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# Physics 992 Final Exam

1.

**Consider the Quantum Electrodynamics(QED) theory, up to one-loop, in the 't-Hooft-Feynman gauge.**

(**a**)

**Write down the bare Lagrangian.**

$$£_0 = -\frac{1}{4}F^0_{\mu\nu}F^0_{\mu\nu} - \bar{\Psi}^0\gamma\partial\Psi^0 - m^0\bar{\Psi}^0\Psi^0 + ie^0\bar{\Psi}^0\gamma_\mu\Psi^0A^0_\mu + \frac{1}{2}(\partial_\mu A^0_\mu)^2$$
$$F^0_{\mu\nu} = \partial_\mu A^0_\nu - \partial_\nu A^0_\mu$$

(**b**)

**Rewrite the bare Lagrangian in terms of the renormalized Lagrangian and the counterterms.**

Define

$$A^0_\mu = \sqrt{Z_3}\,A^R_\mu$$
$$\Psi^0 = \sqrt{Z_2}\,\Psi^R$$
$$e^0 = Z_e\,e^R$$
$$m^0 = Z_m m^R$$

Inserting the fields and parameters into the BARE Lagrangian, and writing $Z_i = 1 + \delta Z_i$, we split the bare Lagrangian into the renormalized Lagrangian $£^R$

$$£^R = -\frac{1}{4}F^R_{\mu\nu}F^R_{\mu\nu} - \bar{\Psi}^R\gamma\partial\Psi^R - m^R\bar{\Psi}^R\Psi^R + ie^R\bar{\Psi}^R\gamma_\mu A^R_\mu + \frac{1}{2}(\partial_\mu A^R_\mu)^2$$
$$F^R_{\mu\nu} = \partial_\mu A^R_\nu - \partial_\nu A^R_\mu$$

and the counterterm Lagrangian $\delta£$

$$£^R = -\delta Z_3\frac{1}{4}F^R_{\mu\nu}F^R_{\mu\nu} - \delta Z_2\bar{\Psi}^R\gamma\partial\Psi^R - (Z_mZ_2 - 1)m^R\bar{\Psi}^R\Psi^R + \delta Z_{1F}\; ie^R\bar{\Psi}^R\gamma_\mu\Psi^RA^R_\mu + \delta Z_3\frac{1}{2}(\partial_\mu A^R_\mu)^2$$

(**C**)

**How to introduce the renormalization scale ($\mu$) dependence in the Lagrangian?**

The action $S = \int £ d^nx$ is dimensionless, and $d^nx$ has dimension $[\mu]^{-n}$ in the $n$-dimension, so

$$(\partial A_\mu)^2 \sim [\mu]^n \Rightarrow A \sim [\mu]^{\frac{n}{2}-1}$$
$$\bar{\Psi}\gamma \bullet \partial\Psi \sim [\mu]^n \Rightarrow \Psi \sim [\mu]^{\frac{n-1}{2}}$$
$$e\bar{\Psi}\gamma\Psi A \sim [\mu]^n \Rightarrow e \sim [\mu]^{\frac{4-n}{2}} = [\mu]^{\varepsilon}, n = 4 - 2\varepsilon$$

In order to keep the renormalized coupling constant dimensionless, we introduce the renormalization scale and replace $e^0$ by $\mu^{\varepsilon}e^0$.

(**D**)

**Calculate the divergent piece and the $\ln(\mu^2)$ dependence using the dimensional regularization scheme for the two point functions of electron and photon, and the three point function of photon-electron-electron**.

The two point functions of electron

$$i\Sigma(p) = i\gamma \bullet p\left\{\frac{e^2}{16\pi^2}\left[\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi) + \ln\left(\frac{\mu^2}{M^2}\right) + O(\varepsilon)\right]\right\}$$

The two point function of photon

$$i\Pi_{\mu\nu}(p) = (p^2 g_{\mu\nu} - p_\mu p_\nu)\left\{\frac{e^2}{16\pi^2}\left(-\frac{4}{3}\right)\left[\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi) + \ln\left(\frac{\mu^2}{M^2}\right) + O(\varepsilon)\right]\right\}$$

The three point function of photon-electron-electron

$$i\Lambda_\mu = \gamma_\mu\left\{\frac{e^2}{16\pi^2}\left[\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi) + \ln\left(\frac{\mu^2}{M^2}\right) + O(\varepsilon)\right]\right\}$$

(**E**)

**Identify the explicit expression of the counterterms using the Minimal Subtraction (MS) scheme**.

$$Z_3 = 1 - \frac{e^2}{12\pi^2}\frac{1}{\varepsilon}$$

$$Z_2 = 1 - \frac{e^2}{16\pi^2}\frac{1}{\varepsilon}$$

$$Z_1 = 1 - \frac{e^2}{16\pi^2}\frac{1}{\varepsilon}$$

(**F**)

**What are the explicit expression of the counterterms in the modified MS scheme**?

$$Z_3 = 1 - \frac{e^2}{12\pi^2}\left[\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi)\right]$$

$$Z_2 = 1 - \frac{e^2}{16\pi^2}\left[\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi)\right]$$

$$Z_1 = 1 - \frac{e^2}{16\pi^2}\left[\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi)\right]$$

(**G**)

**Write down the relation you found among the counterterm**. **What is the reason for that identity to hold**?

We could see that $Z_1 = Z_2$ easily, such that

$$e^R = \frac{Z_2\sqrt{Z_3}}{Z_1}e^0 = \sqrt{Z_3}\, e^0$$

It is due to Ward identity.

(**H**)

**Write down the renormalization group (RG) equation for the electric charge**.

$$\mu \frac{\partial e}{\partial \mu}|_{e_0, m_0} = \beta$$

(**I**)

**What is the explicit expression of the $\beta$ function**?

$$\beta = e \frac{\alpha}{4\pi} \left( \frac{4}{3} \right) = e \frac{\alpha}{3\pi}, \alpha = \frac{e^2}{4\pi}$$

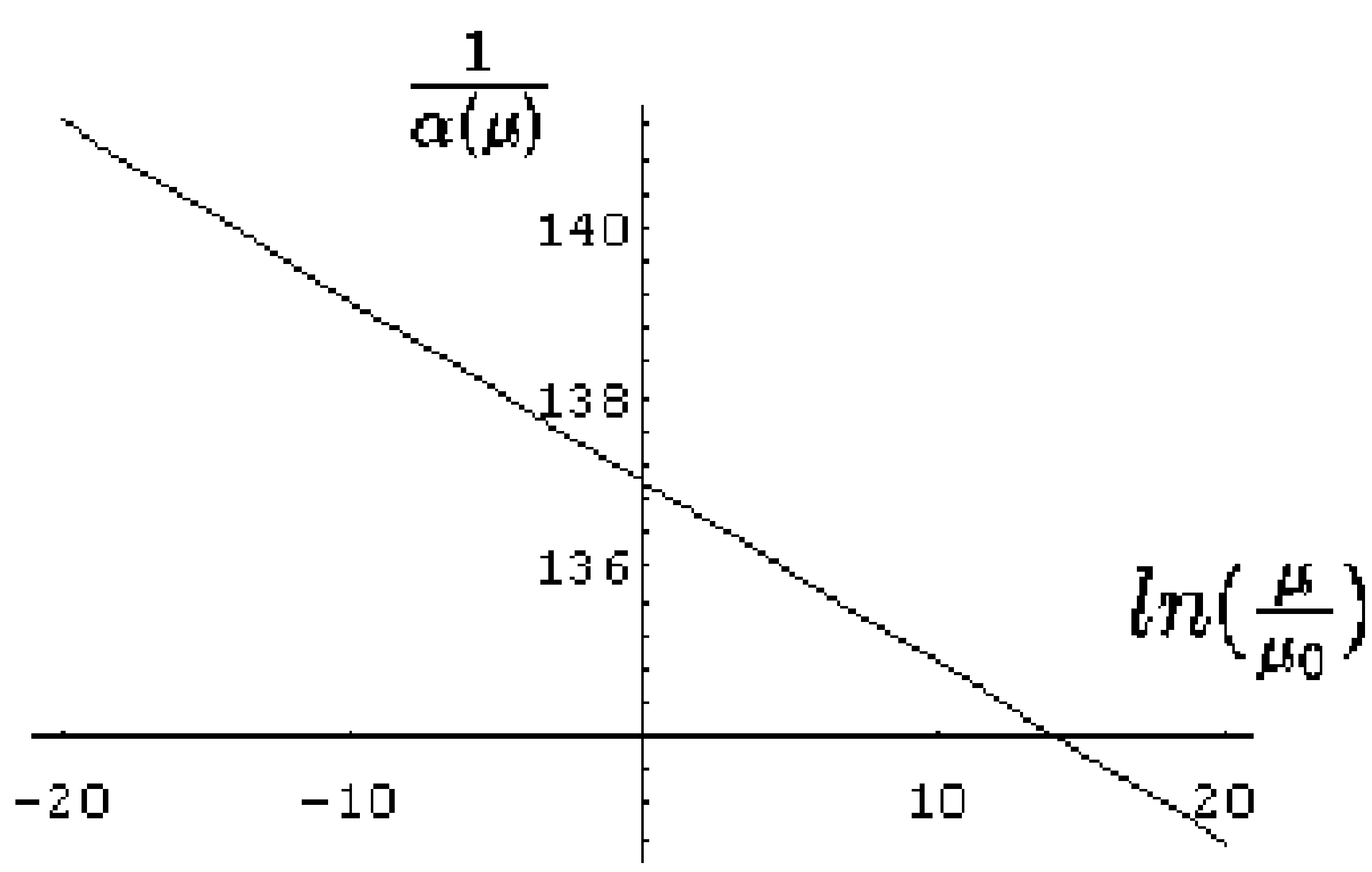
(**J**)

**What is the solution of the above RG equation**?

$$\alpha(\mu) = \frac{\alpha(\mu_0)}{1 - \frac{\alpha(\mu_0)}{4\pi} \left( -\frac{4}{3} \right) \ln\left( \frac{\mu^2}{\mu_0^2} \right)}$$

(**K**)

**In terms of $\alpha_{em}$, fine structure constant, plot $\frac{1}{\alpha_{em}}$ as a function of $\mu$. Is it a straight line on a semi-log scale**?




Yes, It is a straight line on the semi-log scale. For QED, $N_F = 1$.

(**L**)

**Explain the physical meaning of the running coupling $\alpha_{em}(\mu)$**.

The QED running coupling increases with the energy scale, i.e. the electromagnetic charge decreases at large distances. This can be intuitively understood as a screening effect of the virtual fermion-antifermion pairs generated, through quantum effects, around the electron charge. And quantum corrections make QED irrelevant at low energies $\left(\lim_{Q^2 \to 0} \alpha(Q^2) = 0\right)$, which are weak at low energy. The running coupling $\alpha_{em}(\mu)$ includes potentially large logs from higher order loop corrections.

**2.**

**Consider the Quantum Chromodynamics (QCD) theory, up to one-loop, in the 't hooft-Feynman gauge.**

(**a**)

**Write down the bare Lagrangian**

$$\pounds_0 = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) - \frac{1}{2}f_{abc}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)A^{b\mu}A^{c\nu} - \frac{1}{4}g^2 f_{abc}f_{ade}A^b_\mu A^c_\nu A^{d\mu}A^{e\nu}$$
$$+ \bar{\Psi}(i\gamma \bullet \partial - m)\Psi + g\bar{\Psi}T^a\gamma^\mu\Psi A^a_\mu - \frac{1}{2\alpha}(-\partial_\mu A^{a\mu})^2 - (\partial_\mu\bar{\chi}^a)(\partial^\mu\chi^a) + gf_{abc}(\partial_\mu\bar{\chi}^a)\chi^b A^{c\mu}$$

(**b**)

**Rewrite the bare Lagrangian in terms of the renormalized Lagrangian and the counterterms.**

Redefine the fields and parameters

$$A_\mu \to A^0_\mu = \sqrt{Z_3}\,A^R_\mu$$
$$\Psi \to \Psi^0 = \sqrt{Z_2}\,\Psi^R$$
$$m \to m^0 = Z_m m^R$$
$$\chi \to \chi^0 = \sqrt{\tilde{Z}_3}\,\chi^R$$
$$g \to g^0 = Z_g g^R$$
$$a \to \alpha^0 = Z_\alpha \alpha^R$$

Inserting these fields and parameters into the BARE Lagrangian, and writing $Z_i = 1 + \delta Z_i$, we could split the BARE Lagrangian into the renormalized Lagrangian $\pounds^R$

$$\begin{aligned}\pounds^R = &-\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu})\\ &-\frac{1}{2}f_{abc}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)A^{b\mu}A^{c\nu}\\ &-\frac{1}{4}g^2 f_{abc}f_{ade}A^b_\mu A^c_\nu A^{d\mu}A^{e\nu}\\ &+\bar{\Psi}(i\gamma \bullet \partial - m)\Psi\\ &+g\bar{\Psi}T^a\gamma^\mu\Psi A^a_\mu\\ &-\frac{1}{2\alpha}(-\partial_\mu A^{a\mu})^2\\ &-(\partial_\mu\bar{\chi}^a)(\partial^\mu\chi^a)\\ &+gf_{abc}(\partial_\mu\bar{\chi}^a)\chi^b A^{c\mu}\end{aligned}$$

and the counterterm Lagrangian $\delta\pounds^C$

$$
\begin{aligned}
\delta\pounds = &-\delta Z_3 \frac{1}{2} A^{a\mu}\delta_{ab}(\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)A^{b\nu} \\
&- \delta Z_1 \frac{1}{2} g f^{abc}(\partial_\mu\partial_\nu - \partial_\nu A^a_\mu)A^{b\mu}A^{c\nu} \\
&- \delta Z_4 \frac{1}{4} g^2 f^{abc} f^{ade} A^a_\mu A^b_\nu A^{c\mu} A^{d\nu} \\
&+ \delta Z_2 \bar{\Psi}(i\gamma \bullet \partial - m)\Psi - (Z_2 Z_m - 1) m\bar{\Psi}\Psi \\
&+ \delta Z_{1F}\ g\bar{\Psi} T^a \gamma^\mu \Psi A^a_\mu \\
&- \delta Z^3_\alpha \frac{1}{2\alpha}(-\partial_\mu A^{a\mu})^2 \\
&- \delta\tilde{Z}_3(\partial_\mu \bar{\chi}^a)(\partial^\mu \bar{\chi}^a) \\
&+ \delta\tilde{Z}_1 g f^{abc}(\partial_\mu \bar{\chi}^a)\chi^b A^{c\mu}
\end{aligned}
$$

where $Z_1, Z_4, \tilde{Z}_1$ and $Z^3_\alpha$ are defined as follows:

$$
\begin{aligned}
Z_1 &= Z_g Z_3^{3/2} \\
Z_4 &= Z_g^2 Z_3^2 \\
\tilde{Z}_1 &= Z_g \tilde{Z}_3 \sqrt{Z_3} \\
Z_{1F} &= Z_g Z_2 \sqrt{Z_3} \\
Z^3_\alpha &= \frac{Z_3}{Z_\alpha}
\end{aligned}
$$

and

$$
g^R = \frac{1}{Z_g} g^0 = \frac{Z_2\sqrt{Z_3}}{Z_{1F}} g^0
$$

(**C**)

**How to introduce the renormalization scale ($\mu$) dependence in the Lagrangian**?

It is the same method as in the question 1, we introduce

$$
g_s \to g_s \mu^\varepsilon
$$

(**d**)

**Calculate the divergent piece and the $\ln(\mu^2)$ dependence using the dimesional regularization scheme for the two-point functions of quark and gluon**, **and the three-point function of gluon-quark-antiquark**.

Two-point function of quark

$$
i\Sigma = -ip\left[ -\frac{g_s^2}{16\pi^2}(T^b T^b)\left(\frac{4\pi\mu^2}{-p^2}\right)^\varepsilon \frac{1}{\varepsilon}\Gamma(1+\varepsilon)\right]
$$

Two-point function of gluon

$$
i\Pi_{\mu\nu} = -\frac{ig_s^2}{16\pi^2}(-q^2 g_{\mu\nu} + q_\mu q_\nu)\delta_{ab}\left(\frac{5}{3}C_A - \frac{4}{3}T_R N_F\right)\left(\frac{4\pi\mu^2}{-q^2}\right)^e \frac{1}{\varepsilon}\Gamma(1+\varepsilon)
$$

Three-point function of gluon-quark-antiquark

$$i\Gamma = (ig_s\gamma_\mu T^a)\frac{g_{s^2}}{16\pi^2}[C_2(F)+T(A)]\left(\frac{4\pi\mu^2}{-q^2}\right)^{\varepsilon}\frac{1}{\varepsilon}\Gamma(1+\varepsilon)$$

(**E**)

**Identify the explicit expression of the counterterms in the $\overline{MS}$ scheme.**

$$Z_3 = 1+\frac{g_s^2}{16\pi^2}\left(\frac{5}{3}T(A)-\frac{4}{3}T(F)N_F\right)\left(\frac{1}{\varepsilon}-\gamma+\log(4\pi)\right)$$
$$Z_2 = 1-\frac{g_s^2}{16\pi^2}C_2(F)\left(\frac{1}{\varepsilon}-\gamma+\log(4\pi)\right)$$
$$Z_{1F} = 1-\frac{g_s^2}{16\pi^2}[C_2(F)+T(A)]\left(\frac{1}{\varepsilon}-\gamma+\log(4\pi)\right)$$
$$Z_g = \frac{Z_{1F}}{Z_2\sqrt{Z_3}} = 1-\frac{g_s^2}{16\pi^2}\left(\frac{11}{6}T(A)-\frac{2}{3}T(F)N_F\right)\left(\frac{1}{\varepsilon}-\gamma+\log(4\pi)\right)$$

(**F**)

**Write down the RG equation for the strong coupling constant $\alpha$**

$$\frac{\partial\alpha_s(Q)}{\partial\ln(Q^2/\mu^2)} = \beta(\alpha_s(Q))$$

(**G**)

**What is the explicit expression of the $\beta$ function, in terms of the number ($N_F$) of light quarks flavours, whose masses are smaller than the considered energy scale? Is $\beta$ function greater or smaller than zero in the standard model of particle physics?**

$$\beta(\alpha_s) = -\frac{\alpha_s^2}{4\pi}\left[\frac{11}{3}T(A)-\frac{4}{3}T(F)N_F\right]$$

In the Standard Model, $T(A) = 3, T(F) = \frac{1}{2}$ and $N_F \le 6$, so

$$\beta(\alpha_s) < 0$$

(**H**)

**What is the solution of the above RG equation, in terms of $\Lambda_{QCD}$?**

$$\alpha_s(\mu) = \frac{-2}{b_0\ln\left(\frac{\mu^2}{\Lambda^2}\right)}\left\{1+\frac{2b_1}{b_0}\left[\frac{\ln\ln\left(\frac{\mu^2}{\Lambda^2}\right)-\frac{b_0^2C}{b_1}}{\ln\left(\frac{\mu^2}{\Lambda^2}\right)}\right]\right\}$$
$$b_0 = -\frac{1}{2\pi}\left(11-\frac{2}{3}N_F\right)$$
$$b_1 = -\frac{1}{4\pi^2}\left(51-\frac{19}{3}N_F\right)$$

In the $\overline{MS}$ scheme, $C = 0$.

(**I**)

**Explain the physical meaning of $\Lambda_{QCD}$?**

$\Lambda_{QCD}$ represents the scale at which the coupling $\alpha_s(Q^2)$ becomes so strong that the perturbative theory breaks down.

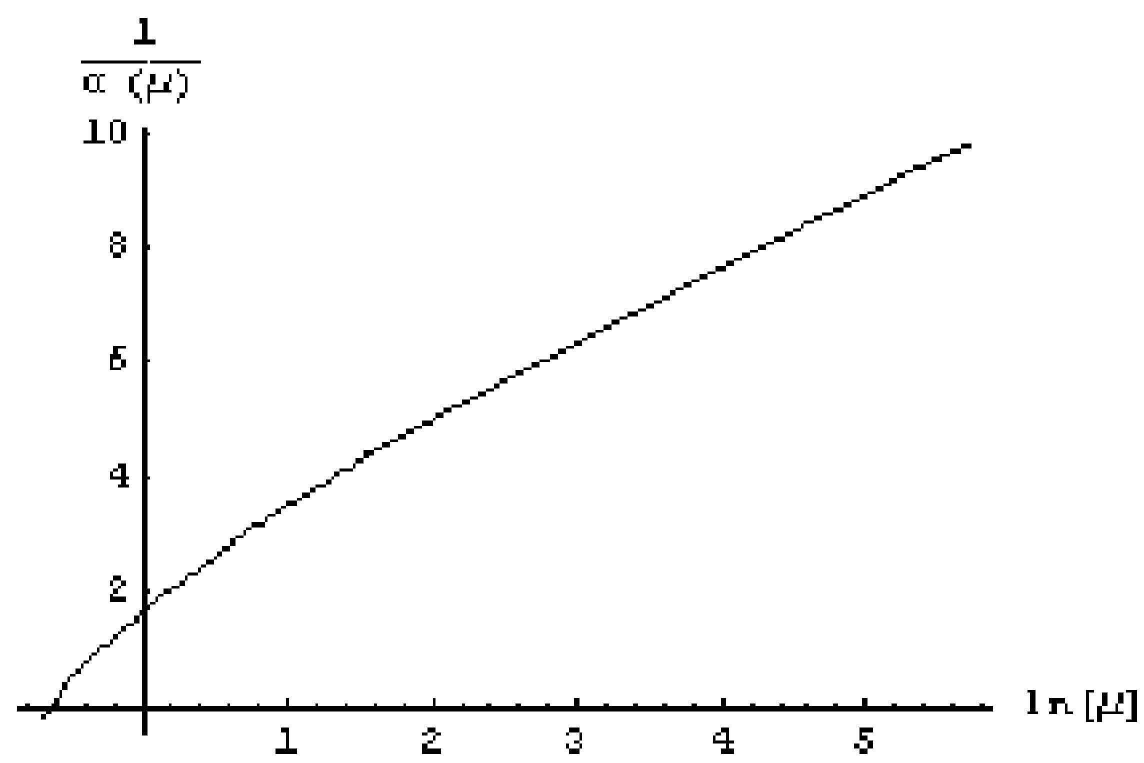
(**J**)

**The mass $(M_Z)$ of $Z$-boson is 91.18$Gev$, and $\alpha_s(at\ M_Z)$ is 0.12. Calculate $\Lambda_{QCD}$ for $N_F = 4$.**

$$\Lambda_{QCD}^{(4,1)} = 0.134206$$
$$\Lambda_{QCD}^{(4,2)} = 0.333842$$

(**K**)

**Plot $\frac{1}{\alpha_s}$ as a function of $\mu$**




(**l**)

**Explain the physical meaning of the running coupling $\alpha_s(\mu)$.**

QCD running coupling decreases at short distance $\left(\lim_{Q^2\to\infty}\alpha_s(Q^2) = 0\right)$, which means the QCD has the required property of asymptotic freedom. The gauge self-interactions of gluon spread out the QCD charge, generating an anti-screening effect. This could not happen in QED, because photons do not carry electric charge. The running coupling $\alpha_s(\mu)$ includes potentially large logs from higher order loop corrections.

**3.**

**Consider the Drell-Yan pair production at a proton-antiproton collider with a center-of mass energy $\sqrt{S}$.**

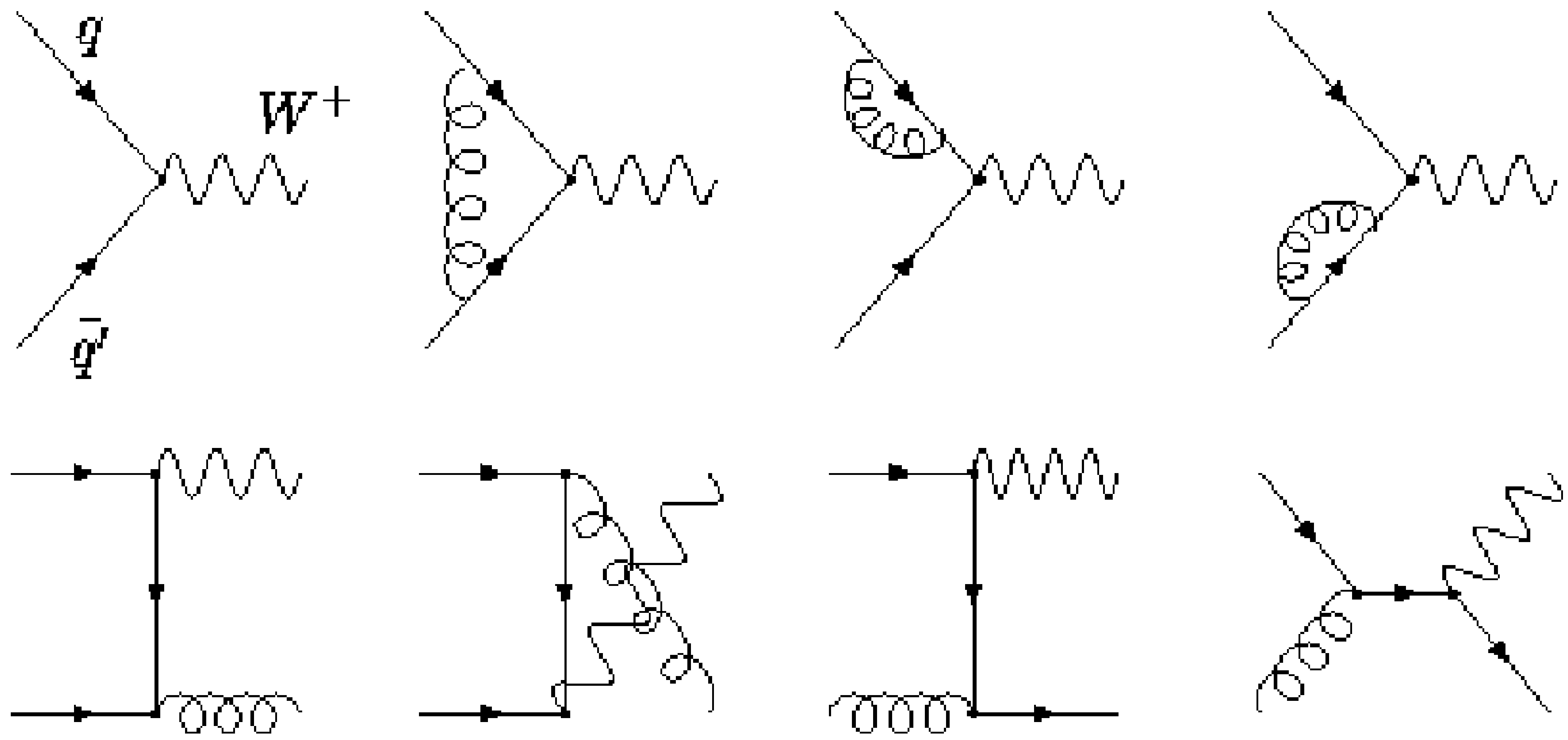
(**a**)

**Denote the mass of the Drell-Yan pair to be $Q$. Write down the relevant formula for calculating the Born level inclusive production rate of the Drell-Yan pair as a function of $Q$, denoted as $d\sigma/dQ^2$, predicted by the standard model.**

$$\frac{d\sigma}{dQ^2} = Q_f^2\frac{4\pi\alpha^2}{3NQ^2\hat{s}}\delta(1-\hat{\tau}),\hat{\tau} = \frac{M_W^2}{\hat{s}}$$

(**b**)

**Consider the next-to-leading order corrections to the Drell-Yan pair production due to QCD interactions. Draw all the relevant Feynman diagrams for calculating the NLO QCD corrections.**




(**C**)

**Write the virtual corrections (after summing over the vertex corrections as well as the wavefunction renormalization contributions) in the 't Hooft-Feynman gauge.**

$$\sigma^{NLO}_{Virtual} = 2\tilde{\sigma}_0 \frac{g_s^2}{16\pi^2}\delta(1-\hat{\tau})\left(\frac{4\pi\mu^2}{M^2}\right)^{\varepsilon}\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\left\{-\frac{2}{\varepsilon^2}-\frac{3}{\varepsilon}-7-\frac{\pi^2}{3}+\delta_{scheme}\right\}$$

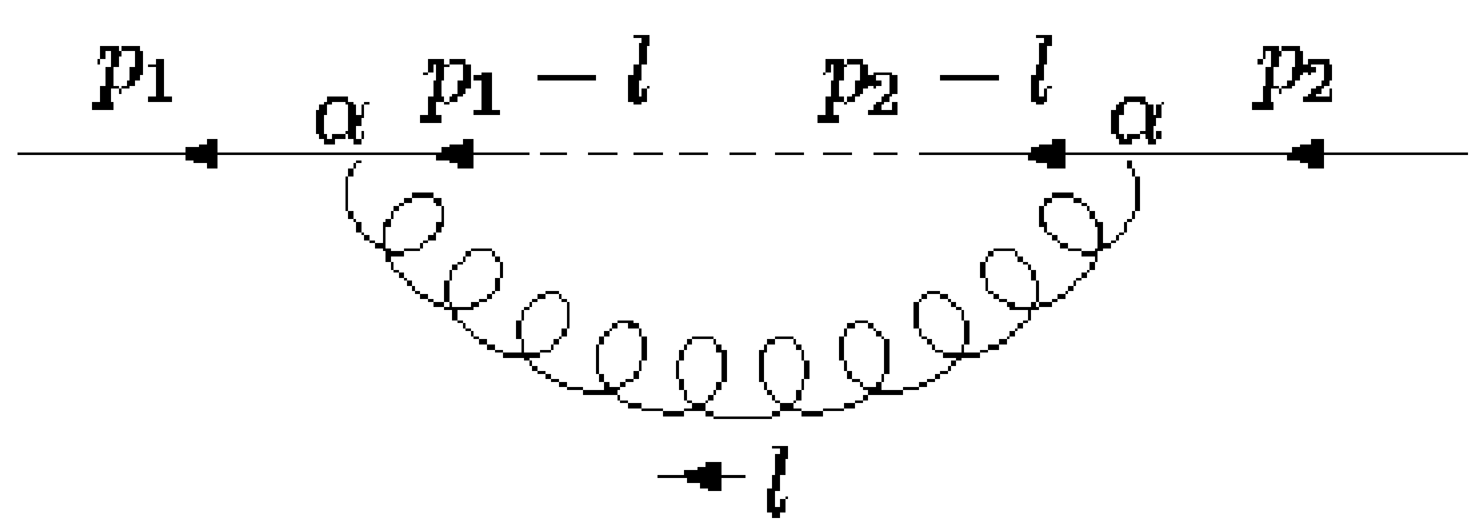
(**D**)

**Is there any ultraviolate singularity remained in the total virtual corrections found in (C)? If no, why not?**

No, there is no ultraviolate singularity remained in the total virtual correction.
The tree level coupling is the electroweak coupling and we are calculating the QCD correction, so the renormalization theory required that all the ultraviolate singularities coming from QCD corrections must cancel among themselves.

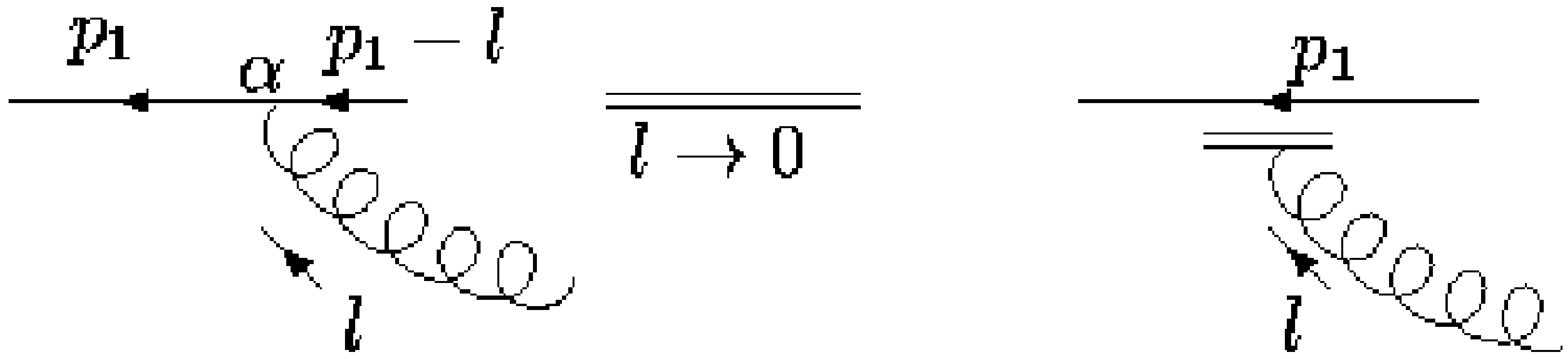
(**E**)

**Identify the sources of the soft singularities in the real-emission (cut-) diagrams. Show that the soft contributions can be factorized into the product of the tree level amplitude square and the radiation antenna.**



$$iM = \bar{v}(p_1)\gamma_\alpha \frac{\gamma.p_1 - \gamma.l}{(p_1 - l)^2} \cdots \frac{\gamma.p_2 - \gamma.l}{(p_2 - l)^2}\gamma^\alpha u(p_2)$$

$$\frac{\gamma.p_2}{(p_2 - l)^2}\gamma^\alpha u(p_2) \underset{l\to 0}{\to} \frac{\gamma.p_2}{(p_2 - l)^2}\gamma^\alpha u(p_2) = \left[\frac{2p_2^\alpha - \gamma^\alpha}{(p_2 - l)^2}\right]u(p_2) = \frac{p_2^\alpha}{(-p_2 \bullet l) + i\varepsilon}u(p_2)$$




$$\bar{v}(p_1)\frac{\gamma_\alpha(\gamma.p_1 - \gamma.l)}{(p_1 - l)^2 + i\varepsilon} \qquad\qquad \bar{v}(p_1)\frac{p_1^\alpha}{(-p_1 \bullet l) + i\varepsilon}$$

After squared the amplitude, we obtain

$$|M|^2 = e^2 \frac{2p_1 \bullet p_2}{(p_1 \bullet l)(p_2 \bullet l)}|M|^2_{Tree}$$

In the $p_1 - p_2$ c.m. frame, we have

$$p_1 = \left(|\vec{p}_1|, \vec{p}_1\right)$$
$$p_2 = \left(|\vec{p}_2|, \vec{p}_2\right)$$
$$l = \left(\left|\vec{l}\right|, \vec{l}\right)$$

and

$$\frac{2p_1 \bullet p_2}{(p_1 \bullet l)(p_2 \bullet l)} = \frac{2(1 - \cos\theta_{12})}{|l|^2(1 - \cos\theta_{l1})(1 - \cos\theta_{l2})}$$

The radiation antenna constant will be divergent when $l \to 0$, this singularity is due to the soft gluon.

(**F**)

**Verify that all the soft singularities cancel after summing contributions from (C) and (E).**

The Real corrections at the soft limit is

$$\sigma^{NLO}_{Soft+Col} = 2\tilde{\sigma}_0 \frac{g_s^2}{16\pi^2}\left(\frac{4\pi\mu^2}{M^2}\right)^\varepsilon \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\left\{\frac{2}{\varepsilon^2}\delta(1-\hat{\tau}) - \frac{4}{\varepsilon}\frac{1}{(1-\hat{\tau})_+} + 8\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+\right\}$$

Sum up the NLO virtual corrections and the real corrections at the soft limit, we immediately see that the double poles $\frac{1}{\varepsilon^2}$ cancel between the two kinds of corrections.

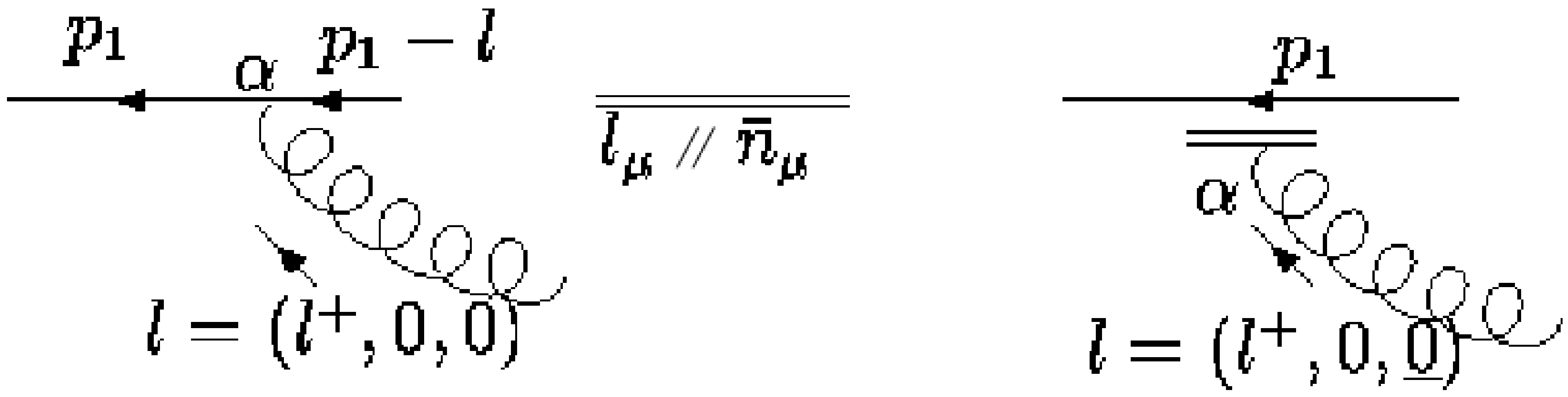
(**G**)

**Is there any remaining singularity in the sum of (C) and (E)? If yes, what is the origin of the remaining singularity?**

There is only the collinear singularity remaining in the sum.

(**H**)

**Identify the sources of the collinear singularities in the real-emission (Cut-) diagrams. Show that the collinear contributions are proportional to the quark splitting kernel defined in the DGLAP parton evolution equation.**




$$\bar{v}(p_1)\frac{\gamma_\alpha(\gamma.p_1-\gamma.l)}{(p_1-l)^2+i\varepsilon} \qquad \bar{v}(p_1)\frac{n_\alpha}{(-n.l)+i\varepsilon}$$

$$with\ n_\alpha = (0,1,\underline{0})$$

We see that the factor $\frac{n_\alpha}{(-n.l)+i\varepsilon}$ is divergent when $l \,//\, \bar{n}_\mu$. It is the collinear singularity.

After considering the real corrections at the collinear limit, we get the NLO cross section with eikonal approximation

$$\sigma^{NLO} = 2\tilde{\sigma}_0\frac{g_s^2}{16\pi^2}\left(\frac{4\pi\mu^2}{\hat{s}}\right)^{\varepsilon}\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}$$
$$\left\{-\frac{2}{\varepsilon}P_{qq}+2(1-\hat{\tau})+\left[8\hat{\tau}-4(1-\hat{\tau})^2\right]\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+-7-\frac{\pi^2}{3}+\delta_{scheme}\right\}$$

where $P_{qq} = C_F\left(\frac{1+\tau^2}{1-\tau}\right)_+$ is the quark splitting kernel.

(**I**)

**Explain how the factorization theorem can be applied in this process.**

The collinear singularity arises when the gluon is emitted parallel to quark $(k_T = 0)$. Realizing that the limit $k_T^2 \to 0$ corresponds to a long-range ('soft') part of the strong interaction which is not calculable in perturbation theory. So we can derive the short-distance cross section from the parton scattering cross section by removing the long-distance pieces and factoring them into the parton distribution functions. The remaining cross section involves only high momentum transfers and is insensitive to the physics of low momentum scales.

(**J**)

**Explain how to cancel the remaining collinear singularities found in (H) by redefining the parton distribution function (PDF).**

Exactly as for the renormalization, we can regard PDF $q^0(x)$ as an unmeasurable, bare distribution. The collinear singularities are absorbed into this bare distribution at a "factorization scale $\mu$", which plays similar role to the renormalization scale. The relation between the scale dependent (renormalized) and bare PDF is

$$f_i^A(x, Q_{PDF}^2) = \int_x^1 \frac{dz}{z}\Big[\delta_{ij}\delta(z-1) + \frac{\alpha_s}{2\pi}R_{i\leftarrow j}(z, Q_{PDF}^2)\Big]f_{j,bare}^A\left(\frac{x}{z}\right)$$

In general, $R$ has the form

$$R_{i\leftarrow j}(z, Q_{PDF}^2) = \frac{-1}{\varepsilon}P_{i\leftarrow j}(z)\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\left(\frac{4\pi\mu^2}{Q_{PDF}^2}\right)^{\varepsilon} + T_{i\leftarrow j}(z)$$

where $P_{i\leftarrow j}(z)$ is the Altarelli-Parisi kernel and $T_{i\leftarrow j}(z)$ is the finite difference between different definition of the PDF. After introducing the PDF 'renormalization', we could get the finite hadron level cross section

$$\begin{aligned}\sigma\left(P\bar{P} \to W^+\right) &= \sum_n \left(\frac{\alpha_s}{\pi}\right)^n \hat{\sigma}_{p\bar{p}}^{(n)} \\ &= q_{i/P,ren}^{(0)} \otimes \hat{\sigma}_{ij}^{(0)} \otimes q_{j/\bar{P},ren}^{(0)} \\ &\quad + q_{i/P,bare}^{(1)} \otimes \hat{\sigma}_{ij}^{(0)} \otimes q_{j/\bar{P},ren}^{(0)} \\ &\quad + q_{i/P,ren}^{(0)} \otimes \hat{\sigma}_{ij}^{(1)} \otimes q_{j/\bar{P},ren}^{(0)} \\ &\quad + q_{i/P,ren}^{(0)} \otimes \hat{\sigma}_{ij}^{(0)} \otimes q_{j/\bar{P},bare}^{(1)}\end{aligned}$$

where

$$f_{i,bare}^A(x) = f_i^A(x, Q_{PDF}^2) - \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z}R_{i\leftarrow j}(z, Q_{PDF}^2)f_{j,bare}^A\left(\frac{x}{z}\right)$$

and the convolution is defined as

$$(f\otimes g)(x,\mu) = \int_x^1 f\left(\frac{x}{\xi},\mu\right)g(\xi,\mu)\frac{d\xi}{\xi}$$

(**K**)

**Write down the relevant formula for calculation $d\sigma/dQ^2$.**

$$\begin{aligned}&\sigma(PP \to W^+) \\ &= \sum_{ij}\int_{\tau_0}^1 dx_1 \int_{\frac{\tau_0}{x_1}}^1 dx_2\left[q_{i/P}(x_1,Q^2)\bar{q}_{j/\bar{P}}(x_2,Q^2) + \bar{q}_{j/\bar{P}}(x_1,Q^2)q_{i/P}(x_2,Q^2)\right]\tilde{\sigma}_0\frac{1}{\hat{s}} \\ &\{\delta(1-\hat{\tau}) \\ &+ \frac{\alpha_s}{2\pi}C_F\left[2\ln\left(\frac{M^2}{Q_{PDF}^2}\right)\left(\frac{1+\hat{\tau}^2}{1-\tau}\right)_+ + 4(1-\hat{\tau}) + 4(1+\hat{\tau})\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+ + \left(-\frac{\pi^2}{3} - 7 + \delta_{scheme}\right)\delta(1-\hat{\tau})\right]\} \\ &+ \frac{\alpha_s}{4\pi}\hat{\sigma}_0\sum_i\int_{\tau_0}^1 dx_1\int_{\tau_0/x_1}^1 dx_2\left[q_{i/P}^{(0)}(x_1)g(x_2) + \bar{q}_{i/P}(x_1)g(x_2) + (x_1 \leftrightarrow x_2)\right]\frac{1}{\hat{s}} \\ &\times\left[\left[\hat{\tau}^2 + (1-\hat{\tau})^2\right]\ln\frac{M_W^2}{Q^2}(1-\hat{\tau}) + \frac{3}{2} - 5\hat{\tau} + \frac{9}{2}\hat{\tau}^2\right]\end{aligned}$$

(**L**)

**If you are asked to perform a similar NLO calculation for the Drell-Yan process due to the QED (not QCD) interactions, do you except ultraviolate singularities from virtual contributions? If yes, how do you handle this new type of singularities so that the final cross section is finite?**

Yes, we do have the ultraviolate singularities from virtual contributions. But the ultraviolate singularities should be canceled after we introduce the corresponding counterterm for the electric charge.

## 4.

**A one-page summary on what you have learned from this class about the QED and QCD theories**.

(1) Helicity amplitude method

(2) Dimensional regularization and Dimensional reduction scheme.

(I)Dimensional regularization: treat momentum and $\gamma_\mu$ both in the $n$-dimension

(II)Dimensional reduction: treat $\gamma_\mu$ in the 4-dimension and momentum in the $n$-dimension

The problem of $\gamma_5$:

(I) naive $\gamma_5$ does not give correct anomaly.

(II) 't-Hooft-Veltman $\gamma_5$ does not respect Ward Identity.

(3) Eikonal Approximation

(4) Factorization theorem and its' application in the Drell-Yan process

(a) Virtual corrections : UV singularities cancel and IR singularities left ($\frac{1}{\varepsilon^2}$ & $\frac{1}{\varepsilon}$)

(b) Real corrections: IR divergences

(I)soft limit: soft singularity and collinear singularity ($\frac{1}{\varepsilon^2}$ & $\frac{1}{\varepsilon}$)

The main contribution of real corrections comes from the soft limit (Sudakov factor)

Soft gluon resummaton (concept)

(II)collinear limit: collinear singularity ($\frac{1}{\varepsilon}$)

Add them up, only collinear singularity left and are proportional to the quark splitting kernel.

(c) Methods: eikonal approximation and phase space integral

(5) Drell-Yan process calculation (Next-to-leading order)

(a) hard part (subprocess) calculation

(b) Redefine the PDF and absorb the collinear singularities into the PDF

(c) Apply the Factorization theorem and get the finite hadron level cross section

(6) Running coupling and Effective field theory

(a) QED running coupling and RG equation

(b) QCD running coupling and RG equation

(I) Renormalization scheme: counterterm approach

(II) $MS$ scheme and $\overline{MS}$ scheme

(III) $\Lambda_{QCD}$