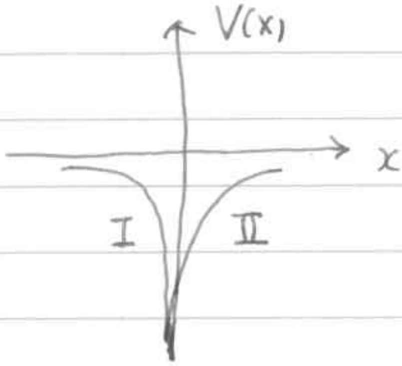
δ 函数势阱



$$V(x) = -\alpha\,\delta(x)$$

$\alpha > 0$ 常数

$$\int_{-\infty}^{+\infty} \delta(x)\,dx = 1 \Rightarrow [\delta(x)] = \frac{1}{L}$$

$$[\alpha] = L^3 M T^{-2}$$

T.I.S.E.:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} - \alpha\,\delta(x)\,\psi(x) = E\,\psi(x)$$

1) 束缚态 $(E<0)$

$x<0$ 时 $V(x)=0$

S.E. $$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = k^2\psi \qquad k = \frac{\sqrt{-2mE}}{\hbar}$$

通解: $$\psi(x) = \cancel{Ae^{-kx}} + Be^{kx} \qquad (x<0)$$

非物理解

$$\Rightarrow \psi_1(x) = Be^{kx}, \quad x<0$$

$x>0$ 时, $V(x)=0$

$$\psi(x) = Fe^{-kx} + Ge^{kx}$$

$$\Rightarrow \psi_2(x) = Fe^{-kx}, \quad x>0$$

$$\left.\frac{d\psi_2}{dx}\right|_{+\varepsilon} = -Bk$$

$$\left.\frac{d\psi_1}{dx}\right|_{-\varepsilon} = Bk$$

$$\Rightarrow \Delta\left(\frac{d\psi}{dx}\right) = -Bk - Bk = -2Bk$$

$$\Rightarrow -2Bk = -\frac{2m\alpha}{\hbar^2}\psi(0) = -\frac{2m\alpha}{\hbar^2}B$$

$$\Rightarrow k = \frac{m\alpha}{\hbar^2}$$

$$\Rightarrow E = -\frac{\hbar^2 k^2}{2m} = -\frac{m\alpha^2}{2\hbar^2}$$

③ "B" 由归一化条件给出

$$\int_{-\infty}^{+\infty} |\psi(x)|^2 dx = 2|B|^2 \int_0^{+\infty} e^{-2kx} dx = \frac{|B|^2}{k} = 1$$

$$\Rightarrow B = \sqrt{k} = \frac{\sqrt{m\alpha}}{\hbar}$$

所以，$\delta(x)$ 势阱中有且仅有1个束缚态

$$\psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-\frac{m\alpha|x|}{\hbar^2}}$$

$$E = -\frac{m\alpha^2}{2\hbar^2}$$

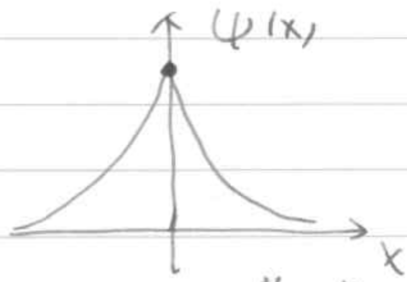
利用 $x=0$ 处边界条件将二者联系起来

① ψ 连续

② $\frac{d\psi}{dx}$：V有限时连续，但$V(x)$无限时未必连续。

① 波函数连续 $\psi_1(0)=\psi_2(0) \Rightarrow F=B$

$$\psi = \begin{cases} B e^{kx}, & x<0 \\ B e^{-kx}, & x>0 \end{cases}$$



$x=0$处波函数导数不连续。

② 波函数导数：$x=0$处 $\Rightarrow$ 能量量化

从S.E.出发，在 $x=0$ 附近有

$$-\frac{\hbar^2}{2m}\int_{-\varepsilon}^{+\varepsilon}\frac{d^2\psi(x)}{dx^2}dx + \int_{-\varepsilon}^{+\varepsilon}V(x)\psi(x)dx = \overset{\text{"0"}}{\cancel{E\int_{-\varepsilon}^{+\varepsilon}\psi(x)dx}}$$

$$\Delta\left(\frac{d\psi}{dx}\right) = \lim_{\varepsilon\to 0}\left(\frac{d\psi_2}{dx}\bigg|_{+\varepsilon} - \frac{d\psi_1}{dx}\bigg|_{-\varepsilon}\right) = \frac{2m}{\hbar^2}\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{+\varepsilon}V(x)\psi(x)dx$$

$$= \frac{2m}{\hbar^2}\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{+\varepsilon}(-\alpha\delta(x))\psi(x)dx$$

$$= -\frac{2m\alpha}{\hbar^2}\psi(0)$$

2) 散射态 $(E>0)$

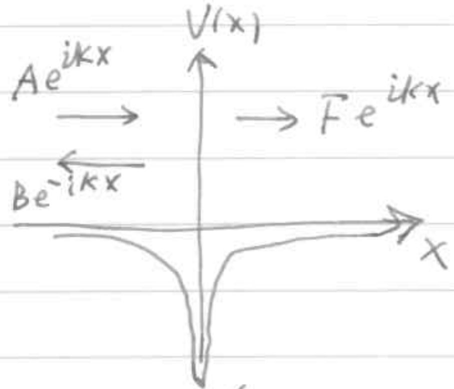
S.E.: ① $\dfrac{d^2\psi}{dx^2} = -\dfrac{2mE}{\hbar^2}\psi(x) = -k^2\psi$, $\quad |x|>0$

$$k = \frac{\sqrt{2mE}}{\hbar}$$

② $|x|\sim 0$.

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) - \alpha\delta(x)\psi(x) = E\psi(x).$$

散射态通解



$$\begin{cases} \psi_- = Ae^{-ikx} + Be^{ikx}, & x<0 \\ \psi_+ = Fe^{ikx} + Ge^{-ikx}, & x>0 \end{cases}$$

设粒子从左边入射 $\Rightarrow G=0$

边界条件:

① $x=0$处 $\psi_+(0) = \psi_-(0) \Longrightarrow \boxed{F = A+B}$

② $x=0$处波函数导数不连续、

$$\psi_+'(0) - \psi_-'(0) = -\frac{2m\alpha}{\hbar^2}\psi(0)$$

$$ik(F-A+B) = -\frac{2m\alpha}{\hbar^2}(A+B)$$

$$\Longrightarrow F = A(1+2i\beta) - B(1-2i\beta), \quad \beta = \frac{m\alpha}{\hbar^2 k}.$$

$$\begin{aligned} F &= A+B, \\ F &= A(1+2i\beta) - B(1-2i\beta) \end{aligned} \Rightarrow \begin{cases} B = \dfrac{i\beta}{1-i\beta} A \\ F = \dfrac{1}{1-i\beta} A \end{cases}$$

反射系数：

$$R = \left|\frac{B}{A}\right|^2 = \frac{\beta^2}{1+\beta^2} = \frac{1}{1+\frac{2\hbar^2 E}{m\alpha^2}}$$

透射系数

$$T = \left|\frac{F}{A}\right|^2 = \frac{1}{1+\beta^2} = \frac{1}{1+\frac{m\alpha^2}{2\hbar^2 E}}$$

$\delta(x)$ 势垒 $V(x) = \alpha\delta(x)$ $(\alpha > 0)$

无束缚态，但有散射态，且反射系数和透射系数不变