Quantum Electrodynamics

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QED Lagrangian

The Lagrangian is

$$\mathcal{L} = \bar{\psi}\left(i\gamma_\mu D^\mu - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \qquad (1)$$

where $iD^\mu = i\partial^\mu - eA^\mu$ (charge of e^-: $e = -|e|$) and $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$.

The equation of motion are

- Dirac equation

$$(i\gamma^\mu D_\mu - m)\psi = 0, \qquad (2)$$

- Maxwell equation

$$\partial_\mu F^{\mu\nu} = e\bar{\psi}\gamma^\nu\psi = ej^\nu, \qquad (3)$$

where j^ν is current density.

Gauge Theory

The Lagrangian is invariant under local gauge transformation

$$\psi(x) \rightarrow e^{i\alpha(x)}\psi(x), \quad (4)$$

$$A_\mu(x) \rightarrow A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x), \quad (5)$$

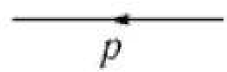
where $\alpha(x)$ depends on space-time x.

Under the above local transformation, we know that

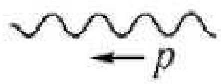
- $D^\mu\psi(x) \rightarrow e^{i\alpha(x)}D^\mu\psi(x)$,
- $\bar{\psi}(x) \rightarrow \bar{\psi}(x)e^{-i\alpha(x)}$,
- $F^{\mu\nu} \rightarrow F^{\mu\nu}$.

Since $\bar{\psi}D^\mu\psi$, $\bar{\psi}\psi$ and $F^{\mu\nu}F_{\mu\nu}$ are invariant, the Lagrangian is invariant.

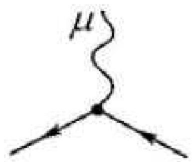
Feynman Rules and Feynman Diagrams

Dirac propagator: 

$$= \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$$

Photon propagator: 

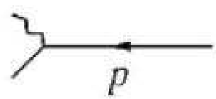
$$= \frac{-ig_{\mu\nu}}{p^2 + i\epsilon}$$

QED vertex: 

$$= iQe\gamma^\mu$$

($Q = -1$ for an electron)

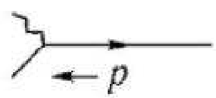
External fermions:



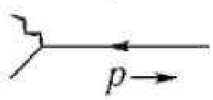
$= u^s(p)$ (initial)

$= \bar{u}^s(p)$ (final)

External antifermions:

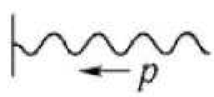


$= \bar{v}^s(p)$ (initial)

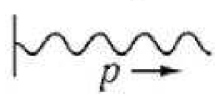


$= v^s(p)$ (final)

External photons:



$= \epsilon_\mu(p)$ (initial)



$= \epsilon^*_\mu(p)$ (final)

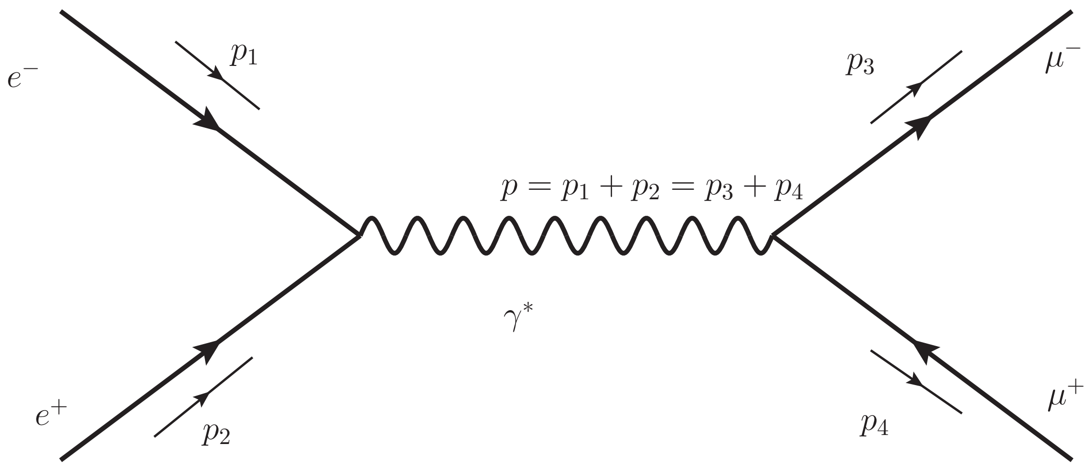
S-matrix and Cross Section

- Given $\mathcal{L}$, one can obtain $\mathcal{H}$, and then construct S-matrix element, which is denoted as $\mathcal{M}$(scattering amplitude).

- The probability is given by taking $\mathcal{M}^\dagger\mathcal{M} = |\mathcal{M}|^2$. Thus one can obtain the scattering cross section.

- We take $e^+e^- \to \mu^+\mu^-$ as an example to calculate the scattering cross section.

$e^+e^- \to \mu^+\mu^-$ scattering



Following the QED Feynman Rules, the invariant amplitude is

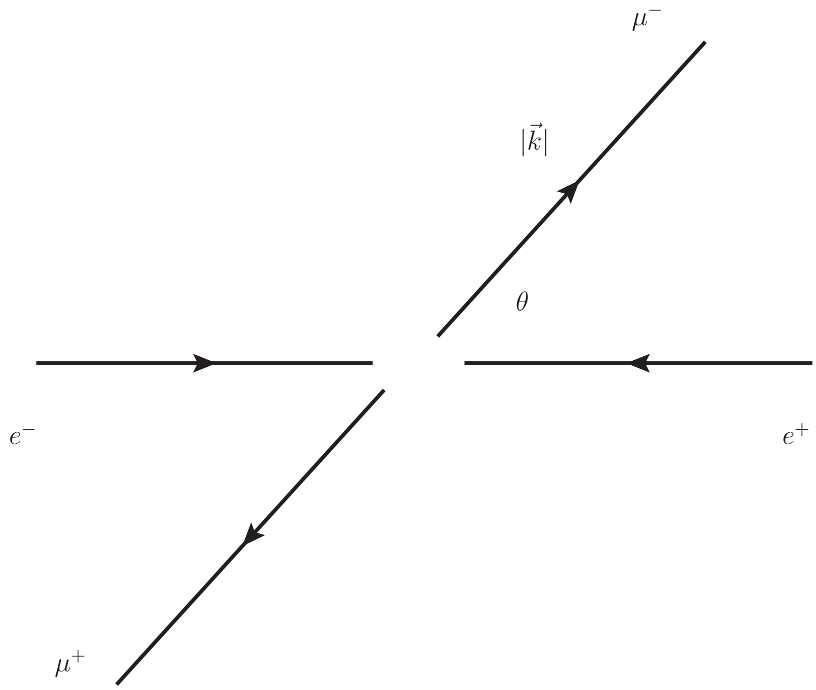
$$-i\mathcal{M} = \left[\bar{u}(p_3)\left(ieQ\gamma^{\beta}\right)v(p_4)\right]\frac{-ig_{\alpha\beta}}{p^2+i\epsilon}\left[\bar{v}(p_2)\left(-ie\gamma^{\beta}\right)u(p_1)\right], \quad (6)$$

where we take the muon charge as Q, and electron charge -1. For an unpolarized e^+ and e^- beam, one needs to average the initial spins and sum over the final spins. So the amplitude square is

$$\begin{aligned}\frac{1}{2}\frac{1}{2}\sum_{spin}|\mathcal{M}|^2 &= \frac{e^4Q^2}{4p^4}\sum_{spin}\left[\bar{v}(p_4)\gamma^{\beta}u(p_3)\bar{u}(p_3)\gamma^{\alpha}v(p_4)\right] \\ &\quad \left[\bar{u}(p_1)\gamma_{\beta}v(p_2)\bar{v}(p_2)\gamma_{\alpha}u(p_1)\right] \\ &= \frac{8e^4}{p^4}\left(p_1\cdot p_4 p_2\cdot p_3 + p_1\cdot p_3 p_2\cdot p_4 + m_{\mu}^2 p_1\cdot p_2\right) \quad (7)\end{aligned}$$

$e^+e^- \to \mu^+\mu^-$ scattering (2)

Specialize to the center of mass frame,



$$p_1 = (E, 0, 0, E) \quad (8)$$

$$p_2 = (E, 0, 0, -E) \quad (9)$$

$$p_3 = (E, |\vec{k}|\sin\theta, 0, |\vec{k}|\cos\theta) \quad (10)$$

$$p_4 = (E, -|\vec{k}|\sin\theta, 0, -|\vec{k}|\cos\theta) \quad (11)$$

$$p^2 = (p_1 + p_2)^2 = (p_3 + p_4)^2 = 4E^2 = s \quad (12)$$

$e^+e^- \to \mu^+\mu^-$ scattering (3)

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\cos\theta d\phi} = \frac{1}{2s}\frac{|\vec{p}_3|}{32\pi^2|\vec{p}_1|}\frac{1}{4}\sum_{spin}|\mathcal{M}|^2$$

$$= \frac{1}{2s}\frac{k}{32\pi^2 E}\frac{1}{4}\sum_{spin}|\mathcal{M}|^2 = \frac{\alpha^2 Q^2\beta}{4s}\left(2-\beta^2+\beta^2\cos^2\theta\right), \quad (13)$$

where $\alpha = \frac{e^2}{4\pi}$ and $\beta = \sqrt{1-\frac{4m_\mu^2}{s}}$.

The total cross section is

$$\sigma = \int \frac{d\sigma}{d\Omega}d\Omega = \frac{4\pi\alpha^2 Q^2\beta}{3s}\frac{3-\beta^2}{2}. \quad (14)$$

In the relativistic limit, $m_\mu \ll \sqrt{s}, \beta \to 1$. The differential cross section will be

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 Q^2}{4s}\left(1+\cos^2\theta\right), \quad (15)$$

The total cross section will be

$$\sigma = \frac{4\pi\alpha^2 Q^2}{3s}. \quad (16)$$