Renormalization and Factorization in QCD

(Notes prepared by Qing-Hong Cao in 2002)

Content

- Renormalization in QCD
- Running coupling in QCD
- Factorization in QCD

Renormalization
in QCD

Renormalization Scheme — counterterm approach

I) One possibility to evaluate predictions of a renormalizable model is the following:
① Calculate physical quantities in terms of the Bare parameters.
② Use the resulting relations as bare parameters exist to express these in terms of physical observables
③ Insert the resulting expressions into the remaining relations.

Thus one arrives at predictions for physical observables in terms of their physical quantities, which have to be determined from experiment.

II) Anothe way is the counterterm approach. Here the UV-divergent bare parameters are expressed by finite renormalized parameters and divergent renormalized constant (counterterm). In additional the bare field may be replaced by renormalized field. The counterterms are fixed through renormalization conditions. These can be chosen arbitrary, but determine the relation between renormalized and physical parameters. The renormalization procedure can be summaried as follows:
① Choose a set of independent parameters
② Seperate the bare parameters (and fields) into renormalized parameters (and fields) and renormalization constant.
③ Choose renormalization condition to fix the counterterms
④ Express physical quantities in terms of the renormalized parameters.
⑤ Choose input data in order to fix the value of the renormalized parameters.
⑥ Evaluate predictions for physical quantities as functions of the input data.

QCD Lagrangian. (C.P.'s Notes labeled "$\mathcal{L}-1$")

$$\mathcal{L} = \mathcal{L}_{kin} + \mathcal{L}_{fer} + \mathcal{L}_{gf} + \mathcal{L}_{FP}.$$

Redefine the fields and parameters.

$$\begin{aligned}
A_\mu &\to A_\mu^0 = \sqrt{Z_3}\, A^\mu{}_r \\
\psi &\to \psi_0 = \sqrt{Z_2}\, \psi_r \\
m &\to m_0 = Z_m\, m_r \\
\chi &\to \chi_0 = \sqrt{\tilde{Z}_3}\, \chi_r \\
g &\to g_0 = Z_g \cdot g_r \\
\alpha &\to \alpha_0 = Z_\alpha \cdot \alpha_r
\end{aligned}$$

These six renormalization constants should be sufficient to make theory UV Finite. Inserting these fields and parameters into the BARE Lagrangian, and writing $Z_i = 1 + \delta Z_i$, we could split the BARE Lagrangian into the renormalized Lagrangian $\mathcal{L}_r$ and the counterterm Lagrangian $\delta\mathcal{L}$.

$$\mathcal{L}_0 = \mathcal{L}_r + \delta\mathcal{L}$$

$\mathcal{L}_r$ has the same form as $\mathcal{L}_0$ but depends on renormalized parameters and field instead of unrenormalized ones. $\delta\mathcal{L}$ yields the couterterm.

$$\begin{aligned}
\mathcal{L}_r = &-\tfrac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) - \tfrac{1}{2} f_{abc}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) A^{b\mu} A^{c\nu} - \tfrac{1}{4} g^2 f_{abc} f_{ade} A_\mu^b A_\nu^c A^{d\mu} A^{e\nu} \\
&+ \bar{\psi}(i\partial\!\!\!/ - m)\psi + g\bar{\psi} T^a \gamma^\mu \psi A_\mu^a - \frac{1}{2\alpha}(-\partial_\mu A^{a\mu})^2 \\
&- (\partial_\mu \bar{\chi}^a)(\partial^\mu \chi^a) + g f_{abc}(\partial_\mu \bar{\chi}^a)\chi^b A^{c\mu}
\end{aligned}$$

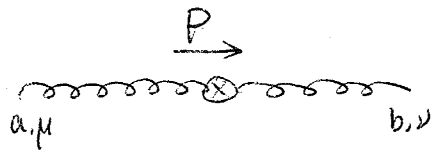
and the counterterm Lagrangian is :

$$\begin{aligned}
\delta\mathcal{L} = & -\frac{1}{4}(Z_3-1)(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) \\
& -(Z_g Z_3^{3/2}-1)\frac{1}{2} g f^{abc}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) A^{b\mu} A^{c\nu} \\
& -(Z_g^2 Z_3^2 - 1)\frac{1}{4} g^2 f^{abc} f^{ade} A_\mu^a A_\nu^b A^{c\mu} A^{d\nu} \\
& +(Z_2-1)\bar{\psi}(i\not{\partial}-m)\psi - (Z_2 Z_m - 1) m\bar{\psi}\psi \\
& +(Z_g Z_2 \sqrt{Z_3}-1) g\bar{\psi} T^a \gamma^\mu \psi A_\mu^a \\
& -\left(\frac{Z_3}{Z_\alpha}-1\right)\frac{1}{2\alpha}(-\partial_\mu A^{a\mu})^2 \\
& -(\tilde{Z}_3 - 1)(\partial_\mu \bar{\chi}^a)(\partial^\mu \bar{\chi}^a) \\
& +(Z_g \tilde{Z}_3 \sqrt{Z_3}-1) g f_{abc}(\partial_\mu \bar{\chi}^a)\chi^b A^{c\mu} \\
= & -\delta Z_3 \frac{1}{2} A^{a\mu}\delta_{ab}(\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2) A^{b\nu} \\
& -\delta Z_1 \cdot \frac{1}{2} g f^{abc}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) A^{b\mu} A^{c\nu} \\
& -\delta Z_4 \frac{1}{4} g^2 f^{abc} f^{ade} A_\mu^a A_\nu^b A^{c\mu} A^{d\nu} \\
& +\delta Z_2 \bar{\psi}(i\not{\partial}-m)\psi - (Z_2 Z_m - 1) m\bar{\psi}\psi \\
& +\delta Z_{1F}\, g\bar{\psi} T^a \gamma^\mu \psi A_\mu^a \\
& -\delta Z_\alpha^3 \frac{1}{2\alpha}(-\partial_\mu A^{a\mu})^2 \\
& -\delta\tilde{Z}_3 (\partial_\mu \bar{\chi}^a)(\partial^\mu \bar{\chi}^a) \\
& +\delta\tilde{Z}_1\, g f_{abc}(\partial_\mu \bar{\chi}^a)\chi^b A^{c\mu}
\end{aligned}$$

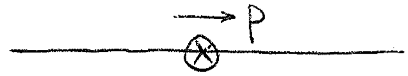
Where Z_1, Z_4, $\tilde{Z}_1$, Z_{1F} and Z_α^3 are defined as follows:

$$Z_1 = Z_g Z_3^{3/2} \qquad Z_4 = Z_g^2 Z_3^2 \qquad \tilde{Z}_1 = Z_g \tilde{Z}_3 \sqrt{Z_3}$$
$$Z_{1F} = Z_g Z_2 \sqrt{Z_3} \qquad Z_\alpha^3 = Z_3 \cdot Z_\alpha^{-1}$$

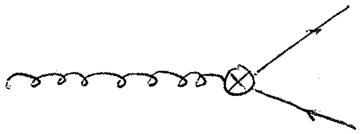
From this couterterm Lagrangian, we immetiately obtain the Feynman rule for couterterm.



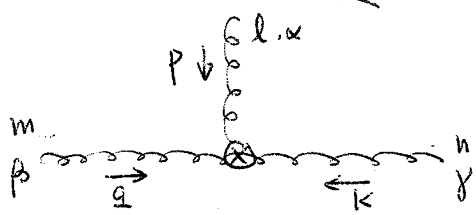
$$i\left[\delta Z_3 (q_\mu q_\nu - q^2 g_{\mu\nu})\delta_{ab} - \frac{\delta Z_\alpha^3}{\alpha} q_\mu q_\nu\right]$$



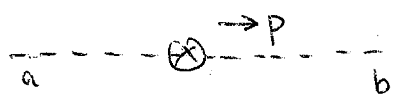
$$i\left[\delta Z_2(\not p - m) - (Z_2 Z_m - 1)m\right]$$



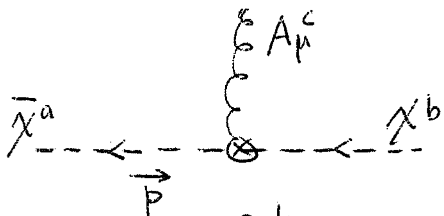
$$i\,\delta Z_{1F}\, g\,\gamma^\mu T^a$$



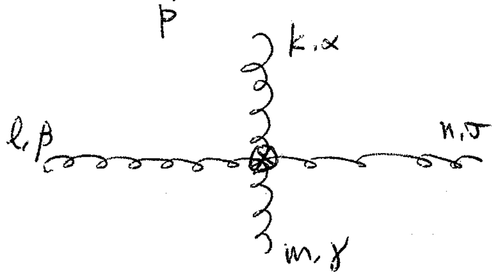
$$\delta Z_1 \cdot g f^{lmn}\{(p-q)_\gamma g_{\alpha\beta} + (q-k)_\alpha g_{\beta\gamma} + (k-p)_\beta g_{\alpha\gamma}\}$$



$$i\,\delta \tilde{Z}_3\, \delta_{ab}\, p^2$$



$$\delta\tilde{Z}_1\, g f^{abc} p_\mu$$



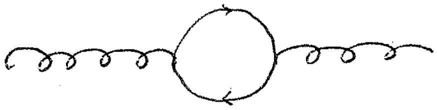
$$-i\,\delta Z_4\, g^2 \Big\{ f^{akl}f^{amn}(g_{\alpha\gamma}g_{\beta\sigma} - g_{\alpha\sigma}g_{\beta\gamma}) + f^{akm}f^{aln}(g_{\alpha\beta}g_{\sigma\gamma} - g_{\beta\gamma}g_{\alpha\sigma}) + f^{akn}f^{alm}(g_{\alpha\beta}g_{\sigma\gamma} - g_{\alpha\sigma}g_{\beta\gamma}) \Big\}$$

From the definition above, it is easy to see that

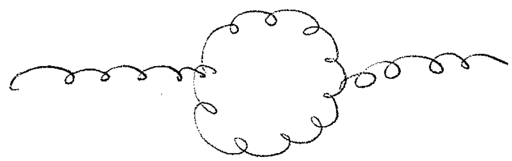
$$g_r = \frac{1}{Z_g} g_0 = \frac{Z_2\sqrt{Z_3}}{Z_{1F}} g_0$$

Then in order to calculate the renormalized coupling constant of QCD, we should deal with three class graphs which are showed as follows.

(a) The vacuum polarization graphs

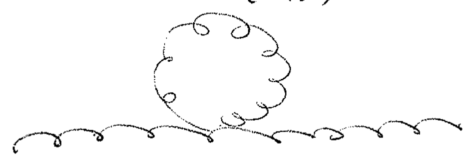


(a1)



(a2)

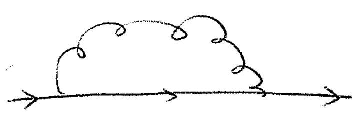
(a3)



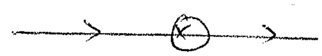
(a4)

(a5)

(b) The self energy graphs

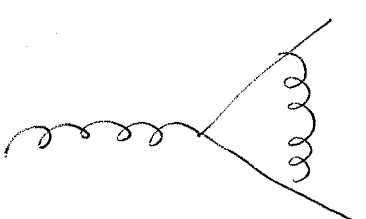


(b1)

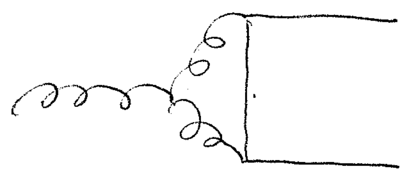


(b2)

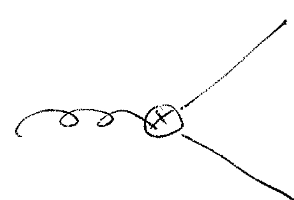
(c) The vertex correction graphs.



(c1)

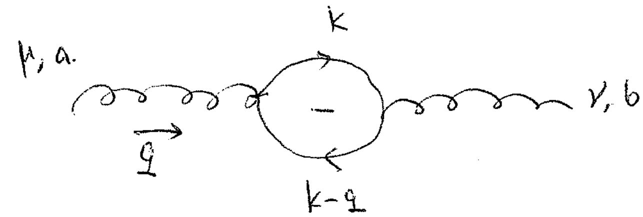


(c2)



(c3)

The vacuum polarization with Fermion Graph



$$i\Pi^{(a1)}_{\mu\nu}(q) = -\sum_{i=1}^{N_F} \mu^{2\epsilon}\int \frac{d^nk}{(2\pi)^n}\,(ig_s\gamma^\mu T^a_{ij})\,\frac{i(\not{k}-\not{q}+m)}{[(k-q)^2-m^2+i\varepsilon]}\,(ig_s\gamma^\mu T^b_{ji})\,\frac{i(\not{k}+m)}{k^2-m^2+i\varepsilon}$$

$$= -g_s^2 N_F \,\mathrm{Tr}(T^aT^b)\int\frac{d^nk}{(2\pi)^n}\cdot\frac{\mathrm{Tr}(\gamma^\mu(\not{k}-\not{q})\gamma^\nu\not{k})}{k^2(k-q)^2}$$

where N_F is the number of quarks in the theory, or more precisely the number of quarks with $m^2 \ll |q^2|$. The whole calculation can be carried through for arbitrary m^2, but one finds that the contribution is suppressed for $m^2 > |q^2|$. Then N_F counts only the light flavors and for simplicity we can set $m=0$.

$$\mathrm{Tr}[\gamma_\mu(\not{k}-\not{q})\gamma_\nu\not{k}] = 4[g_{\mu\nu}(q\cdot k - k^2) + 2k_\mu k_\nu - (k_\mu q_\nu + q_\mu k_\nu)]$$

Use the Feynman parameterization and shift the integration from $k \to l + xq$, then

$$\frac{1}{k^2(k-q)^2} = \int_0^1 dx\,\frac{1}{[l^2-c]^2}\,, \qquad c^2 = -q^2x(1-x)$$

and numerator should be

$$-4g_{\mu\nu}\left[\frac{n-2}{n}l^2 - q^2x(1-x)\right] - 8q_\mu q_\nu x(1-x)$$

By using the integration as follows:

$$\mu^{2\epsilon}\int\frac{d^n k}{(2\pi)^n}\cdot\frac{1}{(k^2-c)^2} = \frac{i}{16\pi^2}\,\frac{1}{\epsilon}\,\Gamma(1+\epsilon)\left(\frac{4\pi\mu^2}{-q^2}\right)^{\epsilon}[x(1-x)]^{-\epsilon}$$

$$\mu^{2\epsilon}\int\frac{d^n k}{(2\pi)^n}\cdot\frac{k^2}{(k^2-c)^2} = \frac{i}{16\pi^2}\,\frac{2}{\epsilon}\,\Gamma(1+\epsilon)\left(\frac{4\pi\mu^2}{-q^2}\right)^{\epsilon}(-q^2)[x(1-x)]^{1-\epsilon}$$

We obtain

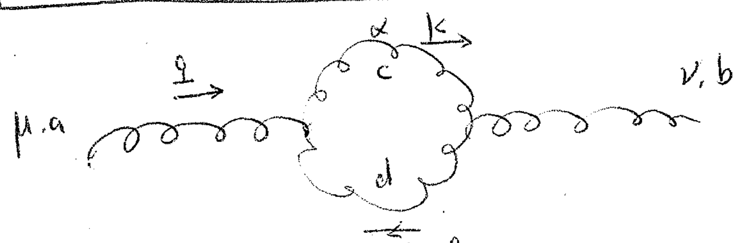
$$i\Pi^{(a1)}_{\mu\nu}(q) = -g_s^2 N_F\cdot \mathrm{Tr}(T^aT^b)\cdot\frac{i}{16\pi^2}\left\{-\frac{4}{3}\cdot\frac{1}{\epsilon}\Gamma(1+\epsilon)\left(\frac{4\pi\mu^2}{-q^2}\right)^{\epsilon}(-q^2 g_{\mu\nu}+q_\mu q_\nu)\right\}$$

$$= \frac{ig_s^2}{16\pi^2}\cdot N_F T(F)\delta_{ab}(q_\mu q_\nu - q^2 g_{\mu\nu})\cdot\frac{4}{3}\,\frac{1}{\epsilon}\,\Gamma(1+\epsilon)\left(\frac{4\pi\mu^2}{-q^2}\right)^{\epsilon}.$$

Where we use

$$\mathrm{Tr}(T^aT^b) = T(F)\,\delta_{ab}$$

$\boxed{\text{The vacuum polarization, the gluon graph}}$



$$i\Pi^{(a2)}_{\mu\nu}(q) = \mu^{2\epsilon}\int\frac{d^n k}{(2\pi)^n}(g_s f_{acd}F_{\mu\alpha\beta})\cdot\frac{-i}{q^2}(g_s f_{cdb}F_{\alpha\beta\nu})\frac{-i}{(k-q)^2}\times\frac{1}{2}$$

$$= -\frac{1}{2}g_s^2\mu^{2\epsilon}\,\mathrm{Tr}(F_aF_b)\int\frac{d^n k}{(2\pi)^n}\cdot\frac{F_{\mu\alpha\beta}(q,-k,k-q)\,F_{\alpha\beta\nu}(k,(q-k),-q)}{q^2(k-q)^2}$$

$$\text{Symmetry Factor} = \frac{1}{2!}\times 6\times 3\times 2\times\frac{1}{6^2} = \frac{1}{2}$$

$$f_{acd}f_{cdb} = (F_a)_{cd}(F_b)_{dc} = \mathrm{Tr}(F_aF_b) = T(A)\,\delta_{ab}$$

$$\text{Numerator} = [g_{\mu\alpha}(q+k)_\beta + g_{\alpha\beta}(q-2k)_\mu + g_{\beta\mu}(k-2q)_\alpha]\cdot[g_{\alpha\beta}(2k-q)_\nu + g_{\beta\nu}(2q-k)_\alpha + g_{\nu\alpha}(-q-k)_\beta]$$

$$= -g_{\mu\nu}(2k^2-2k\cdot q+5q^2) + k_\mu k_\nu(2-1-16+2+2+2-1)$$

$$+ q_\mu q_\nu(-1+2-4+2-1+2+2) + q_\mu k_\nu(2+2+8-1-1-4-1)$$

$$+ k_\mu k_\nu(-1-1+8-4+2-1+2)$$

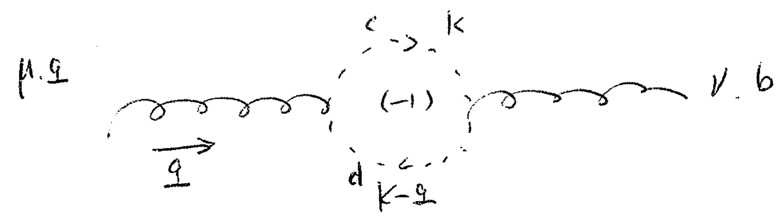
$$= -g_{\mu\nu}(2l^2+2x^2q^2-2xq^2+5q^2) - 10(l_\mu l_\nu + x^2 q_\mu q_\nu) + 2q_\mu q_\nu + 10x\, q_\mu q_\nu$$

where we shift the moment $k \to l + xq$

Thus,

$$i\Pi^{(a2)} = -\frac{ig_s^2}{16\pi^2} T(A)\delta_{ab}\left[-\frac{19}{12} q^2 g_{\mu\nu} + \frac{22}{12} q_\mu q_\nu\right]\cdot\left(\frac{4\pi\mu^2}{-q^2}\right)^\epsilon \cdot \frac{1}{\epsilon}\Gamma(1+\epsilon)$$

THE Vacuum polarization, with ghost graph



$$i\Pi^{(a3)}_{\mu\nu}(q) = -\mu^{2\epsilon}\int\frac{d^nk}{(2\pi)^n}\cdot(-g f_{acd}k_\mu)\frac{i}{k^2}(-g_s f_{cdb})(k-q)_\nu\cdot\frac{i}{(k-q)^2}$$

$$= g_s^2\mu^{2\epsilon}\,\mathrm{Tr}(F^aF^b)\int\frac{d^nk}{(2\pi)^n}\cdot\frac{k_\mu(k-q)_\nu}{k^2(k-q)^2}$$

$$= g_s^2\mu^{2\epsilon}\,T(A)\delta_{ab}\int_0^1 dx\int_0^\infty\frac{d^nl}{(2\pi)^n}\cdot\frac{\frac{1}{n}g_{\mu\nu}l^2 - x(1-x)q_\mu q_\nu}{(l^2-c)^2}$$

$$= -\frac{ig_s^2}{16\pi^2}T(A)\delta_{ab}\left[-\frac{1}{12}q^2g_{\mu\nu} - \frac{2}{12}q_\mu q_\nu\right]\left(\frac{4\pi\mu^2}{-q^2}\right)^\epsilon\cdot\frac{1}{\epsilon}\Gamma(1+\epsilon)$$

Thus,

$$i\Pi^{Gluon} + i\Pi^{Ghost} = -\frac{ig_s^2}{16\pi^2}T(A)\delta_{ab}\left[-q^2g_{\mu\nu} + q_\mu q_\nu\right]\left(\frac{4\pi\mu^2}{-q^2}\right)^\epsilon\cdot\frac{5}{3}\,\frac{1}{\epsilon}\Gamma(1+\epsilon)$$

Here we see that although Π^{gluon} doesn't conserve current, as a current conserving term proportional to $(-q^2 g_{\mu\nu}+q_\mu q_\nu)$. After introducing "ghost" contribution, we find the non-conserving terms are canceled.

THE TADPOLE diagram of 4-gluon coupling

This diagram vanishes in dimensional regularization.

THE COMPLETE ORDER α_s CORRECTION TO THE GLUON PROPAGATOR

$$i\Pi_{\mu\nu}(q) = i\Pi^{(a1)}_{\mu\nu}(q) + i\Pi^{(a2)}_{\mu\nu}(q) + i\Pi^{(a3)}_{\mu\nu}(q) + i\Pi^{(c)}_{\mu\nu}(q) \quad \text{counterterm}$$

$$= -\frac{ig_s^2}{16\pi^2}(-q^2 g_{\mu\nu}+q_\mu q_\nu)\,\delta_{ab}\,(\tfrac{5}{3}C_A-\tfrac{4}{3}T_R N_F)\cdot\left(\frac{4\pi\mu^2}{-q^2}\right)^\epsilon\cdot\frac{1}{\epsilon}\Gamma(1+\epsilon)$$

$$+ i\left[\delta Z_3(-q^2 g_{\mu\nu}+q_\mu q_\nu)\delta_{ab} - \frac{1}{\alpha}\delta Z^3_\alpha\, q_\mu q_\nu\right]$$

In Order to keep the ward-identity, we must set

$$\delta Z^3_\alpha = 0$$

So

$$i\Pi_{\mu\nu}(q) = i(-q^2 g_{\mu\nu}+q_\mu q_\nu)\delta_{ab}\left\{\delta Z_3 - \frac{g_s^2}{16\pi^2}\cdot(\tfrac{5}{3}C_A-\tfrac{4}{3}T_R N_F)\left(\frac{4\pi\mu^2}{-q^2}\right)^\epsilon\cdot\frac{1}{\epsilon}\Gamma(1+\epsilon)\right\}$$

And

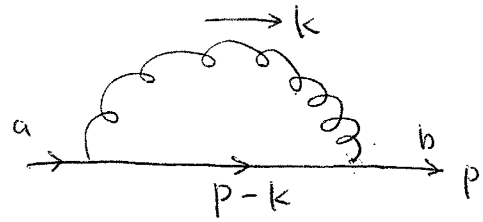
$$\frac{1}{\epsilon}\Gamma(1+\epsilon)\left(\frac{4\pi\mu^2}{-q^2}\right)^\epsilon = \frac{1}{\epsilon}(1-\gamma\epsilon)\cdot\left[1+\epsilon\log\left(\frac{4\pi\mu^2}{-q^2}\right)\right]$$

$$= \frac{1}{\epsilon} - \gamma + \log 4\pi + \log\left(\frac{\mu^2}{-q^2}\right)$$

We choose the $\overline{MS}$ scheme,

$$\boxed{\delta Z_3 = \frac{g_s^2}{16\pi^2}(\tfrac{5}{3}T(A)-\tfrac{4}{3}T(F)N_F)\cdot(\tfrac{1}{\epsilon}-\gamma+\log 4\pi)}$$

THE FERMION SELF-ENERGY GRAPH



$$i\Sigma^{(b1)} = \mu^{2\epsilon}\int \frac{d^nk}{(2\pi)^n}\cdot(ig_s\gamma_\mu T^b_{ij})\,\frac{i(\not p-\not k)}{(p-k)^2}\,(ig_s\gamma_\nu T^a_{ij})\,\frac{-i\delta_{ab}g_{\mu\nu}}{k^2}$$

$$= -\mu^{2\epsilon}\int\frac{d^nk}{(2\pi)^n}\cdot g_s^2\,(T^bT^b)\,(-2)\cdot\frac{\not p-\not k}{k^2(k-p)^2}$$

$$= 2\mu^{2\epsilon}\int_0^1 dx\int\frac{d^nl}{(2\pi)}\cdot\frac{\not p(1-x)}{(l^2-C)^2}$$

$$= -i\not p\left[-\frac{g_s^2}{16\pi^2}(T^bT^b)\left(\frac{4\pi\mu^2}{-p^2}\right)^{\epsilon}\cdot\frac{1}{\epsilon}\Gamma(1+\epsilon)\right]$$

Color matrix identity

identity matrix

$$T^bT^b = \sum_{b=1}^{N^2-1}\left[T^b_{(R)}\right]^2 = C_2(R)\,I$$

R means $C_2(R)$ depends on the representation.

$$Tr[T_a^{(R)}T_b^{(R)}] = T(R)\,\delta_{ab}$$

N-dimensional 'defining' representation.

$$T(F) = \frac{1}{2},\quad C_2(F) = \frac{N^2-1}{2N}$$

the (N^2-1)-dimensional adjoint representation $\boxed{(T_a^{(A)})_{bc} = -iC_{abc}}$

$$T(A) = N \qquad C_2(A) = N$$

Therefore,

$$i\Sigma^{(b1)} = -i\not p\left[-\frac{g_s^2}{16\pi^2}C_2(F)\cdot\left(\frac{4\pi\mu^2}{-p^2}\right)^{\epsilon}\cdot\frac{1}{\epsilon}\Gamma(1+\epsilon)\right]$$

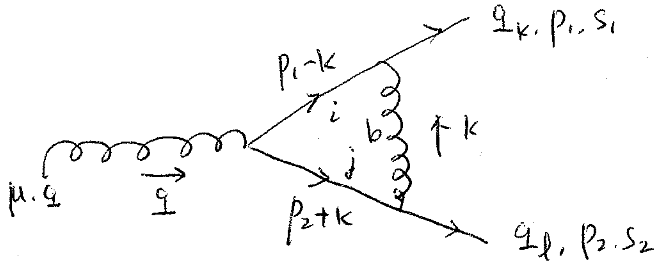
Include the counterterm contribution, the complete fermion self-energy should be

$$
\begin{aligned}
i\Sigma &= i\Sigma^{(b1)} + i\,\delta Z_2 \not p \\
&= i\not p \left\{ \delta Z_2 + \frac{g_s^2}{16\pi^2} C_2(F) \left(\frac{4\pi\mu^2}{-p^2}\right)^{\epsilon} \cdot \frac{1}{\epsilon}\Gamma(1+\epsilon) \right\}
\end{aligned}
$$

$$
\Rightarrow \boxed{\delta Z_2 = -\frac{g_s^2}{16\pi^2} C_2(F) \cdot \left(\frac{1}{\epsilon} - \gamma + \log 4\pi\right)}
$$

VERTEX CORRECTIONS

(1)

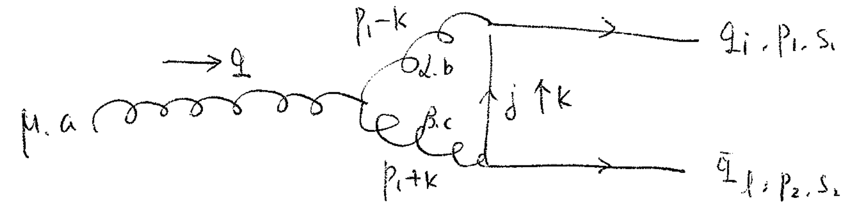


$$
\begin{aligned}
i\Gamma^{(c1)} &= \mu^{2\epsilon}\int \frac{d^n k}{(2\pi)^n} (ig_s \gamma_\alpha T^b_{kj}) \frac{i(\not p_1 - \not k)}{(p_1-k)^2} (ig_s\gamma_\mu T^a_{ij}) \frac{-i(\not p_2+\not k)}{(p_2+k)^2} (ig_s\gamma_\beta T^b_{jl}) \frac{-ig_{\alpha\beta}}{k^2} \\
&= (ig_s)(ig_s^2)(T^bT^aT^b)\int \frac{d^n k}{(2\pi)^n} \cdot \frac{\gamma_\alpha(\not p_1-\not k)\gamma_\mu(\not p_2+\not k)\gamma^\alpha}{k^2(p+k)^2((k-p)^2}
\end{aligned}
$$

$$
\begin{aligned}
T^bT^aT^b &= -\frac{1}{2}\mathrm{Tr}(F_aF_d)T^d + T^a\underbrace{T^bT^b} \\
&= T^a\left(-\frac{1}{2}T(A) + C_2(F)\right)
\end{aligned}
$$

$$
\Rightarrow i\Gamma^{(c1)} = (ig_s\gamma_\mu T^a)\cdot\left\{ \frac{g_s^2}{16\pi^2}\left[-\frac{1}{2}T(A)+C_2(F)\right]\left(\frac{4\pi\mu^2}{-q^2}\right)^{\epsilon}\Gamma(1+\epsilon)\left[\frac{1}{\epsilon_{UV}} - \frac{2}{\epsilon_{IR}^2} - \frac{4}{\epsilon_{IR}}\right]\right\}
$$

(2)



$$i\Gamma^{(2)} = \mu^{2\epsilon}\int\frac{d^nk}{(2\pi)^n}(g_s f_{bac} F_{\alpha\mu\beta})\,\frac{-i}{(p_1-k)^2}\cdot(ig_s\gamma_\alpha T^b_{ij})\cdot\frac{i\not{k}}{k^2}(ig_s\gamma_\beta T^c_{jl})\,\frac{-i}{(p_2+k)^2}$$

$$= ig_s(g_s^2)(f_{bac}T^bT^c)\int\frac{d^nk}{(2\pi)^n}\cdot\frac{F_{\alpha\mu\beta}(-p_1+k,\, q,\, -p_2-k)\,\gamma_\alpha\not{k}\gamma^\beta}{(p_1-k)^2(p_2+k)^2\cdot k^2}$$

$$= ig_s(g_s^2)\left(\frac{i}{2}T(A)T^a\right)\int\frac{d^nk}{(2\pi)^n}\cdot\frac{F_{\alpha\mu\beta}(p_1-k,\, -q,\, p_2+k)\,\gamma_\alpha\not{k}\gamma^\beta}{(p_1-k)^2(p_2+k)^2\cdot k^2}$$

In the same way, we obtain

$$i\Gamma^{(2)} = (ig_s\gamma^\mu T^a)\frac{g_s^2}{16\pi^2}\left(+\frac{1}{2}T(A)\right)\left(\frac{4\pi\mu^2}{-q^2}\right)^\epsilon\Gamma(1+\epsilon)\left[\frac{3}{\epsilon_{UV}}-\frac{4}{\epsilon_{IR}}\right]$$

Then,

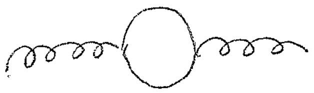
$$i\Gamma^{(1)}+i\Gamma^{(2)} = (ig_s\gamma_\mu T^a)\left\{\frac{g_s^2}{16\pi^2}[C_2(F)+T(A)]\cdot\left(\frac{4\pi\mu^2}{-q^2}\right)^\epsilon\Gamma(1+\epsilon)\cdot\frac{1}{\epsilon_{UV}}\right\}$$

Include the counterterm contribution, the complete vertex correction should be

$$i\Gamma = i\Gamma^{(1)}+i\Gamma^{(2)}+ig\gamma^\mu T^a\,\delta Z_{1F}$$

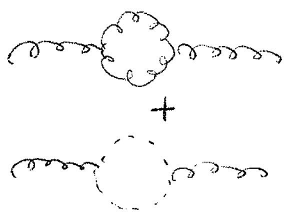
$$= (ig_s\gamma_\mu T^a)\left\{\delta Z_{1F}+\frac{g_s^2}{16\pi^2}[C_2(F)+T(A)]\cdot\left(\frac{4\pi\mu^2}{-q^2}\right)^\epsilon\Gamma(1+\epsilon)\frac{1}{\epsilon_{UV}}\right\}$$

$$\Rightarrow\quad \delta Z_{1F} = \frac{-g_s^2}{16\pi^2}[C_2(F)+T(A)]\cdot\left(\frac{1}{\epsilon}-\gamma+\log 4\pi\right)$$



$$\frac{ig_s^2}{16\pi^2} N_F T(F)\,\delta_{ab}\,(q_\mu q_\nu - q^2 g_{\mu\nu})\cdot\frac{4}{3}\,\Delta$$

$$\left(\Delta = \frac{1}{\varepsilon} - \gamma + \ln 4\pi\right)$$

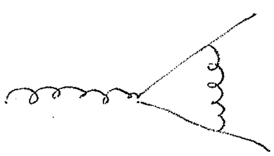


$$-\frac{ig_s^2}{16\pi^2} T(A)\,\delta_{ab}\,(-q^2 g_{\mu\nu} + q_\mu q_\nu)\cdot\frac{5}{3}\,\Delta$$

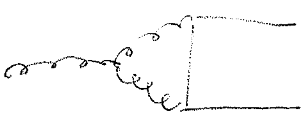
$$\Rightarrow Z_3 = 1 + \frac{g_s^2}{16\pi^2}\left(\frac{5}{3}T(A) - \frac{4}{3}T(F)N_F\right)\cdot\Delta$$

$$-i\cancel{p}\left[-\frac{g_s^2}{16\pi^2}C_2(F)\, i\Delta\right]$$

$$\Rightarrow Z_2 = 1 - \frac{g_s^2}{16\pi^2}C_2(F)\cdot\Delta$$



$$(ig_s\gamma_\mu T^a)\left\{\frac{g_s^2}{16\pi^2}\left[C_2(F) - \frac{1}{2}T(A)\right]\cdot\Delta\right\}$$



$$(ig_s\gamma_\mu T^a)\left\{\frac{g_s^2}{16\pi^2}\left[\frac{3}{2}T(A)\right]\cdot\Delta\right\}$$

$$\Rightarrow Z_{1F} = 1 - \frac{g_s^2}{16\pi^2}\left[C_2(F) + T(A)\right]\cdot\Delta$$

Now we see that if we set $T(A) = 0$ and $C_2(F) = 1$, then

$$Z_{1F} = Z_2.$$

This is the ward identity for QED.

Running Coupling in QCD

$\boxed{Z_g}$

$$Z_g = \frac{Z_{1F}}{Z_2\sqrt{Z_3}}$$

$$= \left(1-\frac{g_s^2}{16\pi^2}[C_2(F)+T(A)]\cdot\Delta\right)\left(1+\frac{g_s^2}{16\pi^2}C_2(F)\cdot\Delta\right)\left(1-\frac{g_s^2}{16\pi^2}\cdot\frac{1}{2}\left[\frac{5}{3}T(A)-\frac{4}{3}T(F)N_F\right]\Delta\right)$$

$$= 1-\frac{g_s^2}{16\pi^2}\left(\frac{11}{6}T(A)-\frac{2}{3}T(F)N_F\right)\cdot\Delta$$

$\boxed{\text{R-G equation}}$

Consider a dimensionless physical observable R. When we calculate R as a perturbation series in the coupling $\alpha_s = g^2/4\pi$, we must introduce another mass scale μ. So R depends on the ratio $\frac{Q^2}{\mu^2}$ for R is dimensionless. But Lagrangian of QCD knows nothing about the mass scale "μ". Therefore, if we hold the bare coupling fixed, physical quantities such as R can not depend on the choice of "μ".

$$\mu^2\frac{d}{d\mu^2}R\left(\frac{Q^2}{\mu^2},\alpha_s\right)=0$$

$$\Rightarrow \left[\mu^2\frac{\partial}{\partial\mu^2}+\mu^2\frac{\partial\alpha_s}{\partial\mu^2}\cdot\frac{\partial}{\partial\alpha_s}\right]R=0$$

Let $t=\ln\left(\frac{Q^2}{\mu^2}\right)$, $\beta(\alpha_s)=\mu^2\cdot\frac{\partial\alpha_s}{\partial\mu^2}$, then

$$\left[-\frac{\partial}{\partial t}+\beta(\alpha_s)\cdot\frac{\partial}{\partial\alpha_s}\right]R(e^t,\alpha_s)=0$$

$$\Rightarrow \quad t = \int_{\alpha_s(\mu)}^{\alpha_s(Q)} \frac{dx}{\beta(x)}$$

Differentiating the equation, we obtain

$$\frac{\partial \alpha_s(Q)}{\partial t} = \beta(\alpha_s(Q)), \quad \frac{\partial \alpha_s(Q)}{\partial \alpha_s} = \frac{\beta(\alpha_s(Q))}{\beta(\alpha_s)}$$

$\alpha_s \equiv \alpha_s(\mu^2)$

So,

$$\left[-\frac{\partial}{\partial t} + \beta(\alpha_s)\frac{\partial}{\partial \alpha_s}\right] R(1, \alpha_s(Q))$$

$$= \left[-\frac{\partial}{\partial t} R(1, \alpha_s(Q)) + \beta(\alpha_s) \cdot \frac{\partial}{\partial \alpha_s} R(1, \alpha_s(Q))\right)$$

$$= -\frac{\partial \alpha_s}{\partial t} \cdot \frac{\partial R}{\partial \alpha_s} + \beta(\alpha_s) \cdot \frac{\partial R}{\partial \alpha_s}$$

$$= \left(-\frac{\partial \alpha_s}{\partial t} + \beta(\alpha_s)\right) \cdot \frac{\partial R}{\partial \alpha_s}$$

$$= 0$$

Thus, $R(\frac{Q^2}{\mu^2}, \alpha_s) = R(1, \alpha_s(Q))$ is a solution. It shows that all of the scale dependence in R enters through the running of coupling constant $\alpha_s(Q)$.

$\boxed{\beta \text{ function}}$

$$g_0 = \mu^\epsilon g Z_g$$
$$\alpha_s^0 = (\mu^2)^\epsilon \alpha_s Z_g^2$$

The renormalization constant Z_g has the form

$$Z_g = 1 + \sum_{i=1}^{\infty} \frac{Z^{(i)}}{\epsilon}$$

The β function determine the scale dependence of the coupling at fixed bare parameters. So,

$$\beta(\alpha_s, \epsilon) = \frac{d\alpha_s}{d\ln\mu^2}\Big|_{\text{fixed } \alpha_s^0}$$

$$\frac{d\alpha_s^0}{d\ln\mu^2} = \mu^2 \frac{d\alpha_s^0}{d\mu^2} = 0$$

$$\Rightarrow \mu^2 \frac{d}{d\mu^2}\left[(\mu^2)^\epsilon \alpha_s Z_g^2\right]$$
$$= \mu^2\left[\epsilon(\mu^2)^{\epsilon-1}\alpha_s Z_g^2 + (\mu^2)^\epsilon \frac{d\alpha_s}{d\mu^2} Z_g^2 + (\mu^2)^\epsilon \alpha_s \frac{dZ_g^2}{d\mu^2}\right]$$
$$= (\mu^2)^\epsilon\left[\epsilon\alpha_s Z_g^2 + \mu^2\frac{d\alpha_s}{d\mu^2} Z_g^2 + \alpha_s \mu^2 \frac{d\alpha_s}{d\mu^2}\cdot\frac{dZ_g^2}{d\alpha_s}\right]$$
$$= (\mu^2)^\epsilon\left[\epsilon\alpha_s Z_g^2 + \beta(\alpha_s,\epsilon) Z_g^2 + 2\alpha_s \beta(\alpha,\epsilon)\frac{Z_g\, dZ_g}{d\alpha_s}\right]$$
$$= (\mu^2)^\epsilon Z_g\left[\beta(\alpha_s,\epsilon) Z_g + \epsilon\alpha_s Z_g + 2\alpha_s \beta(\alpha,\epsilon)\cdot\frac{dZ_g}{d\alpha_s}\right]$$
$$= 0$$

$$\Rightarrow \left[\beta(\alpha_s,\epsilon) + \epsilon\alpha_s + 2\alpha_s\beta(\alpha_s,\epsilon)\frac{d}{d\alpha_s}\right] Z_g = 0$$

Though β function contain no poles in ϵ, but in n-dimension it may contains extra contribution.

$$\beta(\alpha,\epsilon) = \beta(\alpha_s) + \sum_{i=1}^{\infty} \beta^{(i)}(\alpha_s)\,\epsilon^i$$

Inserting it into the above equation we find that

$$\boxed{\begin{aligned} &\beta^{(i)}(\alpha_s) = 0 \quad , \quad i > 1 \\ &\beta^{(1)}(\alpha_s) = -\alpha_s \\ &\beta(\alpha_s) = 2\alpha_s^2 \frac{d}{d\alpha_s} Z^{(1)} \end{aligned}}$$

$$Z_g = 1 - \frac{\alpha_s}{4\pi}\left(\frac{11}{6}T(A) - \frac{2}{3}T(F)N_F\right)\frac{1}{\epsilon}$$

$$\Rightarrow \quad Z^{(1)} = -\frac{\alpha_s}{4\pi}\left(\frac{11}{6}T(A) - \frac{2}{3}T(F)N_F\right)$$

$$\Rightarrow \quad \beta(\alpha_s) = 2\alpha_s^2 \cdot \left(-\frac{1}{4\pi}\right) \times \left[\frac{11}{6}T(A) - \frac{2}{3}T(F)N_F\right]$$

$$= -\frac{\alpha_s^2}{4\pi}\left[\frac{11}{3}T(A) - \frac{4}{3}T(F)N_F\right]$$

So we get one-loop level beta fuction

$$\beta(\alpha_s) = b_0\alpha_s^2 \quad , \quad b_0 = -\frac{1}{4\pi}\left[\frac{11}{3}T(A) - \frac{4}{3}T(F)N_F\right]$$

From $t = \int_{\alpha_s}^{\alpha_s(Q)} \frac{dx}{\beta(x)}$, we obtain

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \frac{1}{4\pi}\left[\frac{11}{3}T(A) - \frac{4}{3}T(F)N_F\right] \cdot \alpha_s(\mu)\log\left(\frac{Q^2}{\mu^2}\right)}$$

In QCD, $T(A) = 3$. $T(F) = \frac{1}{2}$. then

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1+\frac{1}{4\pi}(11-\frac{2}{3}n_f)\alpha_s(\mu^2)\log(\frac{Q^2}{\mu^2})}$$

As $\alpha_s(Q^2)$ is not a seperate function of $\alpha_s(\mu^2)$ and μ^2, it can be rewrited as a function of a single parameter Λ.

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1+\frac{1}{4\pi}(11-\frac{2}{3}n_f)\alpha_s(\mu^2)\log(\frac{\Lambda^2}{\mu^2})+\frac{1}{4\pi}(11-\frac{2}{3}n_f)\alpha_s(\mu^2)\log(\frac{Q^2}{\Lambda^2})}$$

$$= \frac{\alpha_s(\mu^2)}{\frac{1}{4\pi}(11-\frac{2}{3}n_f)\alpha_s(\mu^2)\log(\frac{Q^2}{\Lambda^2})}$$

$$= \frac{4\pi}{(11-\frac{2}{3}n_f)\log(\frac{Q^2}{\Lambda^2})}$$

Where we choose

$$\alpha_s(\mu^2)\frac{1}{4\pi}(11-\frac{2}{3}n_f)\log(\frac{\Lambda^2}{\mu^2}) = -1$$

As $t = \log(\frac{Q^2}{\Lambda^2})$ becomes very Large. the running coupling $\alpha_s(Q)$ decreases to zero, which is called ASYMPTOTIC FREEDOM.

To keep $(11-\frac{2}{3}n_f) > 0$. it requires

$$n_f < 17$$

Two-loop level

$$\frac{\partial \alpha_s(Q^2)}{\partial t} = \beta(\alpha_s(Q^2))$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \ln \frac{Q^2}{\mu^2}} = \frac{1}{2} Q \frac{\partial}{\partial Q}$$

So $$\frac{\partial \alpha_s(Q^2)}{\partial t} = \frac{1}{2} Q \frac{\partial}{\partial Q} \alpha_s(Q^2) = \beta(\alpha_s(Q^2)).$$

In order to be consistent with the notation of last notes, we substitute

$$\boxed{Q \to \mu. \quad 2\beta(\alpha_s(Q^2)) \to \beta(\alpha_s(Q^2))}$$

then we obtain

$$\mu \frac{\partial \alpha_s}{\partial \mu} = \beta(\alpha_s)$$

$$\beta = b_0 \alpha_s^2 + b_1 \alpha_s^3 + O(\alpha_s^4)$$

$$b_0 = -\frac{1}{2\pi}(11 - \frac{2}{3} n_f)$$

$$b_1 = -\frac{1}{4\pi^2}(51 - \frac{19}{3} n_f)$$

Integration yields

$$\int \frac{d\mu}{\mu} = \int \frac{d\alpha_s}{b_0 \alpha_s^2 (1 + \frac{b_1}{b_0}\alpha)} = \int \frac{d\alpha_s}{b_0 \alpha^2} - \int \frac{b_1}{b_0} \frac{d\alpha}{\alpha} + O(1)$$

$$\Rightarrow \ln \mu + c_1 = -\frac{1}{b_0 \alpha} + \frac{b_1}{b_0^2} \ln \frac{1}{\alpha} + c_2 + O(\alpha)$$

(1) Redefine C_2 such that $\frac{b_1}{b_0}\ln(\frac{1}{\alpha}) \rightarrow \frac{b_1}{b_0^2}\ln(\frac{-2}{b_0\alpha})$

(2) Define $\Lambda = e^{-C_1}$

$$\Rightarrow \Lambda = \mu \exp\left\{\frac{1}{b_0\alpha} - \frac{b_1}{b_0^2}\ln\left(\frac{-2}{b_0\alpha}\right) + c + o(\alpha)\right\}$$

So solve the equation, we obtain

$$\ln\left(\frac{\Lambda}{\mu}\right) = \frac{1}{b_0}\left\{\frac{1}{\alpha} - \frac{b_1}{b_0}\ln\left(\frac{-2}{b_0\alpha}\right) + b_0 c + o(\alpha)\right\}$$

$$\Rightarrow \frac{1}{\alpha} = -\frac{b_0}{2}\ln\left(\frac{\mu^2}{\Lambda^2}\right) + \frac{b_1}{b_0}\left[\ln\left(\frac{-2}{b_0\alpha}\right) - \frac{b_0^2 c}{b_1}\right] + o(\alpha)$$

(I) $\rightarrow$
$$= -\frac{b_0}{2}\ln\left(\frac{\mu^2}{\Lambda^2}\right)\left\{1 - \frac{2b_1}{b_0^2}\left[\frac{\ln\left(\frac{-2}{b_0\alpha}\right) - \frac{b_0^2 c}{b_1}}{\ln(\mu^2/\Lambda^2)}\right]\right\} + o\left(\frac{1}{\ln(\mu^2/\Lambda^2)}\right)$$

(II) $\rightarrow$
$$\Rightarrow \alpha_s(\mu) = \frac{-2}{b_0\ln(\mu^2/\Lambda^2)}\left\{1 + \frac{2b_1}{b_0}\left[\frac{\ln\ln(\mu^2/\Lambda^2) - \frac{b_0^2 c}{b_1}}{\ln(\mu^2/\Lambda^2)}\right]\right\} + o\left(\frac{1}{\ln^2(\mu^2/\Lambda^2)}\right)$$

Here we use an approximation in terms of inverse powers of $\ln(\mu^2/\Lambda^2)$. There is a slight difference between the definition of Λ. For the same value of $\alpha_s(Q^2)$, the two Λ's are related by

$$\Lambda_{I} \simeq \left(-\frac{b_0}{2b_1}\right)^{-\frac{b_1}{b_0}}\Lambda_{II} \simeq 1.1\,\Lambda_{II} \quad (n_f = 5)$$

$\boxed{C=0}$

If we set $C=0$, then we define Λ in the $\overline{MS}$ scheme.
Λ depends on the number of active flavours.

From
$$b_0 = -\frac{1}{2\pi}\left(11 - \frac{2n_f}{3}\right)$$
$$b_1 = -\frac{1}{4\pi^2}\left(51 - \frac{19n_f}{3}\right)$$

$$\Rightarrow \quad b_0(n+1) = b_0(n) + \frac{1}{3\pi}$$
$$b_1(n+1) = b_1(n) + \frac{19}{12\pi^2}$$

$$\Lambda^{n} = m_{n+1} \exp\Big\{ \frac{1}{b_0(n)} \Big[-\frac{b_0(n+1)}{2} \ln\Big(\frac{m_{n+1}^2}{\Lambda_{n+1}^2}\Big) + \frac{b_1(n+1)}{b_0(n+1)} \ln\Big(\ln \frac{m_{n+1}^2}{\Lambda_{n+1}^2}\Big) \Big]$$

$$- \frac{b_1(n)}{b_0^2(n)} \ln\Big[\frac{-2}{b_0(n)} \cdot \frac{b_0(n+1)}{-2} \ln\Big(\frac{m_{n+1}^2}{\Lambda_{n+1}^2}\Big) + \frac{b_1(n+1)}{b_0(n+1)} \ln \ln\Big(\frac{m_{n+1}^2}{\Lambda_{n+1}^2}\Big) \Big] \Big\}$$

$$= \Lambda^{n+1} \frac{m_{n+1}}{\Lambda^{n+1}} \cdot \exp\Big\{ -\frac{b_0(n+1)}{b_0(n)} \ln\Big(\frac{m_{n+1}}{\Lambda_{n+1}}\Big) + \frac{b_1(n+1)}{b_0(n)\, b_0(n+1)} \ln\Big(\frac{m_{n+1}^2}{\Lambda_{n+1}^2}\Big)$$

$$- \frac{b_1(n)}{b_0^2(n)} \ln\Big(\ln \frac{m_{n+1}^2}{\Lambda_{n+1}^2}\Big) - \frac{b_1(n)}{b_0^2(n)} \ln\Big(\frac{b_1(n+1)}{b_0(n)}\Big) + \cdots \Big\}$$

$$\approx \Lambda^{n+1} \cdot \Big(\frac{m_{n+1}}{\Lambda^{n+1}}\Big)^{\left(1 - \frac{b_0(n+1)}{b_0(n)}\right)} \cdot \Big[2 \ln\Big(\frac{m_{n+1}}{\Lambda_{n+1}}\Big) \Big]^{\frac{1}{b_0(n)} \left(\frac{b_1(n+1)}{b_0(n+1)} - \frac{b_1(n)}{b_0(n)} \right)}$$

$$\equiv \Lambda^{n+1} \Big(\frac{m_{n+1}}{\Lambda_{n+1}}\Big)^{d_n} \Big[2 \ln\Big(\frac{m_{n+1}}{\Lambda_{n+1}}\Big) \Big]^{g_n}$$

$$d_n = 1 - \frac{b_0(n+1)}{b_0(n)} = \frac{2}{33-2n}$$

$$g_n = \frac{1}{b_0(n)} \Big[\frac{b_1(n+1)}{b_0(n+1)} - \frac{b_1(n)}{b_0(n)} \Big] = \frac{963}{[33-2(n+1)](33-2n)^2}$$

Similarly it may be show that

$$\boxed{\begin{gathered} \Lambda^{n} = \Lambda^{n-1} \Big(\frac{m_n}{\Lambda_{n-1}}\Big)^{-d_n} \Big[2 \ln\Big(\frac{m_n}{\Lambda_{n-1}}\Big) \Big]^{-h_n} \\ d_n = \frac{2}{33-2n}, \quad h_n = \frac{963}{[33-2(n-1)](33-2n)^2} \end{gathered}}$$

$$\boxed{\begin{gathered} \Lambda^{n} = \Lambda^{n+1} \Big(\frac{m_{n+1}}{\Lambda_{n+1}}\Big)^{d_n} \Big[2 \ln\Big(\frac{m_{n+1}}{\Lambda_{n+1}}\Big) \Big]^{g_n} \\ d_n = \frac{2}{33-2n}, \quad g_n = \frac{963}{[33-2(n+1)](33-2n)^2} \end{gathered}}$$

N	b_0	b_1	$\frac{b_1}{b_0}$	$\Lambda_{\overline{MS}}$	$\pi/\alpha_s(\mu)$
3	$-\frac{9}{2\pi}$	$-\frac{96}{12\pi^2}$	$\frac{16}{9\pi}$	$\mu \exp\{-\frac{2}{9}[\frac{\pi}{\alpha_s(\mu)} - \frac{16}{9}\ln(\frac{12\pi}{27\alpha_s(\mu)})]\}$	$\frac{9}{4}\ln(\frac{\mu^2}{\Lambda^2}) + \frac{16}{9}\ln\ln(\frac{\mu^2}{\Lambda^2})$
4	$-\frac{25}{6\pi}$	$-\frac{77}{12\pi^2}$	$\frac{77}{50\pi}$	$\mu \exp\{-\frac{6}{25}[\frac{\pi}{\alpha_s(\mu)} - \frac{77}{50}\ln(\frac{12\pi}{25\alpha_s(\mu)})]\}$	$\frac{25}{12}\ln(\frac{\mu^2}{\Lambda^2}) + \frac{77}{50}\ln\ln(\frac{\mu^2}{\Lambda^2})$
5	$-\frac{23}{6\pi}$	$-\frac{58}{12\pi^2}$	$\frac{29}{23\pi}$	$\mu \exp\{-\frac{6}{23}[\frac{\pi}{\alpha_s(\mu)} - \frac{29}{23}\ln(\frac{12\pi}{23\alpha})]\}$	$\frac{23}{12}\ln(\frac{\mu^2}{\Lambda^2}) + \frac{29}{23}\ln\ln(\frac{\mu^2}{\Lambda^2})$
6	$-\frac{21}{6\pi}$	$-\frac{39}{12\pi^2}$	$\frac{39}{42\pi}$	$\mu \exp\{-\frac{6}{21}[\frac{\pi}{\alpha} - \frac{39}{42}\ln(\frac{12\pi}{21\alpha})]\}$	$\frac{21}{12}\ln(\frac{\mu^2}{\Lambda^2}) + \frac{39}{42}\ln\ln(\frac{\mu^2}{\Lambda^2})$

For all values of the momenta the coupling constant must be both a solution of the renormalization group equation and also a continuous function at the scale $\mu = m$, where m is the mass of the heavy quark.

For $\mu \simeq m_c$,

$$\Lambda^{(3)} = m_c \exp\{-\frac{2}{9}[\frac{25}{12}\ln(\frac{m_c^2}{\Lambda^{(4)2}}) + \frac{77}{50}\ln\ln(\frac{m_c^2}{\Lambda^{(3)2}}) - \frac{16}{9}\ln(\frac{12}{27}\cdot\frac{25}{12}\ln(\frac{m_c^2}{\Lambda^{(4)}}))] + O(\cdot)\}$$

$$= \Lambda^{(4)}\frac{m_c}{\Lambda^{(4)}}\exp\{-\frac{25}{27}\ln(\frac{m_c}{\Lambda^{(4)}}) - \frac{2}{9}[\frac{77}{50} - \frac{16}{9}]\ln\ln(\frac{m_c^2}{\Lambda^{(4)2}}) \underbrace{- \frac{2}{9}[-\frac{16}{9}\ln(\frac{25}{27})]}_{\sim 0.03 \to \text{omit}}\}$$

then.

$$\Lambda^{(3)} = \Lambda^{(4)}\left(\frac{m_c}{\Lambda^{(4)}}\right)^{\frac{2}{27}}\left[\ln\left(\frac{m_c^2}{\Lambda^{(4)2}}\right)\right]^{\frac{107}{2025}}.$$

Similarly.

$$\Lambda^{(4)} = \Lambda^{(3)}\left(\frac{m_c}{\Lambda^{(3)}}\right)^{\frac{2}{25}}\left[\ln\left(\frac{m_c^2}{\Lambda^{(3)2}}\right)^{\frac{-107}{2025}}\right]$$

General Case

$\Lambda^{(n)}_{\overline{MS}}$ for the n quarks in terms of $\Lambda^{(n+1)}_{\overline{MS}}$ and $\Lambda^{(n-1)}_{\overline{MS}}$.

$$\Lambda^{(n)} = m_{n+1}\exp\{\frac{1}{b_0(n)\alpha} - \frac{b_1(n)}{b_0^2(n)}\ln(\frac{-2}{b_0(n)\alpha})\}$$

$$\frac{1}{\alpha_s(n)} = -\frac{b_0(n+1)}{2}\ln\left(\frac{m_{n+1}}{\Lambda_{n+1}}\right)^2 + \frac{b_1(n+1)}{b_0(n+1)}\ln\left(\frac{m_{n+1}^2}{\Lambda_{n+1}^2}\right)$$

Thus,

Numeral results.

$m_t = 175.6$ GeV $\quad m_b = 4.7$ GeV $\quad m_c = 1.88$ GeV

$m_Z = 91.1$ GeV $\quad \alpha(m_Z) = 0.12 \quad m_s = 0.3$ GeV

$\boxed{\dfrac{1}{\alpha_s(\mu)}}$

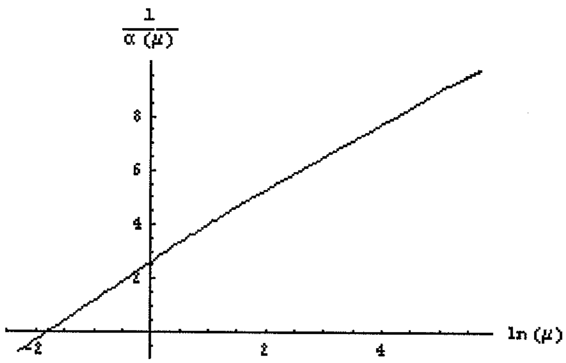
(1) One-loop level

$$\Lambda^{(3,1)} = 0.163188 \qquad \Lambda^{(4,1)} = 0.134206$$
$$\Lambda^{(5,1)} = 0.0985106 \qquad \Lambda^{(6,1)} = 0.0482903$$

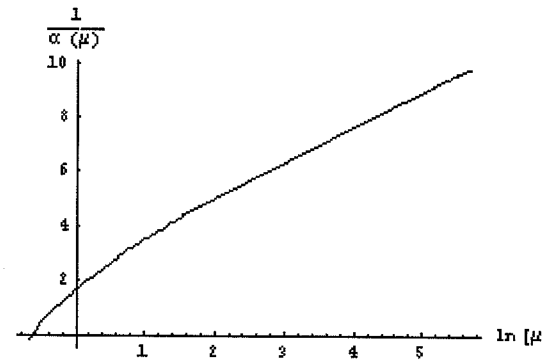
(2) Two-loop level :

$$\Lambda^{(3,2)} = 0.40514 \qquad \Lambda^{(4,2)} = 0.333842$$
$$\Lambda^{(5,2)} = 0.232782 \qquad \Lambda^{(6,2)} = 0.0969044$$

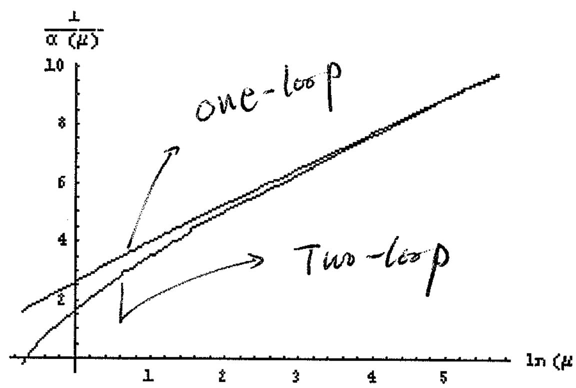
$$\alpha_s(m_t) = 0.109186$$
$$\alpha_s(10^{16}) = 0.022255$$



One-loop



Two-loop



$\ln(m_t) = 5.16479$

$\ln(m_Z) = 4.51196$

$\ln(m_b) = 1.54756$

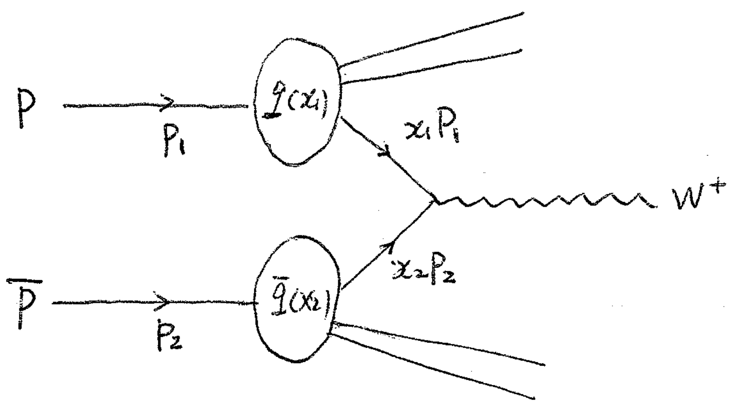
$\ln(m_c) = 0.631272$

$\ln(m_s) = -1.20397$

Factorization
in QCD

(An example: Inclusive W Boson production in hadron collisions)

$P\bar{P} \to W^+ X$



(1) Total cross section (HADRON LEVEL)

$$\sigma(P\bar{P} \to W^+ + X) = \sum_{i,j} \int_{\tau_0}^{1} dx_1 \int_{\tau_0/x_1}^{1} dx_2 \left[q_{i/p}(x_1)\, \bar{q}_{j/\bar{p}}(x_2) + \bar{q}_{j/p}(x_1)\, q_{i/\bar{p}}(x_2) \right] \hat{\sigma}(q_i \bar{q}_j \to W^+)$$

Where $\hat{\sigma}$ is the subprocess cross section

$\tau_0 = M^2/S$. ($\sqrt{S}$ is the c.m. energy of $P\bar{P}$)

$\hat{s} = (p_1+p_2)^2 = 2p_1 \cdot p_2 = x_1 x_2\, 2P_1 \cdot P_2 = x_1 x_2 S$

$\hat{\tau} = \dfrac{M^2}{\hat{s}} = \dfrac{M^2}{x_1 x_2 S} = \dfrac{\tau_0}{x_1 x_2}$

the sum runs over all contributing pair of partons.

(2) The zeroth-order cross section

$$\sigma = \sum_{i,j} \int_{\tau_0}^{1} dx_1 \int_{\tau_0/x_1}^{1} dx_2 \left[q_{i/p}(x_1)\, \bar{q}_{j/\bar{p}}(x_2) + \bar{q}_{j/p}(x_1)\, q_{i/\bar{p}}(x_2) \right] \hat{\sigma}^{(0)}(q_i \bar{q}_j \to W^+)$$

$$= \sum_{i,j} \int_{\tau_0}^{1} dx_1 \int_{\tau_0/x_1}^{1} dx_2 \left[q_{i/p}(x_1)\, \bar{q}_{j/\bar{p}}(x_2) + \bar{q}_{j/p}(x_1)\, q_{i/\bar{p}}(x_2) \right] \tilde{\sigma}_0 (q_i \bar{q}_j \to W^+)\, \delta(\hat{s} - M^2)$$

(using $\hat{s} = x_1 x_2 S$)

$$= \sum_{i,j} \int_{\tau_0}^{1} \frac{dx_1}{x_1} \left[q_{i/p}(x_1)\, \bar{q}_{j/\bar{p}}\left(\tfrac{\tau_0}{x_1}\right) + \bar{q}_{j/p}(x_1)\, q_{i/\bar{p}}\left(\tfrac{\tau_0}{x_1}\right) \right] \hat{\sigma}_0 (q_i \bar{q}_j \to W^+) \frac{1}{S}$$

where we let $\hat{\sigma}^{(0)}(q_i q_j \to W^+) = \tilde{\sigma}_0 (q_i q_j \to W^+)\, \delta(\hat{s} - M^2)$

3) $O(\alpha_s)$ cross sections of the subprocess

* Virtual corrections

$$\hat{\sigma}^{NLO}_{Virt} = \tilde{\sigma}_0 \frac{\alpha_s}{2\pi} \delta(1-\hat{\tau}) \left(\frac{4\pi\mu^2}{M^2}\right)^{\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left\{ -\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 7 - \frac{\pi^2}{3} + \delta_{scheme} \right\} \cdot \frac{1}{\hat{s}}$$

Where $\tilde{\sigma}_0$ is the tree level cross section,

$\alpha_s = g_s^2/4\pi$.

$$\delta_{scheme} = \begin{cases} 0 & , \text{DRED} \\ -1 & , \text{Naive } \gamma_5, \text{HVBM} \end{cases}$$

* Real corrections

$$\hat{\sigma}^{NLO}_{REAL} = \tilde{\sigma}_0 \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{M^2}\right)^{\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left\{ \frac{2}{\epsilon^2} \delta(1-\hat{\tau}) - \frac{2}{\epsilon} \frac{1+\hat{\tau}^2}{(1-\hat{\tau})_+} + 4(1+\hat{\tau}^2)\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+ + 4(1-\hat{\tau}) \right\} \cdot \frac{1}{\hat{s}}$$

* The NLO SUBPROCESS cross section

$$\hat{\sigma}^{(1)} = \hat{\sigma}^{NLO}_{VIRT} + \hat{\sigma}^{NLO}_{REAL}$$

$$= \tilde{\sigma}_0 \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{M^2}\right)^{\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left\{ -\frac{2}{\epsilon} \left(\frac{1+\hat{\tau}^2}{1-\hat{\tau}}\right)_+ + 4(1-\hat{\tau}) + 4(1+\hat{\tau}^2)\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+ \right.$$

$$\left. + \left(-7 - \frac{\pi^2}{3} + \delta_{scheme}\right) \delta(1-\hat{\tau}) \right\} \cdot \frac{1}{\hat{s}}$$

Where we use

$$-\frac{2}{\epsilon}\left[\frac{1+\hat{\tau}^2}{(1-\hat{\tau})_+} + \frac{3}{2}\delta(1-\hat{\tau})\right] = -\frac{2}{\epsilon}\left(\frac{1+\hat{\tau}^2}{1-\hat{\tau}}\right)_+$$

(4) Redefine the Parton Distribution Function (PDF)

The relation between the scale dependent and bare PDF is

$$f_i^A(x, Q_{PDF}^2) = \int_x^1 \frac{dz}{z} \left[\delta_{ij} \delta(1-z) + \frac{\alpha_s}{2\pi} R_{i\leftarrow j}(z, Q_{PDF}^2) \right] f^A_{j,bare}\left(\frac{x}{z}\right)$$

Recall the zero-th order cross section

$$\sigma^{(0)}(p\bar{p}\to W) = \sum_{ij}\int_{\tau_0}^{1}\frac{dx_1}{x_1}\left[q_{i/p}(x_1)\,\bar{q}_{j/\bar{p}}\left(\frac{\tau_0}{x_1}\right) + \bar{q}_{j/p}(x_1)\,q_{i/\bar{p}}\left(\frac{\tau_0}{x_1}\right)\right]\hat{\sigma}_0(q_i\bar{q}_j\to W^+)\frac{1}{S}$$

Now Let us concentrate on the first two parton distribution functions.
Say $q_{i/p}(x_1)\,\bar{q}_{j/\bar{p}}\left(\frac{\tau_0}{x_1}\right)$, then

$$\sigma(p\bar{p}\to W^+) = \sum_n\left(\frac{\alpha_s}{\pi}\right)^n\sigma_{p\bar{p}}^{(n)}$$

$$= q^{(0)}_{i/p,\,\text{bare}}\otimes\hat{\sigma}^{(0)}_{ij}\otimes \bar{q}^{(0)}_{j/\bar{p},\,\text{bare}} \longrightarrow \text{Tree level}$$

$$\left.\begin{aligned}&+ q^{(1)}_{i/p,\,\text{bare}}\otimes\hat{\sigma}^{(0)}_{ij}\otimes\bar{q}^{(0)}_{j/\bar{p},\,\text{bare}}\\ &+ q^{(0)}_{i/p,\,\text{bare}}\otimes\hat{\sigma}^{(0)}_{ij}\otimes\bar{q}^{(1)}_{j/\bar{p},\,\text{bare}}\\ &+ q^{(0)}_{i/p,\,\text{bare}}\otimes\hat{\sigma}^{(1)}_{ij}\otimes\bar{q}^{(0)}_{j/\bar{p},\,\text{bare}}\end{aligned}\right\}\longrightarrow \text{NLO}.$$

$$= q^{(0)}_{i/p,\,\text{ren}}\otimes\hat{\sigma}^{(0)}_{ij}\otimes\bar{q}^{(0)}_{i/\bar{p},\,\text{ren}}$$

$$A\left\{\begin{aligned}&+\Big[-\frac{\alpha_s}{2\pi}R^{(1)}_{i\leftarrow i'}\otimes q^{(0)}_{i/p,\,\text{ren}}\otimes\hat{\sigma}^{(0)}_{ij}\otimes\bar{q}^{(0)}_{j/\bar{p},\,\text{ren}}\\ &\quad-\frac{\alpha_s}{2\pi}q^{(0)}_{i/p,\,\text{ren}}\otimes\hat{\sigma}^{(0)}_{ij}\otimes R^{(1)}_{j\leftarrow j'}\otimes q^{(0)}_{j/\bar{p},\,\text{ren}}\Big]\end{aligned}\right.$$

$$B\ \Big\{+\Big\{q^{(0)}_{i/p,\,\text{ren}}\otimes\hat{\sigma}^{(1)}_{ij}\otimes\bar{q}^{(0)}_{j/\bar{p},\,\text{ren}}\Big\}$$

$A \sim R^{(1)}_{i\leftarrow j}$: the QCD correction to σ_0 due to the correction to the PDF

$B \sim \hat{\sigma}^{(1)}$: the QCD correction to the subprocess cross section.

(5) Consider the first item,

$$\sigma_{1st}(P\bar{P}\to W) = \sum_{ij}\int_{\tau_0}^{1} dx_1 \int_{\frac{\tau_0}{x_1}}^{1} dx_2 \left[q_{i/p}(x_1)\bar{q}_{j/\bar{p}}(x_2) \right] \hat{\sigma}(q_i q_j \to W^+)$$

$$\hat{\sigma}^{(0)}_{ij} = \tilde{\sigma}_0\, \delta(1-\hat{\tau})\cdot\frac{1}{\hat{s}}$$

$$\hat{\sigma}^{(1)}_{ij} = \tilde{\sigma}_0 \left(\frac{\alpha_s}{2\pi} C_F\right)\Big\{ -\frac{2}{\epsilon}\cdot\frac{1}{C_F} P^{(1)}_{q\leftarrow q}(\hat{\tau}) \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left(\frac{4\pi\mu^2}{Q^2_{PDF}}\right)\left(\frac{Q^2_{PDF}}{M^2}\right)^{\epsilon} + 4(1-\hat{\tau})$$

$$+4(1+\hat{\tau}^2)\left(\frac{\log(1-\hat{\tau})}{1-\hat{\tau}}\right)_+ + \left(-7-\frac{\pi^2}{3}+\delta_{scheme}\right)\delta(1-\hat{\tau})\Big\}\cdot\frac{1}{\hat{s}}$$

$$= -\frac{2}{\epsilon}\tilde{\sigma}_0 \frac{\alpha_s}{2\pi} P^{(1)}_{q\leftarrow q}(\hat{\tau}) \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left(\frac{4\pi\mu^2}{Q^2_{PDF}}\right)^{\epsilon}\cdot\frac{1}{\hat{s}}$$

$$+\tilde{\sigma}_0\left(\frac{\alpha_s}{2\pi}C_F\right)\Big\{-2\ln\left(\frac{Q^2_{PDF}}{M^2}\right)\cdot\left(\frac{1+\hat{\tau}^2}{1-\hat{\tau}}\right)_+ + 4(1-\hat{\tau}) + 4(1+\hat{\tau}^2)\left(\frac{\log(1-\hat{\tau})}{1-\hat{\tau}}\right)_+$$

$$+\left(-7-\frac{\pi^2}{3}+\delta_{scheme}\right)\delta(1-\hat{\tau})\Big\}\cdot\frac{1}{\hat{s}}$$

$$= 2\tilde{\sigma}_0 \frac{\alpha_s}{2\pi} R^{(1)}_{q\leftarrow q}(\hat{\tau}, Q^2_{PDF})\frac{1}{\hat{s}} + \hat{\sigma}^{(1)}_{finite}\cdot\frac{1}{\hat{s}}$$

Then,

$$B = q^{(0)}_{i/p,ren}\otimes\hat{\sigma}^{(1)}_{ij}\otimes\bar{q}^{(0)}_{j/\bar{p},ren}$$

$$= 2\cdot\frac{\alpha_s}{2\pi} q^{(0)}_{i/p,ren}\otimes\tilde{\sigma}_0\otimes\bar{q}^{(0)}_{j/\bar{p},ren}\cdot R^{(1)}_{q\leftarrow q}(\hat{\tau}, Q^2_{PDF}) + q^{(0)}_{i/p,ren}\otimes\hat{\sigma}^{(1)}_{finite}\otimes\bar{q}^{(0)}_{j/\bar{p},ren}$$

$$= \sum_{i,j} \int_{\tau_0}^{1} dx_1 \int_{\frac{\tau_0}{x_1}}^{1} dx_2 \Big[\underbrace{-\frac{2}{\epsilon}}_{\text{singular}} \cdot \frac{\alpha_s}{2\pi}\, q_{i/p}^{(0)}(x_1, Q_{PDF}^2)\, q_{j/p}^{(0)}(x_2, Q_{PDF}^2)\, \tilde{\sigma}_0(\hat{\tau}) \cdot P_{q\leftarrow q}^{(1)}(\hat{\tau}) \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left(\frac{4\pi\mu^2}{Q_{PDF}^2}\right)^{\epsilon} \Big] \cdot \frac{1}{\hat{s}}$$

$$+ \sum_{i,j} \int_{\tau_0}^{1} dx_1 \int_{\tau_0/x_1}^{1} dx_2 \left[q_{i/p}^{(0)}(x_1, Q_{PDF}^2)\, q_{j/p}^{(0)}(x_2, Q_{PDF}^2) \cdot \hat{\sigma}_{finite}^{(1)}(\hat{\tau}) \right] \cdot \frac{1}{\hat{s}}$$

where $\hat{\tau} = \frac{\tau_0}{x_1 x_2}$

Calculate A

As mentioned before, A is the QCD corrections due to the corrections to the PDF. So it is easy to obtain from the zeroth order cross section

$$\sigma_0 = \sum_{ij} \int_{\tau_0}^{1} \frac{dx_1}{x_1} \left[q_{i/p}(x_1)\, \bar{q}_{j/p}\!\left(\frac{\tau_0}{x_1}\right) \right] \cdot \tilde{\sigma}_0 (q_i \bar{q}_j \to W^+)\, \frac{1}{S}$$

by replacing $q_{j/p}$ with $q^{(1)}_{j/p}(\frac{\tau_0}{x_1}, Q^2_{PDF})$ or $q_{i/p}$ with $q^{(1)}_{i/p}(x_1, Q^2_{PDF})$ respectively.

$$q_{j/p} \to q^{(1)}_{j/p}\!\left(\frac{\tau_0}{x_1}, Q^2_{PDF}\right) = -\frac{\alpha_s}{2\pi} \int_{\tau_0/x_1}^{1} \frac{dx_2}{x_2} R_{q\leftarrow q}(z, Q^2_{PDF}) \cdot q^{(0)}_{j/p}\!\left(\frac{\tau_0}{z x_1}\right)$$

$$= -\frac{\alpha_s}{2\pi} \int_{\tau_0/x_1}^{1} \frac{dx_2}{x_2} R_{q\leftarrow q}(\hat{\tau}, Q^2_{PDF}) \cdot q^{(0)}_{j/p}(x_2)$$

So,

$$A = -\frac{\alpha_s}{2\pi} \sum_{i,j} \int_{\tau_0}^{1} \frac{dx_1}{x_1} \int_{\tau_0/x_1}^{1} \frac{dx_2}{x_2} \cdot R_{q\leftarrow q}(\hat{\tau}, Q^2_{PDF}) \cdot q^{(0)}_{j/p}(x_2, Q^2_{PDF}) \cdot q^{(0)}_{i/p}(x_1, Q^2_{PDF}) \times \frac{z}{S} \cdot \tilde{\sigma}_0(\hat{\tau})$$

$$= \frac{2\alpha_s}{2\pi} \sum_{i,j} \int_{\tau_0}^{1} dx_1 \int_{\tau_0/x_1}^{1} dx_2 \cdot (-1) \left[-\frac{1}{\epsilon} P^{(1)}_{q\leftarrow q}(\hat{\tau}) \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left(\frac{4\pi\mu^2}{Q^2_{PDF}}\right)^{\epsilon} \right] q^{(0)}_{j/p}(x_2, Q^2_{PDF}) \cdot q^{(0)}_{i/p}(x_1, Q^2_{PDF})$$

$$\times \frac{1}{\underbrace{x_1 x_2 S}_{\hat{s}}} \cdot \tilde{\sigma}_0(\hat{\tau})$$

$$= \sum_{i,j} \int_{\tau_0}^{1} dx_1 \int_{\tau_0/x_1}^{1} dx_2 \left[\underbrace{\frac{2}{\epsilon}}_{\text{Singular}} \cdot \frac{\alpha_s}{2\pi} q^{(0)}_{i/p}(x_1, Q^2_{PDF})\, q^{(0)}_{i/p}(x_2, Q^2_{PDF})\, \tilde{\sigma}_0(\hat{\tau}) \cdot P^{(1)}_{q\leftarrow q}(\hat{\tau}) \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left(\frac{4\pi\mu^2}{Q^2_{PDF}}\right)^{\epsilon} \right] \cdot \frac{1}{\hat{s}}$$

Hence, we could see that

$$A + B = q^{(0)}_{i/p,\,ren} \otimes \hat{\sigma}^{(1)}_{finite} \otimes \bar{q}^{(0)}_{j/p,\,ren}$$

and it is finite. It means the "$\frac{1}{\epsilon}$" poles cancel, or the collinear singularity has been absorbed into the Bare PDF.

(6)

Up to now, we only consider the first item of the hadronic cross section. By ~~considering~~ another item, we obtain the total cross section (Hadron Level) up to the Next-to-leading-order,

$$\sigma(p\bar{p} \to W^+X) = \sum_{ij} \int_{\tau_0}^{1} dx_1 \int_{\tau_0/x_1}^{1} dx_2 \left[q_{i/p}(x_1, Q^2)\, \bar{q}_{j/\bar{p}}(x_2, Q^2) + \bar{q}_{j/\bar{p}}(x_1, Q^2)\, q_{i/p}(x_2, Q^2) \right] \cdot \tilde{\sigma}_0 \cdot \frac{1}{\hat{s}}$$

Tree level

$$\times \Big\{ \delta(1-\hat{\tau})$$

$$\underbrace{+ \frac{\alpha_s}{2\pi} C_F \left[2 \ln\left(\frac{M^2}{Q^2_{PDF}}\right) \left(\frac{1+\hat{\tau}^2}{1-\hat{\tau}}\right)_+ + 4(1-\hat{\tau}) + 4(1+\hat{\tau}^2) \left(\frac{\log(1-\hat{\tau})}{1-\hat{\tau}} \right)_+ + \left(-\frac{\pi^2}{3} - 7 + \delta_{\text{scheme}}\right) \delta(1-\hat{\tau}) \right\}}_{\text{"NLO"}}$$