Notations:

I will use two sets of metric in this lecture.
I will try to be consistent in one complete topic if possible.

(1) Bjorken & Drell ~~notation~~ metric:

$$g^{\mu\nu} = g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$$

$$p_\mu = (E, p_x, p_y, p_z), \qquad p^2 = E^2 - \vec{p}^{\,2} = m^2 > 0$$

for on-shell particle

$$p^\mu = g^{\mu\nu} p_\nu = (E, -p_x, -p_y, -p_z), \qquad \left(p^2 = p^\mu p_\mu = g^{\mu\nu} p_\nu p_\mu \right)$$

(2) Dirac metric:

$$\delta^{\mu\nu} = \delta_{\mu\nu} = (1, 1, 1, 1)$$

$$p^\mu = p_\mu = (p_x, p_y, p_z, iE), \qquad E = p_0$$

$$p^2 = \vec{p}^{\,2} - E^2 = -m^2 < 0$$

for on-shell particle

$$\left(p^2 = \delta_{\mu\nu} p^\mu p_\nu = \delta_{\mu\nu} p_\mu p_\nu \right)$$

0: Notations in Dirac metric

Notation.

[indices $\mu, \nu = 1, 2, 3, 4$. $j, k = 1, 2, 3$.]

I metric

$$g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(1, 1, 1, 1)$$

$$x_\mu = (\vec{x}, x_4) \; ; \quad x_4 = i x_0 = i t \qquad (\hbar = c = 1)$$

$$p^2 \equiv p_\mu p_\mu = (\vec{p}, i p_0)(\vec{p}, i p_0) = \vec{p}^{\,2} + (i p_0)(i p_0) = \vec{p}^{\,2} - p_0^2 = -m^2$$

$$\sum_{\vec{p}} \to \int \frac{d_3 x\, d_3 p}{h^3} \to \frac{V}{(2\pi)^3} \int d_3 p \; ,$$

$$\sum_{\vec{p}_i} \delta_{\vec{p}_i, \vec{p}_j} = \int \delta_3(\vec{p}_i - \vec{p}_j)\, d_3 p_i \; , \quad \left(\text{or} \quad \delta(\vec{p} - \vec{p}\,') = \lim_{V \to \infty} \frac{V}{(2\pi)^3} \delta_{\vec{p}, \vec{p}\,'} \right)$$

* There is no distinguish between up- and down- indices.

e.m

$$A_\mu = (\vec{A}, i\phi) \; ; \qquad p_\mu \equiv \frac{1}{i} \partial_\mu \qquad \mu = 1, 2, 3, 4.$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \; ; \qquad \partial_\mu = (\vec{\nabla}, i \partial_0).$$

II Pauli matrices

$$\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \; , \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[\sigma_i, \sigma_j] = 2i \varepsilon^{ijk} \sigma_k \; ; \quad (\varepsilon^{123} = 1)$$

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij} \; ; \quad (\sigma_1\sigma_2 = -\sigma_2\sigma_1 \;\; \& \;\; \sigma_1\sigma_2 = i\sigma_3)$$

$$\mathrm{Tr}(\sigma_i \sigma_j) = 2\delta_{ij}$$

$$\sum_i (\sigma_i)_{ab} (\sigma_i)_{cd} = 2\left(\delta_{bc}\delta_{ad} - \frac{1}{2}\delta_{ab}\delta_{cd}\right)$$

$$\sigma_j^+ = \sigma_j \; ; \quad \sigma_j^2 = 1$$

$$(\vec{\sigma} \cdot \vec{A})(\vec{\sigma} \cdot \vec{B}) = \mathbb{1}(\vec{A} \cdot \vec{B}) + i \vec{\sigma} \cdot (\vec{A} \times \vec{B})$$

III Dirac matrices

$$\gamma^j = \sigma_2 \otimes \sigma_j = \begin{pmatrix} 0 & -i\sigma_j \\ +i\sigma_j & 0 \end{pmatrix} \qquad j = 1, 2, 3$$

$$\gamma^4 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

$$\gamma^5 \equiv \gamma^1\gamma^2\gamma^3\gamma^4 = \begin{pmatrix} 0 & -\mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} = \frac{1}{4!} \varepsilon_{\alpha\beta\gamma\delta} \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta$$

$$(\gamma^\mu)^+ = \gamma^\mu \; ; \quad (\gamma^\mu)^2 = \mathbb{1} \; ; \quad \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\delta_{\mu\nu}\mathbb{1}$$

$$\gamma^{5+} = \gamma^5 \; , \quad (\gamma^5)^2 = 1 \qquad \gamma^5\gamma^\sigma = -\gamma^\sigma\gamma^5 \; , \quad \sigma = 1, 2, 3, 4$$

$$\sigma_{\mu\nu} \equiv \frac{1}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) \; ; \qquad \sigma_{jk} = \frac{i}{2}\epsilon_{jkl}\begin{pmatrix}\sigma_l & 0\\ 0 & \sigma_l\end{pmatrix}$$

$$\sigma_{\mu\nu} = -\sigma_{\nu\mu} \qquad \sigma_{j4} = -\sigma_{4j} = \frac{i}{2}\begin{pmatrix}0 & \sigma_j\\ \sigma_j & 0\end{pmatrix}$$

passive viewpoint
$$\left[\; \psi'(x') = X\psi(x) \; ; \quad X \equiv e^{\frac{1}{2}\alpha_{\mu\nu}\sigma_{\mu\nu}} \; ; \right.$$
$$x' = Lx \; ; \quad L \equiv e^{\frac{1}{2}\alpha_{\mu\nu}k_{\mu\nu}} \; ; \quad (k_{\mu\nu})_{\alpha\beta} = +\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha} \left.\right]$$
$$X^+\gamma^4 = \gamma^4 X^{-1} \; ; \quad X^{-1}\gamma^\mu X = L_{\mu\nu}\gamma^\nu \; ; \quad X^{-1}\gamma^5 X = (\det L)\gamma^5$$

(1) Dirac eq
$$(i\gamma_\mu P_\mu + m)\psi(x) = 0 \; ; \qquad P_\mu \equiv \frac{1}{i}\partial_\mu \; , \; \mu = 1, 2, 3, 4$$

at rest frame

$$u^1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad u^2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \; ; \quad u^3 = \begin{pmatrix}0\\0\\-1\\0\end{pmatrix} \quad u^4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

e^- up — e^- down — e^+ down (v^2) — e^+ up (v^1)

[Define $v^1(\vec{p}) = u^4(-\vec{p})$, $v^2(\vec{p}) = u^3(-\vec{p})$] ;

$$\psi^{1,2} = e^{ip\cdot x}u^{1,2}$$
$$\psi^{3,4} = e^{-ip\cdot x}u^{3,4} \; , \quad p_0 = +\sqrt{\vec{p}^2 + m^2} > 0$$

$$\Rightarrow \quad (i\gamma^\mu p_\mu + m)\,u^s(\vec{p}) = 0 \; ; \quad s = 1, 2$$
$$(-i\gamma^\mu p_\mu + m)\,u^s(-\vec{p}) = 0 \; ; \quad s = 3, 4$$

(2)
$$\bar{u}(\vec{p})u(\vec{p}) = \frac{m}{p_0} \quad \text{for } u^1 \;\&\; u^2 \; ;$$

$\bar{u} = u^* \gamma^4$

$\bar{u}u$ is Lorentz inv.

$$\bar{u}(-\vec{p})u(-\vec{p}) = -\frac{m}{p_0} \quad \text{for } u^3 \;\&\; u^4$$

$$u^*_\alpha u_\beta = \delta_{\alpha\beta} \quad \text{for all } u$$

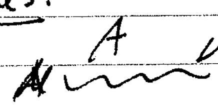
$$\sum_{j=1}^{2} u^j_\beta(\vec{p})\,\bar{u}^j_\alpha(\vec{p}) = \frac{1}{2p_0}(-i\gamma p + m)_{\beta\alpha}$$

$$\sum_{j=3}^{4} u^j_\beta(-\vec{p})\,\bar{u}^j_\alpha(-\vec{p}) = \frac{1}{2p_0}(-i\gamma p - m)_{\beta\alpha}$$

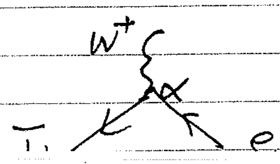
Translation Table for Feynman Rules between Bjorken-Drell & Dirac notations

Bjorken-Drell	Dirac
$p^2 - m^2 + i\varepsilon$	$p^2 + m^2 - i\varepsilon$
$\not{p} + m$	$-i\not{p} + m$
γ^μ	$i\gamma^\mu$
$1 - \gamma^5$ (or $-\gamma^5$)	$1 + \gamma_5$ (or γ^5)
$\int \frac{d^n k}{(2\pi)^n}$	$\int d^n k$
(i) (propagator)	$\left(\frac{1}{(2\pi)^4 i}\right)$ (propagator)
(i) (vertex)	$\left((2\pi)^4 i\right)$ (vertex)
$g_{\mu\nu}$	$-\delta_{\mu\nu}$
p_μ $(\mu = 0,1,2,3)$	p_μ $(\mu = 1,2,3,4)$ $(p_4 = i p_0)$
$\frac{p_\mu p_\nu}{p^2}$	$\frac{-p_\mu p_\nu}{p^2}$
$p \cdot q$	$-(p \cdot q)$
$i\epsilon^{\mu\nu\alpha\beta}$ $\left(\mu\nu\alpha\beta = 0,1,2,3 \quad \epsilon_{0123} = 1 = -\epsilon^{0123}\right)$	$-\epsilon^{\mu\nu\alpha\beta}$ $\left(= -\epsilon_{\mu\nu\alpha\beta}\right)$ $(\mu\nu\alpha\beta = 1,2,3,4)$

Examples:



$$(i)\,\frac{-g_{\mu\nu}}{p^2 + i\varepsilon} \longrightarrow \frac{1}{(2\pi)^4 i}\,\frac{\delta_{\mu\nu}}{p^2 - i\varepsilon}$$



$$(i)\,\frac{1}{2\sqrt{2}}\, g\, \gamma^\alpha (1 - \gamma_5) \longrightarrow \left((2\pi)^4 i\right) \frac{i}{2\sqrt{2}}\, g\, \gamma^\alpha (1 + \gamma_5)$$

I. Standard Model ($SU(3) \times SU(2) \times U(1)$)
Lagrangian in R_ξ gauge

1. QCD Lagrangian

In Dirac metric,
Gauge-fixing term,
Ghost term,
R_ξ gauge
Light-cone gauge (axial gauge)

Feynman rules in $\begin{pmatrix} \text{Dirac} \ \& \\ \text{B-J} \end{pmatrix}$ notations

I.-1 QCD $\mathcal{L}$ p1

The Complete Lagrangian of QCD
in Dirac Metric

1. For an SU(3) group structure, there are 8 generators, which are chosen to be $\frac{\lambda^a}{2}$, [illegible] and λ^a's are Gell-mann matrices.

$$\left[\frac{\lambda^a}{2}, \frac{\lambda^b}{2}\right] = i f_{abc} \frac{\lambda^c}{2}$$

$$Tr(\lambda^a \lambda^b) = 2\delta_{ab}$$

(f_{abc} are odd under the permutation of any pair of indices.)

1) The Lagrangian for the fermion fields is

$$\mathcal{L}_f = -\bar{\psi} \not\partial \psi$$

Under an SU(3) transformation X, with

$$X = e^{-i\frac{\lambda^a}{2}\beta^a} \qquad (\beta^a(x) \text{ is real})$$

then $\psi \to X\psi$

In other word, under a local gauge transformation

$$\psi \to X\psi,$$

therefor

$$\partial_\mu \psi \to X \partial_\mu \psi + (\partial_\mu X)\psi$$

The idea is to introduce a <u>covariant derivative</u> $D_\mu \psi$ which transforms like ψ, ie

$$D_\mu \psi \to X D_\mu \psi$$

Then $\mathcal{L}_f$ will be invariant under local gauge transformation

2) The covariant derivative D_μ is found to be

$$D_\mu = \partial_\mu - ig\frac{\lambda^a}{2}A_\mu^a$$

where A_μ^a are the vector boson fields

3) The SU(3) vector boson Lagrangian is

$$\mathcal{L}_{Kin} = \frac{-1}{4}F_{\mu\nu}^a F_{\mu\nu}^a,$$

with

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f_{abc} A_\mu^b A_\nu^c$$

If we define the matrices

$$\mathbb{A}_\mu \equiv \frac{-i\lambda^a}{2}A_\mu^a,$$

and

$$\mathbb{F}_{\mu\nu} \equiv \frac{-i\lambda^a}{2}F_{\mu\nu}^a,$$

then

$$\mathbb{F}_{\mu\nu} = \partial_\mu \mathbb{A}_\nu - \partial_\nu \mathbb{A}_\mu + g[\mathbb{A}_\mu, \mathbb{A}_\nu]$$
$$= \frac{1}{g}[D_\mu, D_\nu],$$

$$\left(\begin{aligned} D_\mu &= \partial_\mu - ig\frac{\lambda^a}{2}A_\mu^a \\ &= \partial_\mu + g\mathbb{A}_\mu \end{aligned} \right)$$

Since $Tr\left(\frac{\lambda^a}{2}\frac{\lambda^b}{2}\right) = \frac{1}{2}\delta_{ab}$,

therefore

$$\mathcal{L}_{Kin} = \frac{-1}{4}F_{\mu\nu}^a F_{\mu\nu}^a$$
$$= \frac{-1}{2}Tr(\mathbb{F}_{\mu\nu}\mathbb{F}_{\mu\nu})$$

(*) For short-hand, we will denote $\frac{\lambda^a}{2}$ as T^a

2. The Lagrangian is

$$\mathcal{L}_0 = \mathcal{L}_{kin} + \mathcal{L}_f$$
$$= \frac{-1}{4} F^a_{\mu\nu} F^a_{\mu\nu} - \bar{\psi}(\not{D} + m)\psi ,$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A_\nu$$
$$D_\mu = \partial_\mu - i g T^a A^a_\mu .$$

(Note: Here we put in the fermion mass by hand. But it should be understood that it comes from the symmetry breaking.)

1)

$$F^a_{\mu\nu} F^a_{\mu\nu} = \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu\right)\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{ab'c'} A^{b'}_\mu A^{c'}_\nu\right)$$
$$= \partial_\mu A^a_\nu \partial_\mu A^a_\nu - \partial_\nu A^a_\mu \partial_\mu A^a_\nu + g f_{abc} A^b_\mu A^c_\nu \partial_\mu A^a_\nu$$
$$- \partial_\mu A^a_\nu \partial_\nu A^a_\mu + \partial_\nu A^a_\mu \partial_\nu A^a_\mu - g f_{abc} A^b_\mu A^c_\nu \partial_\nu A^a_\mu$$
$$+ g f_{ab'c'} (\partial_\mu A^a_\nu) A^{b'}_\mu A^{c'}_\nu - g f_{ab'c'} (\partial_\nu A^a_\mu) A^{b'}_\mu A^{c'}_\nu$$
$$+ g^2 f_{abc} f_{ab'c'} A^b_\mu A^c_\nu A^{b'}_\mu A^{c'}_\nu$$

Obviously,

$$\partial_\mu A^a_\nu \partial_\mu A^a_\nu = \partial_\nu A^a_\mu \partial_\nu A^a_\mu$$
$$\partial_\nu A^a_\mu \partial_\mu A^a_\nu = \partial_\mu A^a_\nu \partial_\nu A^a_\mu$$

Since we express all the terms in terms of group components (a, b, c) not matrices, <u>therefore</u> we can interchange the order of A-<u>fields</u>. Hence

$$f_{abc} A^b_\mu A^c_\nu \partial_\mu A^a_\nu = f_{abc} (\partial_\mu A^a_\nu) A^b_\mu A^c_\nu \quad , etc.$$

$$F^a_{\mu\nu}F^a_{\mu\nu} = 2\left[\partial_\mu A^a_\nu \partial_\mu A^a_\nu - \partial_\mu A^a_\nu \partial_\nu A^a_\mu\right]$$
$$+2g\left[f_{abc}(\partial_\mu A^a_\nu)A^b_\mu A^c_\nu - f_{abc}(\partial_\nu A^a_\mu)A^b_\mu A^c_\nu\right]$$
$$+ g^2 f_{abc} f_{ab'c'} A^b_\mu A^c_\nu A^{b'}_\mu A^{c'}_\nu$$

2) $\bar\psi(\not{D}+m)\psi = \bar\psi(\not{\partial}+m)\psi + \bar\psi(-igT^a\not{A}^a)\psi$

3) The complete Lagrangian of QCD is ~~therefore~~

$$\mathcal{L}_0 = \frac{-1}{4} F^a_{\mu\nu}F^a_{\mu\nu} - \bar\psi(\not{D}+m)\psi$$

$$= \frac{-1}{2}\left[\partial_\mu A^a_\nu \partial_\mu A^a_\nu - \partial_\mu A^a_\nu \partial_\nu A^a_\mu\right]$$
$$\frac{-1}{2} g f_{abc}\left[\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right] A^b_\mu A^c_\nu$$
$$\frac{-1}{4} g^2 f_{abc} f_{ade} A^b_\mu A^c_\nu A^d_\mu A^e_\nu$$
$$- \bar\psi(\not{\partial}+m)\psi$$
$$+ igT^a A^a_\mu \bar\psi \gamma_\mu \psi$$

3. Calculate the propagator of vector boson field A_μ (ie gluon)

1) From the Lagrangian, the quadratic terms of A_μ are

$$\frac{-1}{2}\left[\partial_\mu A^a_\nu \partial_\mu A^a_\nu - \partial_\mu A^a_\nu \partial_\nu A^a_\mu\right]$$
$$= \frac{-1}{2}\left[- A^a_\nu \partial_\mu\partial_\mu A^a_\nu + A^a_\nu \partial_\mu \partial_\nu A^a_\mu\right]$$
$$= \frac{-1}{2}\left[- A^a_\mu \partial\partial A^a_\mu + A^a_\mu \partial_\nu \partial_\mu A^a_\nu\right]$$
$$= \frac{-1}{2} A^a_\mu\left[-\partial^2 \delta_{\mu\nu}\delta_{ab} + \partial_\mu\partial_\nu \delta_{ab}\right] A^b_\nu$$

Note that because of gauge invariance, there is no inverse of this operator. Therefore propagator doesn't exist until we break the gauge invariance of this Lagrangian (Here we assume $m=0$.)

2) The gauge fixing term

$$\mathcal{L}_{gf} = \frac{-1}{2\alpha}(\partial_\mu A^a_\mu)^2 \quad \underline{\text{or}} \quad \boxed{\mathcal{L}_{gf} = \frac{-1}{2\alpha}(-\partial_\mu A^a_\mu)^2}$$

another form

Take this sign to agree with convention

$$= \frac{-1}{2\alpha}\partial_\mu A^a_\mu \partial_\nu A^a_\nu$$

$$= \frac{-1}{2\alpha}\left(-A^a_\mu \partial_\mu \partial_\nu A^a_\nu\right)$$

$$= \frac{-1}{2\alpha} A^a_\mu \left(-\partial_\mu \partial_\nu \delta_{ab}\right) A^b_\nu$$

Now the quadratic terms ~~of~~ A_μ in $(\mathcal{L}_0 + \mathcal{L}_{gf})$ is

$$\mathcal{L}_2 = \frac{-1}{2} A^a_\mu \left[-\partial^2 \delta_{\mu\nu}\delta_{ab} + \left(1-\frac{1}{\alpha}\right)\partial_\mu \partial_\nu \delta_{ab}\right] A^b_\nu$$

The propagator of A_μ is obtained through the derivative

$$\begin{pmatrix} \partial^2 \to -k^2 \\ \partial_\mu \to +ik_\mu \end{pmatrix}$$

$$P^{-1} \equiv \frac{-\partial^2 \mathcal{F}(\mathcal{L}_2)}{\partial A^c_\alpha \partial A^d_\beta},$$

($\mathcal{F}(\mathcal{L}_2)$ means taking the Fourier transformation of $\mathcal{L}_2$)

$$= k^2 \delta_{\alpha\beta}\delta_{cd} - \left(1-\frac{1}{\alpha}\right) k_\alpha k_\beta \delta_{cd}$$

Then take the inverse of the above quantity.

Define propagator as

$$P = \left(A\,\delta_{\mu\nu} + B\,\frac{k_\mu k_\nu}{k^2}\right)\delta_{ab}$$

Thus

$$\left(A\delta_{\mu\nu}+B\frac{k_\mu k_\nu}{k^2}\right)\left(k^2\delta_{\nu\sigma}-\left(1-\frac{1}{\alpha}\right)k_\nu k_\sigma\right)\delta_{ab}=\delta_{\mu\sigma}\delta_{ab}$$

i.e.

$$Ak^2\delta_{\mu\sigma}+B\,k_\mu k_\sigma - A\left(1-\frac{1}{\alpha}\right)k_\mu k_\sigma - B\left(1-\frac{1}{\alpha}\right)k_\mu k_\sigma=\delta_{\mu\sigma}$$

So

$$Ak^2=1 \quad \text{and}$$

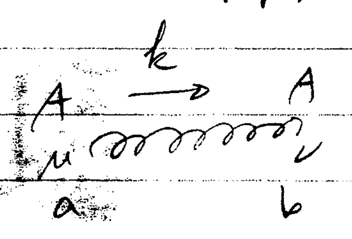
$$B-A\left(1-\frac{1}{\alpha}\right)-B\left(1-\frac{1}{\alpha}\right)=0$$

Thus

$$A=\frac{1}{k^2},\qquad \frac{B}{\alpha}=\left(1-\frac{1}{\alpha}\right)\frac{1}{k^2},$$

$$B=(\alpha-1)\frac{1}{k^2}$$

Hence the propagator of A_μ^a is



$$\frac{\delta_{ab}}{k^2}\left[\delta_{\mu\nu}-\frac{(1-\alpha)k_\mu k_\nu}{k^2}\right]$$

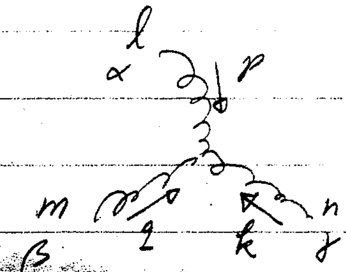
4. Calculate the 3-vertex gluon self-couplings.

1) The relevant term is

$$\mathcal{L}_3=\frac{-1}{2}g\,f_{abc}\left[\partial_\mu A_\nu^a-\partial_\nu A_\mu^a\right]A_\mu^b A_\nu^c$$

2) To obtain its Feynman rule, use

$$\Gamma^{(3)}=\frac{\partial^3\mathcal{F}(\mathcal{L}_3)}{\partial A_\alpha^l\,\partial A_\beta^m\,\partial A_\gamma^n}$$



Assign the array (ℓ, m, n), then the possible permitations are ($3! = 3\times2\times1 = 6$)

(abc), (a, cb), (bca), (b, a, c)
(cab) and (cba)

However, because of the propety of f_{abc} [illegible] only have to consider $\frac{3!}{2} = 3$ cases.

(1) Consider $(\ell, m, n) \to (a, b, c)$, we have

$$\frac{-1}{2} g f_{abc}\, \delta_{mb}\delta_{\beta\mu}\delta_{nc}\delta_{\gamma\nu}\left[\delta_{a\ell}\delta_{\alpha\nu}(ip_\mu) - \delta_{\mu\alpha}\delta_{a\ell}(ip_\nu)\right]$$

$$= \frac{-1}{2} g f_{\ell mn}\left[(ip_\beta)\delta_{\alpha\gamma} - (ip_\gamma)\delta_{\alpha\beta}\right]$$

(2) Consider $(\ell, m, n) \to (b, c, a)$

$$\frac{-1}{2} g f_{abc}\, \delta_{\ell b}\delta_{mc}\delta_{\alpha\mu}\delta_{\nu\beta}\left[\delta_{an}\delta_{\nu\gamma}(ik_\mu) - \delta_{an}\delta_{\mu\gamma}(ik_\nu)\right]$$

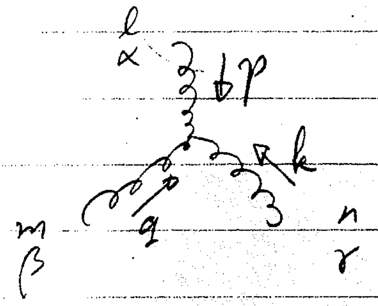
$$= \frac{-1}{2} g f_{n\ell m}\left[(ik_\alpha)\delta_{\beta\gamma} - (ik_\beta)\delta_{\alpha\gamma}\right]$$

Note. $f_{n\ell m} = -f_{\ell nm} = -(-f_{\ell mn}) = f_{\ell mn}$

(3) Consider $(\ell, m, n) \to (c, a, b)$

$$\frac{-1}{2} g f_{\ell mn}\left[(iq_\gamma)\delta_{\alpha\beta} - (iq_\alpha)\delta_{\beta\gamma}\right]$$

3) Hence its Feynman rule is



$$+ig f_{lmn}\left\{ (q-k)_\alpha \delta_{\beta\gamma} + (k-p)_\beta \delta_{\gamma\alpha} + (p-q)_\gamma \delta_{\alpha\beta} \right\}$$

5. Calculate the 4-vertex gluon self couplings.

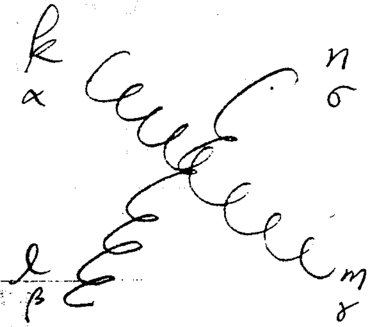
1) The relevant term is

$$\mathcal{L}_4 = \frac{-1}{4} g^2 f_{abc} f_{ade} A^b_\mu A^c_\nu A^d_\mu A^e_\nu$$

2) To obtain its Feynman rule, use

$$\Gamma^{(4)} = \frac{\partial^4 \mathcal{F}(\mathcal{L}_4)}{\partial A^k_\alpha \partial A^l_\beta \partial A^m_\gamma \partial A^n_\sigma}$$

Assign the array (k, l, m, n), then there are $4! = 4\times3\times2\times1 = 24$ possible permutations.



But due to the symmetry that (k, l, m, n) is related to (k, l, n, m),
(k, m, l, n)
and (k, n, m, l)

(1) Consider $(k,l,m,n) \to (b,c,d,e)$,

$$\frac{-1}{4} g^2 f_{abc} f_{ade}\, \delta_{bk}\delta_{\mu\alpha}\delta_{cl}\delta_{\nu\beta}\delta_{dm}\delta_{\mu\gamma}\delta_{en}\delta_{\nu\sigma}$$

$$= \frac{-1}{4} g^2 f_{akl} f_{amn}\, \delta_{\alpha\gamma}\delta_{\beta\sigma}$$

Consider $(k,l,n,m) \to (bcde)$

$$\frac{-1}{4} g^2 f_{akl} f_{anm}\, \delta_{\alpha\sigma}\delta_{\beta\gamma} = \frac{-1}{4} g^2 f_{akl} f_{amn} \left(-\delta_{\alpha\sigma}\delta_{\beta\gamma}\right)$$

Consider $(k,m,l,n) \to (bcde)$

$$\frac{-1}{4} g^2 f_{akm} f_{aln}\, \delta_{\alpha\beta}\delta_{\sigma\gamma} \checkmark$$

Consider $(k,n,m,l) \to (bcde)$

$$\frac{-1}{4} g^2 f_{akn} f_{aml}\, \delta_{\alpha\sigma}\delta_{\beta\gamma} \checkmark$$

$(k,m,n,l) \Rightarrow f_{akm} f_{anl}\, \delta_{\alpha\sigma}\delta_{\gamma\beta}$ ✓

$(k,n,l,m) \Rightarrow f_{akn} f_{alm}\, \delta_{\alpha\beta}\delta_{\sigma\gamma}$ ✓

(2) Consider $(l,m,n,k) \to (b,c,d,e)$

$$\frac{-1}{4} g^2 f_{alm} f_{ank}\, \delta_{\beta\sigma}\delta_{\alpha\gamma}$$

$(l,m,k,n) \Rightarrow \frac{-1}{4} g^2 f_{alm} f_{akn}\, \delta_{\alpha\beta}\delta_{\sigma\gamma}$

$(l,n,m,k) \Rightarrow \frac{-1}{4} g^2 f_{aln} f_{amk}\, \delta_{\beta\gamma}\delta_{\alpha\sigma}$ ✓

$(l,k,n,m) \Rightarrow \frac{-1}{4} g^2 f_{alk} f_{anm}\, \delta_{\beta\sigma}\delta_{\alpha\gamma}$ ✓

$(l,n,k,m) \Rightarrow f_{aln} f_{akm}\, \delta_{\alpha\beta}\delta_{\sigma\gamma}$

$(l,k,m,n) \Rightarrow f_{alk} f_{amn}\, \delta_{\beta\gamma}\delta_{\alpha\sigma}$ ✓

(3) Consider $(m,n,k,l) \to (b,c,d,e)$

$(m,n,k,l) \Rightarrow f_{amn} f_{akl}\, \delta_{\alpha\gamma}\delta_{\beta\sigma}$ ✓

$(m,n,l,k) \Rightarrow f_{amn} f_{alk}\, \delta_{\beta\gamma}\delta_{\alpha\sigma}$ ✓

$(m,k,n,l) \Rightarrow f_{amk} f_{anl}\, \delta_{\sigma\gamma}\delta_{\alpha\beta}$ | $(m,k,l,n) \Rightarrow f_{akl} f_{aln}\, \delta_{\beta\gamma}\delta_{\alpha\sigma}$

$(m,l,k,n) \Rightarrow f_{aml} f_{akn}\, \delta_{\alpha\gamma}\delta_{\beta\sigma}$ | [illegible] $f_{ank}\, \delta_{\sigma\gamma}\delta_{\alpha\beta}$

(4) Consider (n,k,l,m).

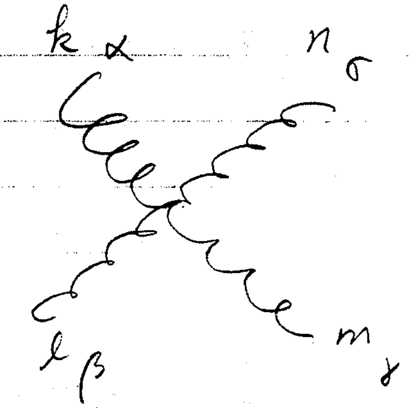
$(n,k,l,m) \Rightarrow f_{ank} f_{alm}\, \delta_{\beta\sigma}\delta_{\alpha\gamma}$

$(n,k,m,l) \Rightarrow f_{ank} f_{aml}\, \delta_{\sigma\gamma}\delta_{\alpha\beta}$

$(n,l,k,m) \Rightarrow f_{anl} f_{akm}\, \delta_{\alpha\sigma}\delta_{\beta\gamma}$ | $(n,l,m,k) \Rightarrow f_{anl} f_{amk}\, \delta_{\sigma\gamma}\delta_{\alpha\beta}$

$(n,m,l,k) \Rightarrow f_{anm} f_{alk}\, \delta_{\sigma\beta}\delta_{\alpha\gamma}$ ✓ | $(n,m,k,l) \Rightarrow f_{anm} f_{akl}\, \delta_{\alpha\sigma}\delta_{\beta\gamma}$

Hence, its Feynman rule is



$$\frac{-1}{4} g^2 f_{akl} f_{amn}\left(\delta_{\alpha\gamma}\delta_{\beta\sigma} - \delta_{\alpha\sigma}\delta_{\beta\gamma}\right)\cdot 4$$

$$\frac{-1}{4} g^2 f_{akm} f_{aln}\left(\delta_{\alpha\beta}\delta_{\sigma\gamma} - \delta_{\beta\gamma}\delta_{\alpha\sigma}\right)\cdot 4$$

$$\frac{-1}{4} g^2 f_{akn} f_{alm}\left(\delta_{\alpha\beta}\delta_{\sigma\gamma} - \delta_{\alpha\sigma}\delta_{\beta\gamma}\right)\cdot 4$$

$$= -g^2 \left\{ \begin{aligned} & f_{akl} f_{amn}\left(\delta_{\alpha\gamma}\delta_{\beta\sigma} - \delta_{\alpha\sigma}\delta_{\beta\gamma}\right) \\ & + f_{akm} f_{aln}\left(\delta_{\alpha\beta}\delta_{\sigma\gamma} - \delta_{\beta\gamma}\delta_{\alpha\sigma}\right) \\ & + f_{akn} f_{alm}\left(\delta_{\alpha\beta}\delta_{\sigma\gamma} - \delta_{\alpha\sigma}\delta_{\beta\gamma}\right) \end{aligned} \right\}$$

6. Fermion Propagator

$$\mathcal{L}_f = -\bar{\psi}(\not\partial + m)\psi$$

Its propagator is

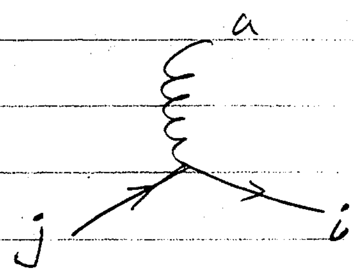
[redacted]

$$\mathbb{P} = \left(\frac{-\partial^2 \mathcal{F}(\mathcal{L}_f)}{\partial \bar{\psi}\, \partial \psi} \right)^{-1}$$

$$= (i\not p + m)^{-1}$$

$$= \frac{1}{i\not p + m} = \frac{-i\not p + m}{p^2 + m^2}$$

7. Fermion-Gluon vertex



The interaction term is

$$+ig T^a A^a_\mu \bar{\psi} \gamma_\mu \psi$$

$$= ig\, \bar{\psi}_i \gamma_\mu (T^a)_{ij} \psi_j A^a_\mu$$

$a = 1, 2 \cdots 8$
$i, j = 1, 2, 3$

Its Feynman rule is

$$ig (T^a)_{ij} \gamma_\mu$$

<u>Note</u>: The order of (ij) is important because T^a is not a symmetric matrix.

8. Ghost.

The ghost Lagrangian is obtained as follows:

1) The gauge fixing term is expressed as

$$\mathcal{L}_{gf} = \frac{-1}{2\alpha}\left(C^a\right)^2,$$ ($C^a =$ [redacted] case)

Then the ghost Lagrangian is [redacted]

$$\bar{\chi}^a M_{ab} \chi^b \quad ; \qquad a, b = 1, 2, \cdots 8$$

where M_{ab} is obtained from

$$M_{ab} = \frac{\delta C^a}{\delta \Lambda^b}$$

2) Since under a local gauge transformation,

this is another option

$$\boxed{(-)\, C^a = \partial_\mu A^a_\mu} \longrightarrow (-)\,\partial_\mu\left(A^a_\mu + \delta A^a_\mu\right) \equiv C^a + \delta C^a$$

so

$$\begin{aligned}
(-)\,\delta C^a &= \partial_\mu\left(\delta A^a_\mu\right) \\
&= \partial_\mu\left(-\partial_\mu \Lambda^a + g f_{abc} \Lambda^b A^c_\mu\right) \\
&= -\partial^2 \Lambda^a + g f_{abc}\, \partial_\mu\left(\Lambda^b A^c_\mu\right) \\
&= \left\{ -\partial^2 \delta_{ab} + g f_{abc}\left[A^c_\mu \partial_\mu + \partial_\mu A^c_\mu\right] \right\} \Lambda^b
\end{aligned}$$

Thus

$$\begin{aligned}
(-)\,\frac{\delta C^a}{\delta \Lambda^b} &= -\partial^2 \delta_{ab} + g f_{abc}\left[A^c_\mu \partial_\mu + \partial_\mu A^c_\mu\right] \\
&= M_{ab}
\end{aligned}$$

3) The resulting ghost Lagrangian is

$$(-)\,\mathcal{L}_{gh} = \bar{\chi}^a M_{ab}\,\chi^b$$

$$= \bar{\chi}^a \left\{ -\partial^2 \delta_{ab} + g f_{abc}\left[A^c_\mu \partial_\mu + \partial_\mu A^c_\mu\right]\right\}\chi^b$$

$$= -\bar{\chi}^a \partial^2 \delta_{ab}\,\chi^b$$

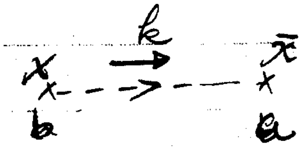
$$+ g f_{abc}\,\bar{\chi}^a\,\partial_\mu\left(A^c_\mu \chi^b\right) \quad = -\left\{\bar{\chi}\,\partial_\mu\left(D_\mu\right)\chi\right\}$$

$$= \left(\partial_\mu \bar{\chi}^a\right)\left(\partial_\mu \chi^a\right) - g f_{abc}\left(\partial_\mu \bar{\chi}^a\right) A^c_\mu\,\chi^b$$

4) The propagator of ghost is

$$P = (-)\left(\frac{-\partial^2 \mathcal{F}\left(-\bar{\chi}^a \partial^2 \delta_{ab}\,\chi^b\right)}{\partial \bar{\chi}^a\;\partial \chi^b}\right)^{-1},$$

$$= (-)\,\frac{-1}{k^2}\,\delta_{ab}$$

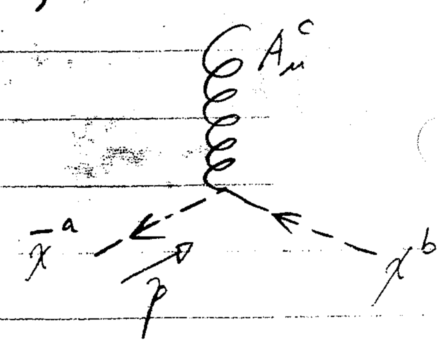


5) The vertex of gluon-ghost

Notice that $\bar{\chi}$ is <u>not</u> the anti-particle of χ.
Hence its Feynman rule is

$$(-) - g f_{abc}\left(+i p_\mu\right)$$

$$= (-) - i g f_{abc}\, p_\mu$$



Summary:

The complete Lagrangian of QCD

$$\mathcal{L} = \mathcal{L}_{Kin} + \mathcal{L}_f + \mathcal{L}_{gf} + \mathcal{L}_{gh}$$

$$\mathcal{L}_{Kin} = \frac{-1}{2}\left(\partial_\mu A_\nu^a \partial_\mu A_\nu^a - \partial_\mu A_\nu^a \partial_\nu A_\mu^a\right) - \frac{1}{2} g f_{abc}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right) A_\mu^b A_\nu^c$$
$$- \frac{1}{4} g^2 f_{abc} f_{ade} A_\mu^b A_\nu^c A_\mu^d A_\nu^e$$

$$\mathcal{L}_f = -\bar{\psi}(\not{\partial} + m)\psi + ig\frac{\lambda^a}{2} A_\mu^a \bar{\psi}\gamma_\mu \psi$$

$$\mathcal{L}_{gf} = \frac{-1}{2\alpha}\left(\partial_\mu A_\mu^a\right)^2 \quad \underline{\underline{or}} \quad \mathcal{L}_{gf} = \frac{-1}{2\alpha}\left(-\partial_\mu A_\mu^a\right)^2$$

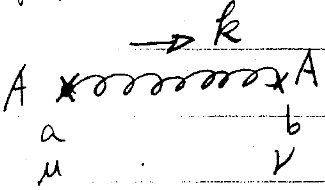
$$(-)\ \mathcal{L}_{gh} = \left(\partial_\mu \bar{\chi}^a\right)\left(\partial_\mu \chi^a\right) - g f_{abc}\left(\partial_\mu \bar{\chi}^a\right)\chi^b A_\mu^c$$

Note:

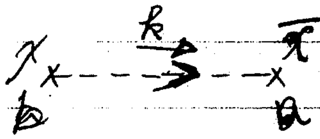
$$Tr\left(\lambda^a \lambda^b\right) = 2\delta_{ab}$$
$$\left[\frac{\lambda^a}{2}, \frac{\lambda^b}{2}\right] = i f_{abc}\frac{\lambda^c}{2}$$

Feynman rules for QCD in Dirac matric.
We will suppress $(2\pi)^4 i$ factor and the δ-function for each vertex, also $\frac{1}{(2\pi)^4 i}$ factor for each propagator.
(All the momenta are flowing into the vertex)

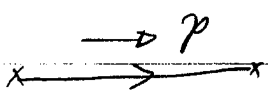
Propagators.



$$\frac{\delta_{ab}}{k^2}\left[\delta_{\mu\nu} - \frac{(1-\alpha)k_\mu k_\nu}{k^2}\right]$$

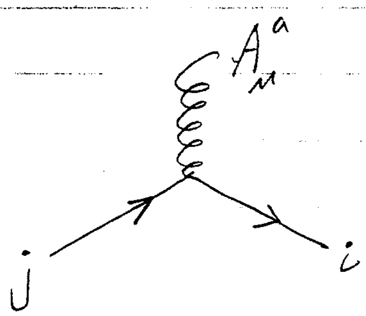


$$\boxed{(-)}\ \frac{-1}{k^2}\delta_{ab}$$

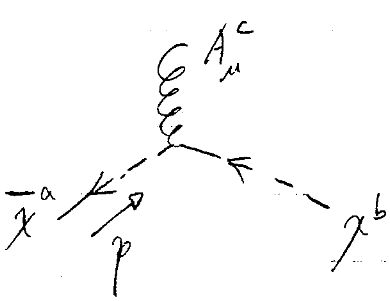


$$\frac{1}{i\not{p}+m} = \frac{-i\not{p}+m}{p^2+m^2}$$

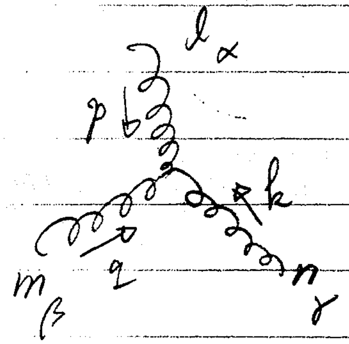
Vertices:



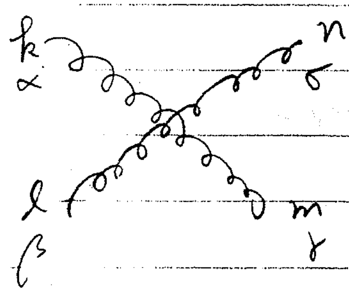
$$ig\gamma_\mu\left(\frac{\lambda^a}{2}\right)_{ij}$$



$$\boxed{(-)}\ -ig f_{abc}\, p_\mu$$



$$+ig f_{lmn}\left\{(q-k)_\alpha \delta_{\beta\gamma} + (k-p)_\beta \delta_{\gamma\alpha} + (p-q)_\gamma \delta_{\alpha\beta}\right\}$$



$$\begin{aligned}-g^2\Big\{ & f_{akl} f_{amn} (\delta_{\alpha\gamma}\delta_{\beta\sigma} - \delta_{\alpha\sigma}\delta_{\beta\gamma}) \\ & + f_{akm} f_{aln} (\delta_{\alpha\beta}\delta_{\sigma\gamma} - \delta_{\beta\gamma}\delta_{\alpha\sigma}) \\ & + f_{akn} f_{alm} (\delta_{\alpha\beta}\delta_{\sigma\gamma} - \delta_{\alpha\sigma}\delta_{\beta\gamma}) \Big\}\end{aligned}$$

In B-D notation of QCD Feynman rules:

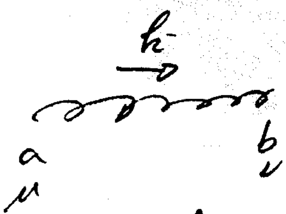
1. λ^a's are defined the same as in Dirac notation

$$Tr\left(\frac{\lambda^a}{2}\frac{\lambda^b}{2}\right) = \frac{1}{2}\delta_{ab}$$

$$\left[\frac{\lambda^a}{2}, \frac{\lambda^b}{2}\right]_- = i f_{abc}\frac{\lambda^c}{2}$$

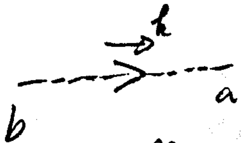
(A factor i is included in both propagator and vertex.)

2.

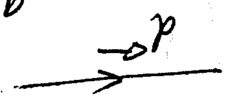


$$(i)\frac{\delta_{ab}}{k^2}\left[-g_{\mu\nu} + \frac{(1-\alpha)k_\mu k_\nu}{k^2}\right]$$

$$\mathcal{L}_{gf} = \frac{-1}{2}(\partial_\mu A^\mu)^2$$



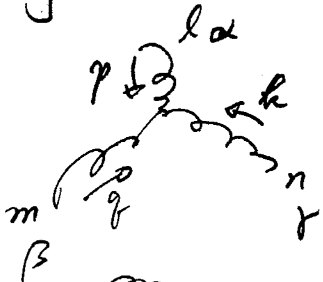
$$(i)\frac{-1}{k^2}\delta_{ab} \quad (-1)$$ for



$$(i)\frac{1}{\not p - m} = (i)\frac{\not p + m}{p^2 - m^2}$$

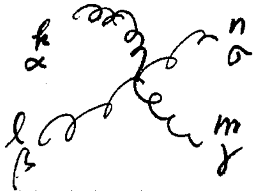
$$(i) g_s \gamma_\mu \left(\frac{\lambda^a}{2}\right)_j^i$$

(Incoming quark is in (3) representation ⇒ lower index. Outgoing quark is in ($\bar{3}$) representation so upper index.)



$$g_s f^{lmn}\{(p-q)_\gamma g_{\alpha\beta} + (q-k)_\alpha g_{\beta\gamma} + (k-p)_\beta g_{\alpha\gamma}\}$$

↑ no (i) here



$$-(i)g_s^2\left\{ f^{akl}f^{amn}(g_{\alpha\gamma}g_{\beta\sigma} - g_{\alpha\sigma}g_{\beta\gamma}) + f^{akm}f^{aln}(g_{\alpha\beta}g_{\sigma\gamma} - g_{\beta\gamma}g_{\alpha\sigma}) + f^{akn}f^{alm}(g_{\alpha\beta}g_{\sigma\gamma} - g_{\alpha\sigma}g_{\beta\gamma}) \right\}$$

$$g_s f^{abc} p_\mu \quad (-1)$$

↑ no (i) here

Feynman rules in Axial gauge

The Lagrangian is

$$\mathcal{L} = \frac{-1}{4} F^a_{\mu\nu} F^a_{\mu\nu} - \bar\psi(\not{D} + m)\psi - \frac{1}{2\lambda}\left(n_\mu A^a_\mu\right)^2,$$

where we have chosen the gauge fixing terms as $C^a = n_\mu A^a_\mu$.

1) To find the propagator of the gluon field, we only need the quadratic part of $\mathcal{L}$, which gives

$$\mathcal{L}_2 \equiv \frac{-1}{2}\left[\partial_\mu A^a_\nu \partial_\mu A^a_\nu - \partial_\mu A^a_\nu \partial_\nu A^a_\mu\right] - \frac{1}{2\lambda} n_\mu n_\nu A^a_\mu A^a_\nu$$

$$= \frac{-1}{2} A^a_\mu \left[-\partial^2 \delta_{\mu\nu}\delta_{ab} + \partial_\mu\partial_\nu \delta_{ab} + \frac{1}{\lambda} n_\mu n_\nu \delta_{ab}\right] A^b_\nu$$

The propagator of A_μ is obtained through the derivative

$$P^{-1} \equiv \frac{-\partial^2 \mathcal{J}(\mathcal{L}_2)}{\partial A^c_\alpha \, \partial A^d_\beta} \qquad \begin{pmatrix} \partial^2 \to -k^2 \\ \partial_\mu \to +ik_\mu \end{pmatrix}$$

$$= k^2 \delta_{\alpha\beta}\delta_{cd} - k_\alpha k_\beta \delta_{cd} + \frac{1}{\lambda} n_\alpha n_\beta \delta_{cd}$$

$$= \delta_{cd}\left[k^2 \delta_{\alpha\beta} - k_\alpha k_\beta + \frac{1}{\lambda} n_\alpha n_\beta\right]$$

In order to get P, we need to inverse this matrix

$$M_{\mu\nu} = k^2 \delta_{\mu\nu} - k_\mu k_\nu + \frac{1}{\lambda} n_\mu n_\nu$$

Assume:

$$(M^{-1})_{\mu\nu} = a\,\delta_{\mu\nu} + b\,k_\mu k_\nu + c\,(k_\mu n_\nu + k_\nu n_\mu) + d\,n_\mu n_\nu,$$

then

$$\left(k^2 \delta_{\mu\tau} - k_\mu k_\tau + \frac{1}{\lambda} n_\mu n_\tau\right)\left(a\,\delta_{\tau\nu} + b\,k_\tau k_\nu + c\,(k_\tau n_\nu + k_\nu n_\tau) + d\,n_\tau n_\nu\right) = \delta_{\mu\nu}$$

$$\begin{aligned} = \; & a\left(k^2 \delta_{\mu\nu} - k_\mu k_\nu + \frac{1}{\lambda} n_\mu n_\nu\right) \\ & + b\left(\cancel{k^2 k_\mu k_\nu - k^2 k_\mu k_\nu} + \frac{1}{\lambda}(k\cdot n)\, n_\mu k_\nu\right) \\ & + c\left(\cancel{k^2 k_\mu n_\nu} + k^2 k_\nu n_\mu - \cancel{k_\mu (k\cdot k) n_\nu} - k_\mu k_\nu (n\cdot k) + \frac{1}{\lambda} n_\mu n_\nu (n\cdot k) + \frac{1}{\lambda} n^2 (n_\mu k_\nu)\right) \\ & + d\left(k^2 n_\mu n_\nu - k_\mu (n\cdot k) n_\nu + \frac{1}{\lambda} n^2 n_\mu n_\nu\right) \end{aligned}$$

Coefficient of $\delta_{\mu\nu}$: $a k^2 = 1$, $\Rightarrow$ $a = \frac{1}{k^2}$

coeff. of $k_\mu k_\nu$: $-a - c(n\cdot k) = 0 \Rightarrow c = \frac{-a}{n\cdot k} = \frac{-1}{k^2 (n\cdot k)}$

coeff. of $n_\mu n_\nu$: $\underbrace{\frac{1}{\lambda} a + \frac{1}{\lambda} c\,(n\cdot k)}_{0} + d k^2 + d\frac{n^2}{\lambda} = 0 \Rightarrow \boxed{d = 0 \text{ if } k^2 + \frac{n^2}{\lambda} \neq 0}$

coeff of $n_\mu k_\nu$: $\frac{1}{\lambda} b\,(n\cdot k) + c\left(k^2 + \frac{n^2}{\lambda}\right) = 0 \Rightarrow b = \frac{-(\lambda k^2 + n^2) c}{(n\cdot k)} = \frac{\lambda k^2 + n^2}{k^2 (n\cdot k)^2}$

~~...~~

Hence, the propagator of the gluon field is

$$\mathbb{P} = \frac{\delta_{ab}}{k^2}\left[\delta_{\mu\nu} - \frac{n_\mu k_\nu + k_\mu n_\nu}{(n\cdot k)} + \frac{(\lambda k^2 + n^2) k_\mu k_\nu}{(n\cdot k)^2}\right]$$

If we choose n_μ, so that ~~...~~

$n^2 = 0$ (This is the light-cone gauge), then $(\lambda k^2 + n^2) \to \lambda k^2$.

If $\lambda = 0$, then $(\lambda k^2 + n^2) \to n^2$, if in addition in light-cone gauge, then $n^2 \to 0$.

2) Under gauge transformation

$$C^a = n_\mu A_\mu^a \longrightarrow n_\mu (A_\mu^a + \delta A_\mu^a) = C^a + \delta C^a,$$

$$\begin{aligned}\delta C^a &= n_\mu\, \delta A_\mu^a \\ &= n_\mu(-\partial_\mu \Lambda^a + g f_{abc} \Lambda^b A_\mu^c) \\ &= \{-(n\cdot\partial) + g f_{abc} (n\cdot A^c)\}\Lambda^b\end{aligned}$$

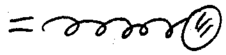
$$\frac{\delta C^a}{\delta \Lambda^b} = \left[-(n\cdot\partial) + g f_{abc} (n\cdot A^c)\right] \equiv M_{ab}$$

The ghost Lagrangian

$$\begin{aligned}\mathcal{L}_{gh} &= \bar{\chi}^a M_{ab} \chi^b \\ &= \bar{\chi}^a\left[-(n\cdot\partial) + g f_{abc} (n\cdot A^c)\right]\chi^b\end{aligned}$$

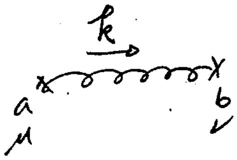
Hence, ghost field does <u>not</u> have propagator. Therefore, it will not contribute to any Green's function.

3) The Ward identity does not imply

=  $= 0$ because

$$(ik_\mu)\cdot\left(\frac{1}{(2\pi)^4 i}\right)\frac{\delta_{ab}}{k^2}\left[\delta_{\mu\nu} - \frac{n_\mu k_\nu + k_\mu n_\nu}{(n\cdot k)} + \frac{(\lambda k^2+n^2)k_\mu k_\nu}{(n\cdot k)^2}\right] \neq 0$$

4) In Bjorken-Drell notation, we get the propagator

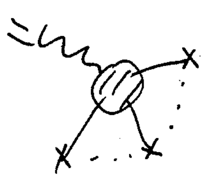


$$\frac{(i)\,\delta_{ab}}{k^2}\left[-g_{\mu\nu} + \frac{n_\mu k_\nu + k_\mu n_\nu}{(n\cdot k)} - \frac{(\lambda k^2+n^2)k_\mu k_\nu}{(n\cdot k)^2}\right]$$

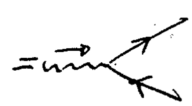
In light-cone gauge ($n^2=0$), and in the limit $\lambda=0$, we have

$$\frac{(i)\,\delta_{ab}}{k^2}\left[-g_{\mu\nu} + \frac{n_\mu k_\nu + k_\mu n_\nu}{(n\cdot k)}\right]$$

5) If all the other current sources are physical, then

 $= 0$ is always true in QCD.

Namely, $k_\mu \cdot J^\mu = 0$

For instance, in Feynman gauge  $= 0$.

<u>Note</u>. In light-cone gauge, the propagator of gluon is

$$\not{P} = \frac{\delta_{ab}}{k^2}\left(\delta_{\mu\nu} - \frac{n_\mu k_\nu + k_\mu n_\nu}{(n\cdot k)}\right),$$

where the ~~denominator~~ numerator is

$$\delta_{\mu\nu} - \frac{n_\mu k_\nu + k_\mu n_\nu}{(n\cdot k)} = \sum_{\lambda=+,-} \varepsilon^{\lambda}_{\mu}(k)\, \varepsilon^{\lambda *}_{\nu}(k)$$

for $k = (0, 0, K, iK)$ $\left(\text{or}, \quad \begin{array}{l} k = (k_+, 0, \underline{0}) \\ n = (0, 1, \underline{0}) \end{array}\right)$

& $n = \frac{1}{K}(0, 0, -K, iK)$

Also, there is no ghost contribution ~~from~~ in this gauge.

Therefore, this is the "physical" gauge.

Colour Sums

1. Some identities, generalized to $SU(N)$.
The dummy indices mean summation over these indices.

In $SU(3)$ group, we will use $T_a \equiv \frac{\lambda^a}{2}$, where λ^a are Gell-mann matrices.

2. $$[T_a, T_b] = i f_{abc} T_c$$
$$\{T_a, T_b\} = \frac{1}{N}\delta_{ab}\mathbb{1}_{(N)} + d_{abc} T_c ,$$

where $\mathbb{1}_{(N)}$ is the N-dimensional unit matrix. The f_{abc} are anti-symmetric and the d_{abc} symmetric under the interchange of any two indices. There are $(2N-1)\times(2N-1)$ nonvanishing f_{abc} or d_{abc}. (In $SU(3)$ there are 25 nonvanishing f_{abc} or d_{abc}.)

In $SU(2)$, the quantities analogous to (T_a, f_{abc}, d_{abc}) are ([illegible], ϵ_{ijk}, 0)

Note [illegible] $SU(3)$ group. In QCD, N is the number of quarks of gluon.

$N^2-1=8$ in $SU(3)$ group is the no. of gluons.

3\.

$$T_a T_a = \frac{N^2-1}{2N} \mathbb{1}_{(N)} \equiv C_F \mathbb{1}_{(N)}$$

Define $C_F = \frac{N^2-1}{2N}$ $\left(C_F = \frac{4}{3} \text{ in } SU(3) \right)$

$$T_a T_b = \frac{1}{2}\left\{ \frac{1}{N}\delta_{ab}\mathbb{1}_{(N)} + (d_{abc} + i f_{abc}) T_c \right\}$$

$$T_a^{ij} T_a^{kl} = \frac{1}{2}\left\{ \delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl} \right\}$$

$$\mathrm{Tr}(T_a) = 0$$

$$\mathrm{Tr}(T_a T_b) = \frac{1}{2}\delta_{ab}$$

$$\mathrm{Tr}(T_a T_b T_c) = \frac{1}{4}(d_{abc} + i f_{abc})$$

$$\mathrm{Tr}(T_a T_b T_a T_c) = \frac{-1}{4N}\delta_{bc}$$

$$\mathrm{Tr}(T_b T_a T_a T_c) = C_F\left(\frac{1}{2}\delta_{bc}\right) = \frac{N^2-1}{2N}\left(\frac{1}{2}\delta_{bc}\right)$$

Note: $\delta_{aa} = N^2 - 1$ for $SU(N)$ group.

$$f_{abe}f_{ecd} + f_{cbe}f_{aed} + f_{dbe}f_{ace} = 0$$

$$f_{abe}d_{ecd} + f_{cbe}d_{aed} + f_{dbe}d_{ace} = 0$$

$$f_{abe}f_{cde} = \frac{2}{N}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) + (d_{ace}d_{bde} - d_{bce}d_{ade})$$

[illegible]

$$f_{acd}f_{bcd} = N\delta_{ab}$$

$$f_{acd}d_{bcd} = 0$$

$$d_{acd}d_{bcd} = \frac{N^2-4}{N}\delta_{ab}$$

$$f_{ade}f_{bef}f_{afg}f_{cgd} = \frac{N^2}{2}\delta_{bc}$$

6. Examples:

1) $\mathrm{Tr}(T_b T_a T_a T_c) = \mathrm{Tr}\left(T_b^{ij} T_a^{jk} T_a^{kl} T_c^{li} \right)$

Since
$$T_a^{jk} T_a^{kl} = \frac{N^2-1}{2N}\,\delta^{jl},$$
So it's
$$\mathrm{Tr}\left(T_b^{ij} T_c^{ji} \right)\frac{N^2-1}{2N} = \frac{N^2-1}{2N}\cdot\left(\frac{1}{2}\delta_{bc}\right)$$

Thus $\mathrm{Tr}(T_b T_a T_a T_b) = \dfrac{4}{3}\left(\dfrac{1}{2}\,8\right) = \dfrac{16}{3}$

2) $-\mathrm{Tr}(T_b T_a T_a T_c) = \mathrm{Tr}\left(\{T_a, T_b\} T_a T_c \right) - \mathrm{Tr}\left(T_a T_b T_a T_c \right)$
$$= i f_{abe}\,\mathrm{Tr}\left(T_e T_a T_c\right) + \frac{1}{4N}\delta_{bc}$$
$$= i f_{abe}\left\{\frac{1}{4}\left(d_{eac} + i f_{eac}\right)\right\} + \frac{1}{4N}\delta_{bc}$$

Since
$$f_{abe} d_{eac} = f_{eab} d_{eac} = 0$$
$$f_{abe} f_{eac} = f_{eab} f_{eac} = N\delta_{bc}$$
So
$$\mathrm{Tr}(T_b T_a T_a T_c) = \frac{N}{4}\delta_{bc} - \frac{1}{4N}\delta_{bc}$$
$$= \frac{N^2-1}{4N}\delta_{bc}$$

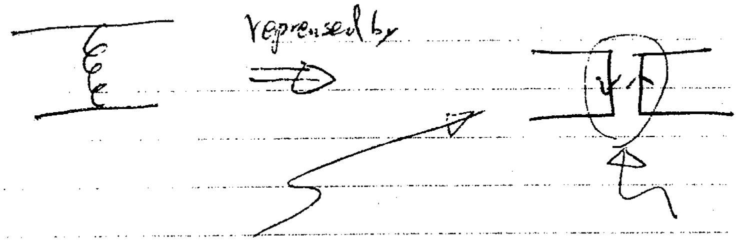
This gives the same answer as above.

1-24

4.

Some "cute" rules for Color factors

1. Replace gluon line by $\rightleftarrows$. ($\frac{1}{N_c}$ expansion due to 't Hooft.)



(arrow indicates the color flow)

$3 \times 3 = 1 \oplus 8$ ← gluon

① ~~8 colors~~

② $\left(\frac{1}{2}\right)$ for each vertex due to $Tr(T_a T_b) = \frac{1}{2}\delta_{ab}$

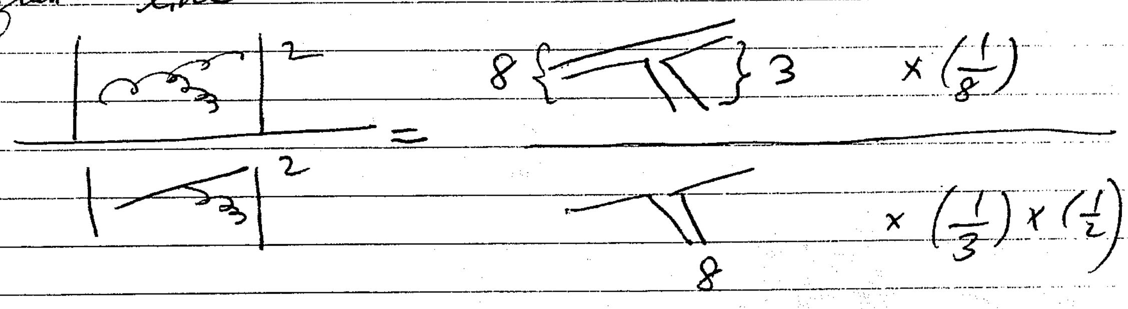
$\Rightarrow \left(\frac{1}{2}\right)\cdot\left(\frac{1}{2}\right)$ for two vertices

③ Averaging over the initial state colors,

$\Rightarrow$ Color factor is $\frac{1}{3} \times \frac{1}{3} \times 8 \times \left(\frac{1}{2}\right)^2 = \frac{2}{9}$

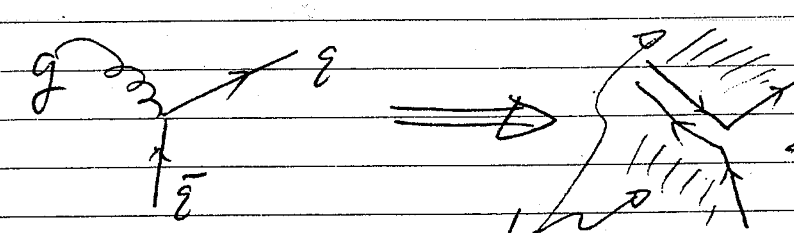
for the square of this diagram.

2. The ratio of probabilities for emitting a gluon from a gluon or quark line



$= \frac{3}{\left(\frac{4}{3}\right)} = \frac{9}{4}$

3. Hadron multiplicity: Consider $e^-e^+ \to q\bar{q}g$ in the e^-e^+ c.m. frame,



This region will have less hadrons due to $\frac{1}{N_c}$.

hadrons like to form in these two regions due to the color flow structure. Also gluon jet is broader.