不确定关系第二种推导方法

令 $|\phi\rangle = (\hat{A}-\bar{A})|\psi\rangle + i\lambda(\hat{B}-\bar{B})|\psi\rangle$ ，λ是实数

$$\langle\phi| = \langle(\hat{A}-\bar{A})\psi| - i\lambda\langle(\hat{B}-\bar{B})\psi|$$

则有 $0 \le \langle\phi|\phi\rangle = \langle(\hat{A}-\bar{A})\psi|(\hat{A}-\bar{A})\psi\rangle$

$$+\lambda^2\langle(\hat{B}-\bar{B})\psi|(\hat{B}-\bar{B})\psi\rangle$$

$$+i\lambda\left[\langle(\hat{A}-\bar{A})\psi|(\hat{B}-\bar{B})\psi\rangle - \langle(\hat{B}-\bar{B})\psi|(\hat{A}-\bar{A})\psi\rangle\right]$$

定义 $I(\lambda) = \langle\phi|\phi\rangle$

$$= \langle\psi|(\hat{A}-\bar{A})^2|\psi\rangle + \lambda^2\langle\psi|(\hat{B}-\bar{B})^2|\psi\rangle$$

$$+i\lambda\langle\psi|(\hat{A}-\bar{A})(\hat{B}-\bar{B}) - (\hat{B}-\bar{B})(\hat{A}-\bar{A})|\psi\rangle$$

$$= (\Delta A)^2 + \lambda^2(\Delta B)^2 + \underbrace{i\lambda\langle\psi|[\hat{A},\hat{B}]|\psi\rangle}_{\equiv\lambda\langle\psi|\hat{F}|\psi\rangle}$$

$$I(\lambda) = \langle\phi|\phi\rangle \geq 0$$

求极值：$0 = \left.\dfrac{dI(\lambda)}{dx}\right|_{\lambda_{min}} = 2\lambda_{min}(\Delta B)^2 + \langle\psi|\hat{F}|\psi\rangle$

$$\Rightarrow \lambda_{min} = -\frac{\langle\psi|\hat{F}|\psi\rangle}{2(\Delta B)^2}$$

将λ_{min}代入到$I(\lambda)$中可得.

$$I(\lambda_{min}) = (\Delta A)^2 + \left(-\frac{\langle\psi|\hat{F}|\psi\rangle}{2(\Delta B)^2}\right)^2(\Delta B)^2 - \frac{\langle\psi|\hat{F}|\psi\rangle}{2(\Delta B)^2}\langle\psi|\hat{F}|\psi\rangle$$

$$= (\Delta A)^2 + \frac{1}{4}\frac{(\langle\psi|\hat{F}|\psi\rangle)^2}{(\Delta B)^2} - \frac{1}{2}\frac{(\langle\psi|\hat{F}|\psi\rangle)^2}{(\Delta B)^2}$$

$$= (\Delta A)^2 - \frac{(\langle\psi|\hat{F}|\psi\rangle)^2}{4(\Delta B)^2} \geq 0$$

$$\Rightarrow (\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}(\langle\psi|\hat{F}|\psi\rangle)^2$$

最小不确定关系 $\Delta x \cdot \Delta p = \frac{\hbar}{2}$ 需要上面等式成立

也即 $\langle\phi|\phi\rangle = 0$，此仅当 $\phi = 0$ 时成立

$$|0\rangle = |\phi\rangle = \left(-i\hbar\frac{d}{dx} - \bar{p}\right)\psi(x) + i\lambda(x - \bar{x})\psi(x)$$

$$\lambda_{min} = -\frac{\hbar}{2(\Delta x)^2}$$

$$\frac{d}{dx}\psi(x) = \frac{i\bar{p}}{\hbar}\psi(x) - \frac{(x-\bar{x})}{2(\Delta x)^2}\psi(x)$$

$$\Rightarrow \psi(x) \propto e^{\frac{i\bar{p}x}{\hbar}} e^{-\frac{(x-\bar{x})^2}{4(\Delta x)^2}}$$

又因为 x 和 p 地位相当，故 $\Delta x \cdot \Delta p = \frac{\hbar}{2} = (\Delta x)^2 = (\Delta$

$$\Rightarrow \psi(x) \propto e^{\frac{i\bar{p}x}{\hbar}} e^{-\frac{(x-\bar{x})^2}{2\hbar}}$$ 高斯波包