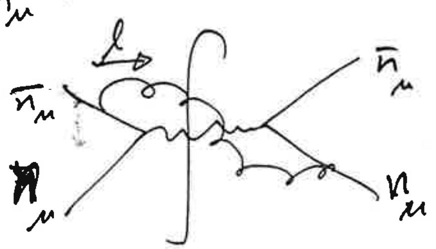
we need to consider the collinear limit for

$l \parallel \bar{n}_\mu$, which results in

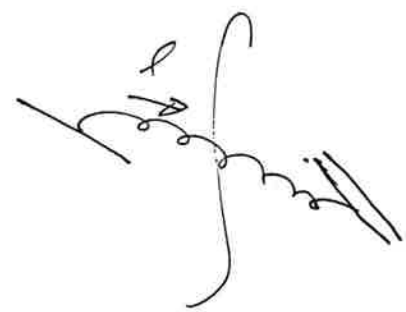


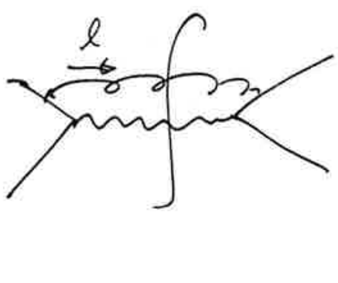
$\overline{\overline{(l \parallel \bar{n}_\mu)}}$





$\overline{\overline{(l \parallel \bar{n}_\mu)}}$

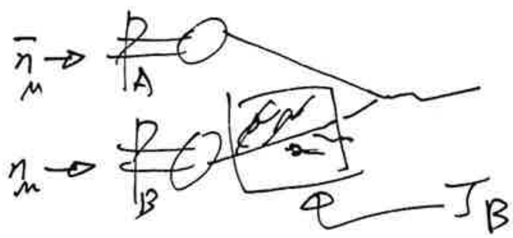




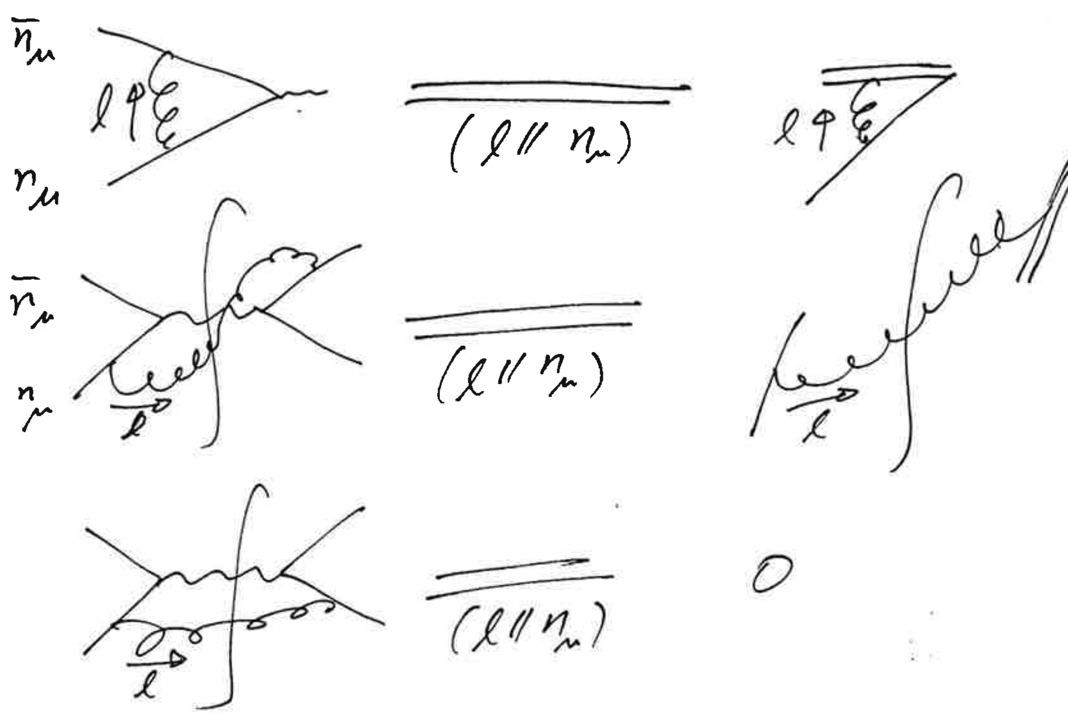
$\overline{\overline{(l \parallel \bar{n}_\mu)}}$ 0

Note that the only condition here is $(l \parallel \bar{n}_\mu)$,
namely l can be either a hard gluon or a soft gluon as long as it is parallel to $\bar{n}_\mu$.
$(l \to 0)$

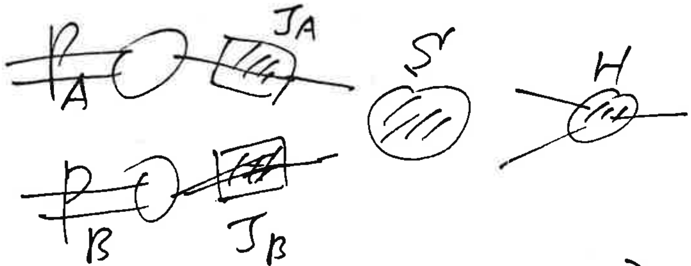
2) Similarly, to define the incoming jet of J_B in



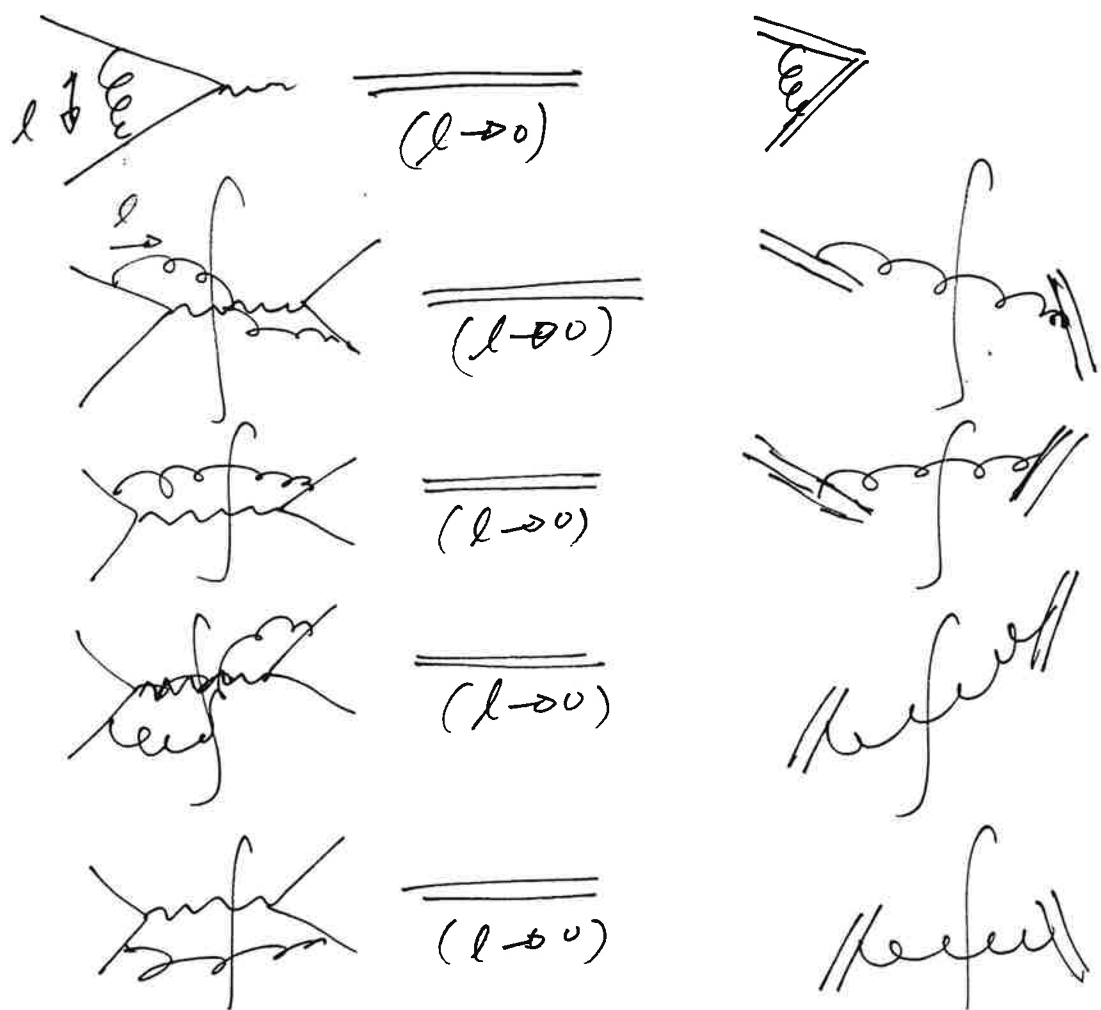
We need to consider the collinear limit ~~of~~ for
$(\ell \parallel n_\mu)$, which results in



3) We also need to construct the soft function S to collect all the soft gluon contribution which are not included in J_A or J_B. Namely, the gluons are soft, but not collinear to either $\bar{n}_\mu$ or n_μ.



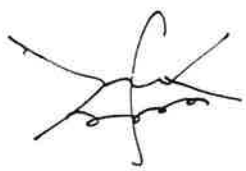
This is the limit of $(l_\mu \to 0)$, which results in



5. To derive the A and B functions in the Sudakov factor, we need to sum over the contributions from all the above Eikonalized diagrams in J_A, J_B and S.

Note that the above results were derived in the 't Hooft-Feynman gauge (where the gluon propagator is $\frac{-g_{\mu\nu}}{\ell^2+i\varepsilon}$).

In the ~~light~~ axial gauge, the classification of the diagrams are different. For instance, the self-energy diagram  should be included, and

 $\neq 0$ in the collinear limit, etc.

$$\boxed{\begin{aligned}\gamma^+ &\equiv \frac{1}{\sqrt{2}}(\gamma^0+\gamma^3)\\ \gamma^- &\equiv \frac{1}{\sqrt{2}}(\gamma^0-\gamma^3)\end{aligned}}$$

$$k^+ = \frac{1}{\sqrt{2}}(k^0+k^3)$$
$$k^- = \frac{1}{\sqrt{2}}(k^0-k^3)$$

$$\not{k} = k^\mu \gamma_\mu = k^+\gamma^- + k^-\gamma^+ - \underline{k}_T\cdot\underline{\gamma}_T$$

$\underline{\gamma}_T$:
$$\gamma(R) = \frac{-1}{\sqrt{2}}(\gamma^1 + i\gamma^2)$$
$$\gamma(L) = \frac{1}{\sqrt{2}}(\gamma^1 - i\gamma^2)$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}, \quad j = 1, 2, 3$$

$$\gamma^0\gamma^0 = 1, \quad \gamma^j\gamma^j = -1,$$

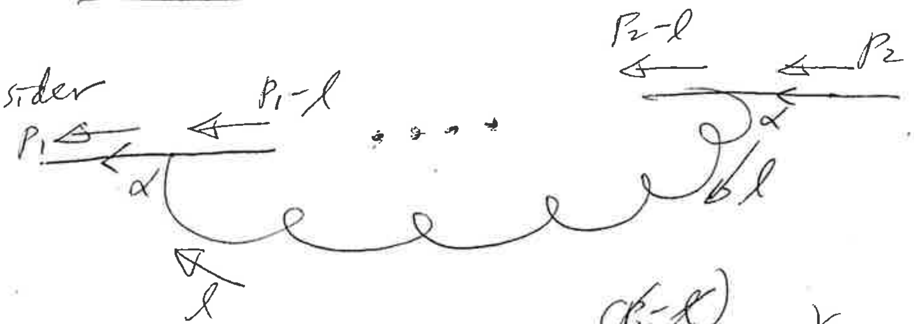
$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\gamma^-\gamma^- = \frac{1}{2}(\gamma^0\gamma^0 + \gamma^3\gamma^3) = 0,$$
etc.

Also, $\gamma^0\gamma^3 + \gamma^3\gamma^0 = 2g^{30} = 0$

Another look of the Collinear Eikonalization

Consider



$$\gamma^\alpha \frac{(\not{p}_1 - \not{l})}{(p_1-l)^2 + i\varepsilon} \cdots \frac{(\not{p}_2 - \not{l})}{(p_2-l)^2 + i\varepsilon} \gamma_\alpha$$

where $p_2^\mu = (p_2^+, 0, \underline{0}) = p_2^+ u_A^\mu$, with $u_A = (1, 0, \underline{0})$

$p_1^\mu = (0, p_1^-, \underline{0}) = p_1^- u_B^\mu$, with $u_B = (0, 1, \underline{0})$

(1) $$(\not{p}_2 - \not{l})\gamma_\alpha = \left(p_2^+\gamma^- - \left(l^+\gamma^- + l^-\gamma^+ \pm \underline{l}^T \underline{\gamma}^T\right)\right)\gamma_\alpha$$

For $p_2^+ \gg |l^+|, |l^-|, |l^T|$ then

$$(\not{p}_2 - \not{l})\gamma_\alpha \sim p_2^+ \gamma^- \gamma_\alpha$$

Since $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, so $\{\gamma^-, \gamma^-\} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\{\gamma^0 - \gamma^3, \gamma^0 - \gamma^3\}$

$= 0 = 2\gamma^-\gamma^-$

$\Rightarrow \gamma^-\gamma^- = 0$

Thus,

$$(\not{p}_2 - \not{l})\gamma_\alpha \approx (\not{p}_2 - \not{l})\gamma^+ \equiv A^+$$

(2)

Similarly,

$$\gamma^\alpha(\not{p}_1 - \not{l}) = \gamma^\alpha\left(p_1^-\gamma^+ - \left(l^+\gamma^- + l^-\gamma^+ \pm \underline{l}^T\underline{\gamma}^T\right)\right)$$

For $|p_1^-| \gg |l^+|, |l^-|, |l^T|$

$$\gamma^\alpha(\not{p}_1 - \not{l}) \sim \gamma^\alpha p_1^- \gamma^+$$

Thus $$\gamma^\alpha(\not{p}_1 - \not{l}) \sim \gamma^-(\not{p}_1 - \not{l}) \equiv B^-$$

(3) Consider $B^- A^+ = \frac{B^- l^+ A^+}{l^+} = \frac{(B\cdot l) A^+}{l^+}$, (assume $|l^+| > |l^-|, |l_T|$)

$$B\cdot l = \not l(\not p_1 - \not l) = (\not l + \not p_1 - \not p_1)(\not p_1 - \not l)$$

Thus

$$\frac{B\cdot l}{(p_1-l)^2+i\varepsilon} = \frac{(p_1-l)^2 - \not p_1(\not p_1-\not l)}{(p_1-l)^2+i\varepsilon} = 1 - \not p_1 \frac{(\not p_1-\not l)}{(p_1-l)^2+i\varepsilon}$$

$\to 0$ (on-shell condition, either $\bar u(p_1)\not p_1 = 0$ or $\mathrm{Tr}(\not p_1 \not p_1 \cdots) = 0$)

Hence, we have

$(|l^+| > |l^-|, |l_T|)$

$$\frac{1}{l^+}\cdots\cdots\frac{(\not p_2-\not l)\gamma^+}{(p_2-l)^2+i\varepsilon}$$

$$= \frac{1}{l\cdot u_B}\cdots\frac{(\not p_2-\not l)\gamma_\alpha\cdot u_B^\alpha}{(p_2-l)^2+i\varepsilon}$$

which is the eikonal line

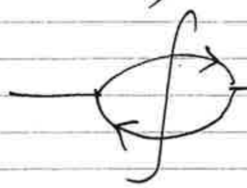
for $(l \parallel p_2)$

Note

(1) Since $|l^+|$ is a large quantity, we can drop the $(i\varepsilon)$ in the denomiter of $\left(\frac{-1}{-l\cdot u_B + i\varepsilon}\right)$.

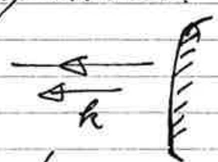
(2) In the above derivation, we assume $|p_1^-| \gg |l^+|$, hence, this approximation (eikonalization) don't work for $|l^+| > |p_1^-|, |p_2^+|$. Namely, the gluon here is still "soft" as compared to $|p_1^-|$ in this case

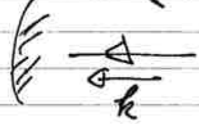
3. Unitarity cut-line in B-D notation

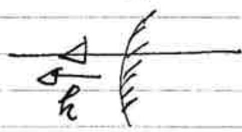
 $= 2\,\mathrm{Im}\,T$, where $\mathrm{Im}\,T$ is defined via  $= i(\mathrm{Re}\,T + i\,\mathrm{Im}\,T)$

Note. The fermion loop factor (−1) should be included in calculation

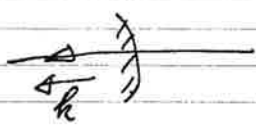
Feynman rules:

 $(i)\ \dfrac{(\not k + m)}{k^2 - m^2 + i\varepsilon}$

 $(-i)\ \dfrac{(\not k + m)}{k^2 - m^2 - i\varepsilon}$

 $(+2\pi)\ (\not k + m)\ \theta(-k_0)\ \delta(k^2 - m^2)$,

$$\boxed{(i)(-i\,2\pi) = +2\pi}$$

 $(+2\pi)\ (\not k + m)\,\theta(k_0)\,\delta(k^2 - m^2)$

vertex in unshadow region $(i)\ g$
vertex in shadow region $(-i)\ g$

γ-matrices Γ need care when taking $\bar\Gamma \equiv \gamma^0 \Gamma^\dagger \gamma^0$

For a scalar, the Feynman rules are the same without $(\not k + m)$

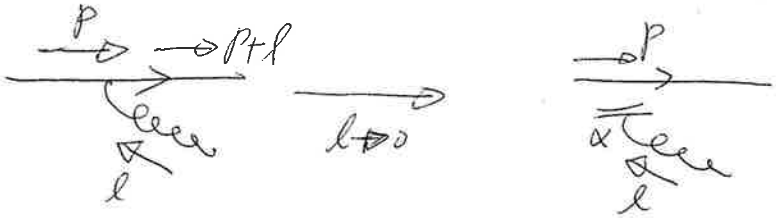
For a spin-1 object, replace $(\not k + m)$ by $\left(-g_{\mu\nu} + \dfrac{(1-\alpha) k_\mu k_\nu}{k^2 - \alpha m^2 + i\varepsilon}\right)$

Loop integral: $\displaystyle\int \frac{d^n k}{(2\pi)^n}$

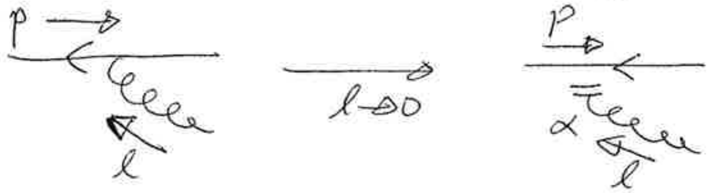
Landau gauge $\alpha = 0$
Unitary gauge $\alpha = \infty$
Feynman 't Hooft $\alpha = 1$

Note. $\dfrac{1}{p^2 - m^2 + i\varepsilon} = P\left(\dfrac{1}{p^2 - m^2}\right) - i\,2\pi\,\delta(p^2 - m^2)$ ✓ in B-D notation

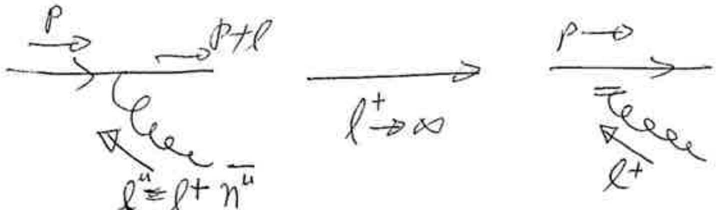
Feynman Rules for eikonal lines



$$(i)\left(\frac{p^\alpha}{p\cdot l+i\varepsilon}\right)(ig_s)(T^a)$$



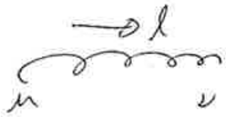
$$(i)\left(\frac{-p^\alpha}{p\cdot l+i\varepsilon}\right)(ig_s)(T^a)$$



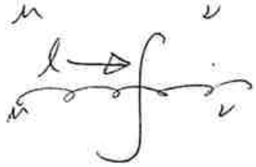
$$(i)\left(\frac{n^\alpha}{n\cdot l+i\varepsilon}\right)(ig_s)\cdot(T^a)$$

where $\bar{n}^\mu = (1, 0, \underline{0})$
$n^\mu = (0, 1, \underline{0})$

$$(i)\left(\frac{-n^\alpha}{n\cdot l+i\varepsilon}\right)(ig_s)(T^a)$$

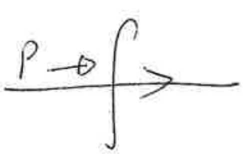


$$(i)\,\frac{-g^{\mu\nu}}{l^2+i\varepsilon}$$



$$(i)(-g^{\mu\nu})\left(-i2\pi\,\delta^+(l^2)\right) = (-g^{\mu\nu})\left(2\pi\,\delta^+(l^2)\right)$$

$$(i)\,\frac{\not p+m}{p^2-m^2+i\varepsilon}$$



$$(i)(\not p+m)\left(-i2\pi\,\delta^+(p^2-m^2)\right)$$

Note: If the above (non-cut) diagrams are in the right-hand side ($\mathcal{M}^+$) of the cut diagram, then change (i) to $(-i)$ everywhere.