PHY905-004

Running Couplings

and

Higher Order Calculations.

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- Regularization
- Renormalization
- Factorization

- Coupling Constant and Perturbative Calculation

— Consider a gauge theory with a single coupling constant.

e in QED $\quad \left(\alpha = \frac{e^2}{4\pi} \right)$

g_s in QCD $\quad \left(\alpha_s = \frac{g_s^2}{4\pi} \right)$

— In perturbative calculation,

Any physical observable $= \langle \mathcal{O} \rangle_{\text{tree}} \times \{$

$$1 + c_1 \frac{\alpha}{\pi} + c_2 \left(\frac{\alpha}{\pi}\right)^2 + c_3 \left(\frac{\alpha}{\pi}\right)^3 + \cdots \} .$$

e.g.

$$R \equiv \frac{e^- e^+ \to \text{hadrons}}{e^- e^+ \to \mu^- \mu^+} ,$$

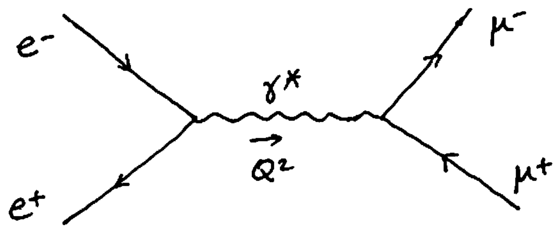
$$= R^{(0)} \left\{ 1 + c_1 \frac{\alpha_s}{\pi} + c_2 \left(\frac{\alpha_s}{\pi}\right)^2 + c_3 \left(\frac{\alpha_s}{\pi}\right)^3 + \cdots \right\}$$

$$= R^{(0)} \left\{ 1 + \underbrace{\frac{\alpha_s}{\pi} + 1.411 \left(\frac{\alpha_s}{\pi}\right)^2 - 12.8 \left(\frac{\alpha_s}{\pi}\right)^3 + \cdots}_{} \right\}$$

$\uparrow$

QCD corrections.

- Determine α from $e^-e^+ \to \mu^-\mu^+$ (in QED)



Tree level :
$$\sigma = \frac{4\pi\alpha^2}{3s} \quad , \qquad s \equiv Q^2$$

— Need to determine α from one experiment,

then can predict all rates in different experiments.

— Let's consider

α is determined at Q_1^2 from σ_1,

what's the rate σ_2 at Q_2^2 ?

• 1) At tree-level, (Born-level, leading order)

— ① $\sigma_1 = \frac{4\pi\alpha^2}{3\,Q_1^2}$

measured — Known

$\Rightarrow$ α is determined. $\alpha^2 = \frac{3\,Q_1^2}{4\pi}\sigma_1$

— ② At Q_2^2 :

$$\frac{\sigma_2}{\sigma_1} = \frac{\alpha^2/Q_2^2}{\alpha^2/Q_1^2} = \frac{Q_1^2}{Q_2^2}$$

(Okay, but not good enough.)

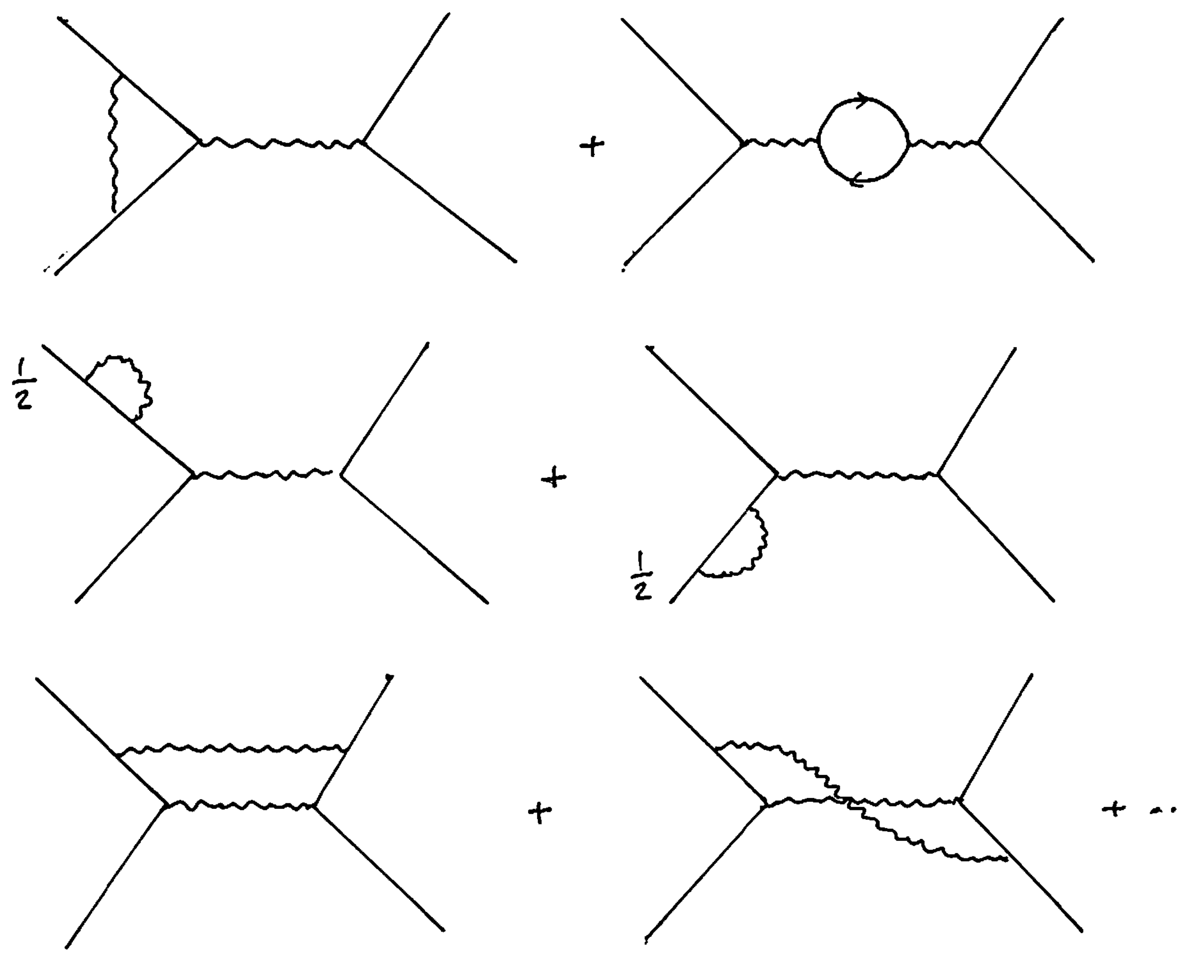
$\Rightarrow$ $\sigma_2 = \sigma_1 \frac{Q_1^2}{Q_2^2}$ (prediction.)

— <u>Note</u> : In reality, for large s, also need to include Z-boson.

$\Rightarrow$ Complicate Considerations.

- 2) At one-loop level (NLO)

— Can improve prediction power by doing higher-oder calculations.



For any physical observables, the theoretical prediction has to be finite.

$\Rightarrow$ Renormalizable Theory!

— 1 In loop calculations :

Two kinds of divergencies :

$$\begin{cases} \text{Ultravidate :} & p \to \infty \\ \text{Infrared :} & p \to 0 \quad : \begin{cases} \text{Soft} \\ \text{Collinear} \end{cases} \end{cases}$$

where p is loop momentum : $\int d^4p$.

Need to " regularize " the divergence ;

* Momentum cutoff scheme.

(Introducing $\Lambda^2_{\text{cut-off}}$ in $\int^{\Lambda^2} d^4p$)

* Dimensional Regularization

(Introducing μ to restore proper dimension

of coupling in n-dimesion.)

Note : How is μ introduced in loop calculations ?

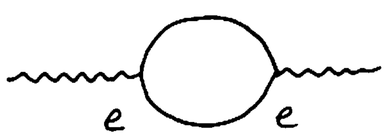
In n-dimension, coupling e has the dimension ~~eq~~ $[M]^{2-\frac{n}{2}}$,

so we replace e by $e \cdot [\mu^{2-\frac{n}{2}}]$ in the Lagrangian,

where, e remains ~~de~~ dimensionless.

e.g.

$n = 4-2\varepsilon$



$$\sim \mu^{4-n} \cdot e^2 \int \frac{d^n p}{(2\pi)^n}$$

$$\sim \left(\frac{\pi^2}{16\pi^4}\right) \cdot (e^2 \mu^{2\varepsilon}) \cdot \frac{1}{\varepsilon}$$

where π^2 comes from $\int_4 d\Omega \sim \pi^2(\cdots)$.

$\left[\frac{1}{16\pi^2}\right]$ ← loop factor

$$\lim_{\varepsilon \to 0} \mu^{2\varepsilon} \cdot \frac{1}{\varepsilon} = \lim_{\varepsilon \to 0} e^{\varepsilon \ln \mu^2} \cdot \frac{1}{\varepsilon}$$

$$= (1 + \varepsilon \ln \mu^2) \frac{1}{\varepsilon}$$

$$= \frac{1}{\varepsilon} + \underline{\underline{\ln \mu^2}}.$$

— ② Any physical observable has to be finite :

In loop calculation :

$$\lim_{n \to 4} \frac{1}{4-n} \to \infty \qquad \text{(pole term)}$$

$\Rightarrow$ Need to add "counter terms" to the Lagrangian

to cancel the divergence in loop calculation

and obtain finite prediction.

Renormalization :

Counter term = "pole term" + finite piece

$\uparrow$ Universial

$\uparrow$ scheme dependent

- Renormalization Procedure for a Renormalizable Theory

—. $\mathcal{L} = \mathcal{L}(g_{bare}, \psi_{bare}, \cdots)$

—. First, define the renormalized coupling g_{ren}. via

$$g_{bare} = g_{ren}(1 + \delta g)$$

where δg is the counterterm in general proportional to $\frac{1}{4-n}$ in n-dimensional regularization, which cancels $\frac{1}{(4-n)}$ terms from loop diagram calculation.

—. g_{ren} is finite, which is to be compared to data.

— (3) loop-level result:

$$\sigma = \frac{4\pi}{3Q^2}[\alpha(\mu)]^2 \left\{ 1 + 2\frac{\alpha}{4\pi}\beta_1 \ln\frac{\mu^2}{Q^2} + \cdots \right\}$$

physical observable (Independent of μ)

α. depends on μ.

may contain other kinds of log's.

$$\mu \frac{d\sigma}{d\mu} = 0$$

$$\sigma = \frac{4\pi}{3Q^2}\left\{ \alpha(\mu)\left(1 + \frac{\alpha}{4\pi}\beta_1 \ln\frac{\mu^2}{Q^2}\right)\right\}^2 \{1 + \cdots\}$$

$$= \frac{4\pi}{3Q^2}\{\alpha(Q)\}^2 \cdot \{1 + \cdots\}$$

running coupling / effective coupling.

Use tree-level relation to obtain 1-loop result by replacing α_{tree} by $\alpha(Q)$:

$\sigma \sim \frac{\alpha^2}{s}$ (tree) , $\sigma \sim \frac{\alpha^2}{s}\left(1 + \frac{\alpha}{4\pi}\beta_1 \ln\frac{\mu^2}{Q^2}\right) \sim \frac{\alpha^2_{loop}}{s}$ (1-loop)

Measure $\alpha(\mu)$ from σ_1 at a chosen μ. Then,

can predict σ_2 from the above.

$$\frac{\sigma_2}{\sigma_1} = \frac{Q_1^2\,[\alpha(Q_2)]^2}{Q_2^2\,[\alpha(Q_1)]^2}$$

$$\frac{\alpha(Q_2)}{\alpha(Q_1)} = \frac{\frac{1}{\alpha(Q_1)}}{\frac{1}{\alpha(Q_2)}} = \frac{Q_2}{Q_1}\sqrt{\frac{\sigma_2}{\sigma_1}}$$

$$= \frac{\frac{1}{\alpha(Q_1)}}{\frac{1}{\alpha(Q_1)} + \frac{\beta_1}{4\pi}\ln\frac{Q_2^2}{Q_1^2}}$$

$$= 1 - \frac{\beta_1}{4\pi}\,\alpha(Q_1)\,\ln\frac{Q_2^2}{Q_1^2}$$

$$\Rightarrow \frac{\sigma_1}{\sigma_2} = \left(\frac{Q_1}{Q_2}\right)^2 \cdot \left\{ 1 - 2\cdot \frac{\alpha(Q_1)}{4\pi}\,\beta_1 \ln\frac{Q_2^2}{Q_1^2} \right\}$$

This theory prediction agrees better with data.

- . β - function .

$$\beta \equiv \mu \frac{\partial g(\mu)}{\partial \mu} \equiv -g \left\{ \frac{\alpha_g}{4\pi} \beta_1 + \left(\frac{\alpha_g}{4\pi}\right)^2 \beta_2 + \cdots \right\}$$

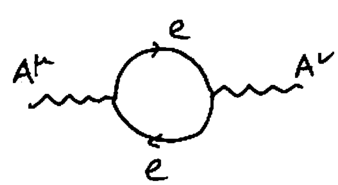
$$\alpha_g \equiv \frac{g^2}{4\pi} .$$

In one loop.

$$\Rightarrow \quad \frac{1}{\alpha_g(\mu)} - \frac{1}{\alpha_g(\mu_0)} = \frac{\beta_1}{4\pi} \ln \frac{\mu^2}{\mu_0^2} \qquad (\underline{\underline{\text{running}}})$$

* In QED

$$g = e , \quad \alpha_g = \alpha = \frac{e^2}{4\pi}$$



$$\beta_1 = -\frac{4}{3}$$

* In QCD,

$$g = g_s , \quad \alpha_g = \alpha_s = \frac{g_s^2}{4\pi}$$

$$\beta_1 = 11 - \frac{2n_f}{3} ,$$

(n_f is number of light quarks)

i.e. $m_q \le \mu$

Note : $\overline{MS}$ scheme .

— $\alpha_s(M_Z)$ is physical observable.

$$\frac{1}{\alpha_s(\mu)} = \frac{1}{\alpha_s(M_Z)} + \frac{\beta_1}{4\pi}\ln\frac{\mu^2}{M_Z^2}$$

$$= \underbrace{\frac{1}{\alpha_s(M_Z)} - \frac{\beta_1}{4\pi}\ln M_Z^2} + \frac{\beta_1}{4\pi}\ln \mu^2$$

$$= -\frac{\beta_1}{4\pi}\ln\Lambda_{QCD}^2 + \frac{\beta_1}{4\pi}\ln\mu^2$$

$$= \frac{\beta_1}{4\pi}\ln\frac{\mu^2}{\Lambda_{QCD}^2}$$

Λ_{QCD} ~~is~~ can be determined from. $\alpha_s(M_Z^2)$ and M_Z^2.

$$\alpha_s(\mu) = \frac{4\pi}{\beta_1 \ln\frac{\mu^2}{\Lambda_{QCD}^2}} \quad , \quad \beta_1 = 11 - \frac{2n_f}{3} \quad , \quad \overset{n_f}{(m_f \le \mu)}$$

Given a $\Lambda_{QCD}^{(n_f,\,loop)}$, then α (at all μ) is determined!

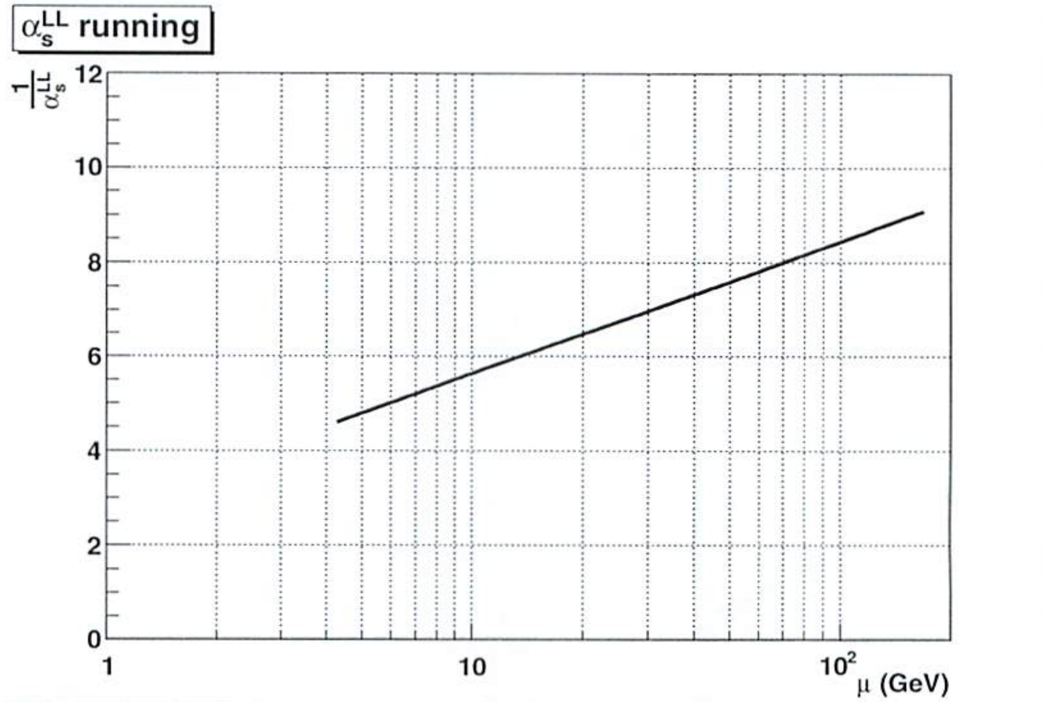
Running α_s in QCD

$$\beta_1 = 11 - \frac{2n_f}{3}$$

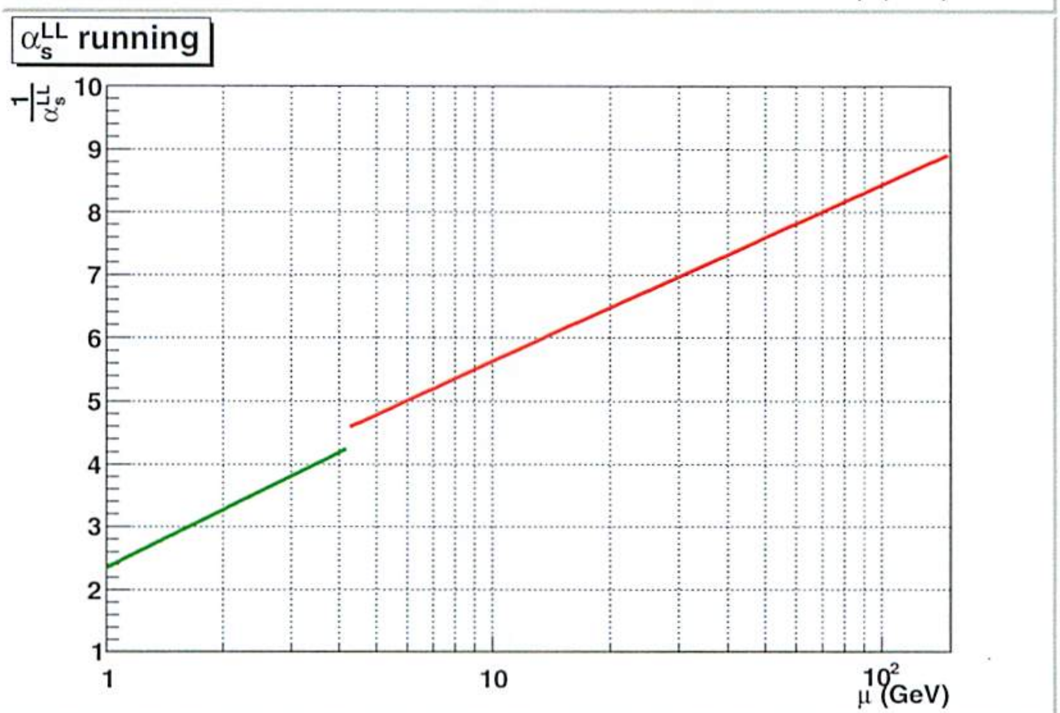
$$\alpha_s(M_Z) = 0.12$$

$$\Lambda_{QCD}(n_f = 5,\ one-loop) = 99\,MeV$$

$$\Lambda_{QCD}(n_f = 4,\ one-loop) = 170\,MeV$$



α_s^{LL} running
$\frac{1}{\alpha_s^{LL}}$
12
10
8
6
4
2
0
1
10
10^2
μ (GeV)



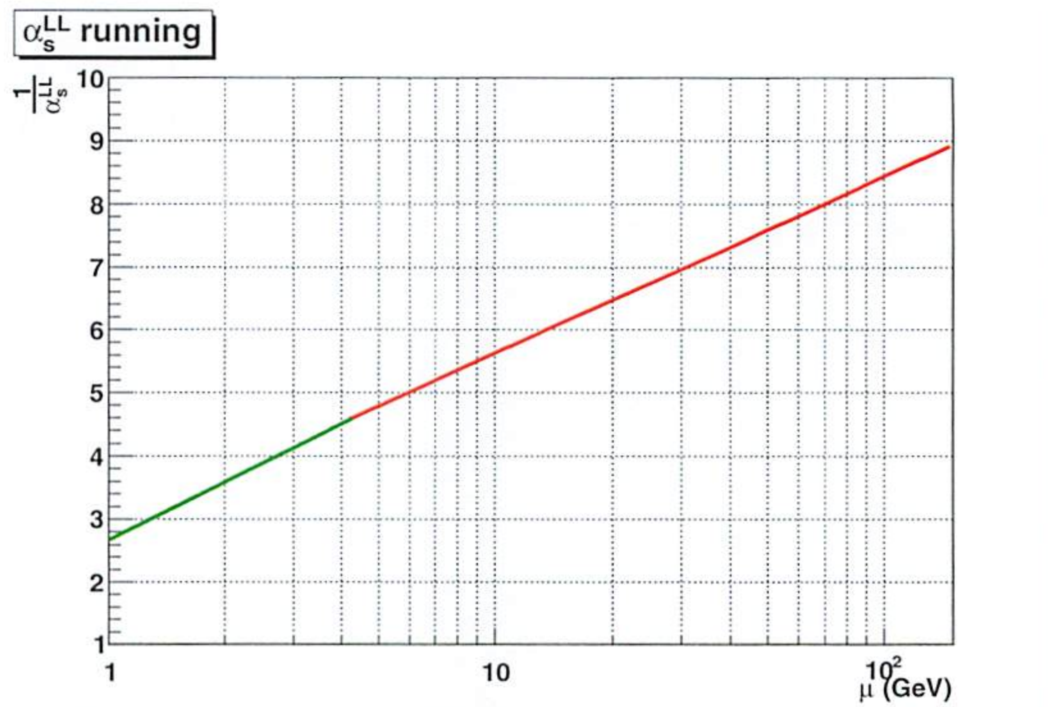
α_s^{LL} running
$\frac{1}{\alpha_s^{LL}}$
10
9
8
7
6
5
4
3
2
1
1
10
10^2
μ (GeV)

Matching at m_b

$$\Lambda^{(4)} = \Lambda(n_f = 5)\left(\frac{m_b}{\Lambda(n_f = 5)}\right)^{2/25}$$

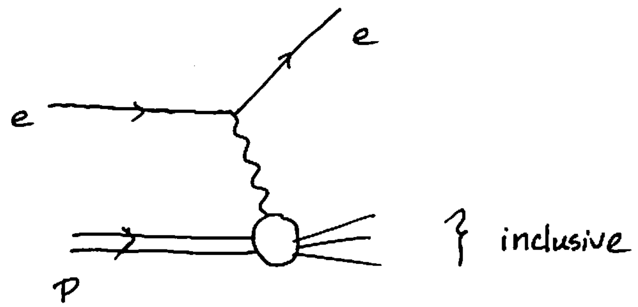
$$m_b = 4.3 GeV \quad (MS - bar\ bottom\ quark\ mass)$$

$$\Lambda^{(4)} = 134\, MeV$$


α_s^{LL} running
$\frac{1}{\alpha_s^{LL}}$
10
9
8
7
6
5
4
3
2
1
1
10
10^2
μ (GeV)

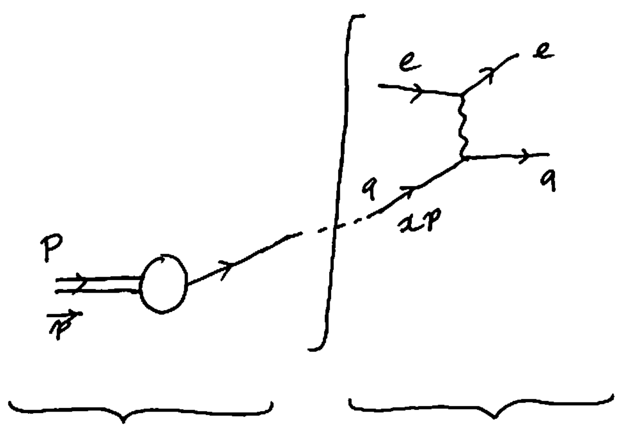
• Factorization

— DIS process :



Extract F_2 (structure function)

— In paton model



long distance

↑

Parton distribution function

$f(x)$

(not calculable)

short distance

↑

Calculated using theory

factorization !

$$F_2 = \hat{f} \otimes \hat{\sigma}$$

↑ pdf

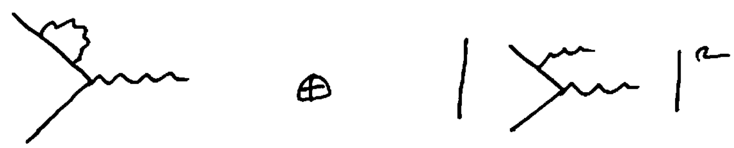
↑ hard part cross section

- Infrared Singularity in loop calculations

— Soft div. cancelled after including all diagram contributions.

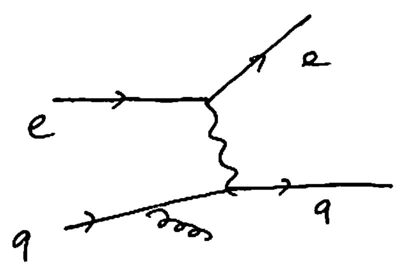
loop $\oplus$ Real emission

e.g.



KNL Theorem.

— Collinear div.



$m_q \to 0 \Rightarrow$ Collinear div.

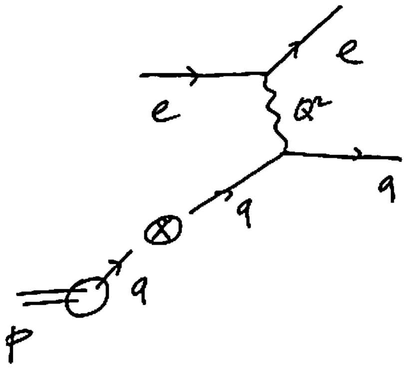
* Physical obs. has to be infrared finite !

$\Rightarrow$ All the collinear div. from loop calculations has to disappear !

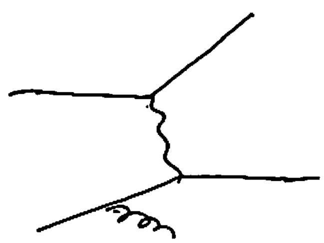
Need Factorization Theorem to show they can be absorbed into the "definition" of PDF.

— · Feynman Diagrams

* Tree level.



* loop cantribution to hard part ;

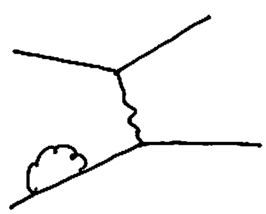


Ultra. → Cancel (Renormalizaton)

Soft → Cancel (KNL)

Collinear → left.

+



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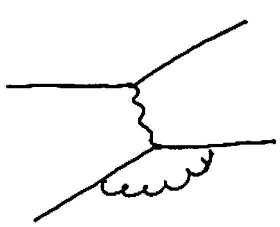


+ . . .

— • UV and IR divergences

* Scale μ to regularize Ultra and Infrared divergence in dim. reg. :

$$\begin{cases} \mu = \mu_V & \text{for ultra div.} \\ \mu = \mu_I & \text{for Infrared div.} \end{cases}$$

* To regularize UV div., use

$$\varepsilon = \varepsilon_V = 4-n > 0 \qquad \text{e.g.} \quad \int \frac{d^n p}{(p^2+m^2)^2}$$

* To regularize IR div., use

$$\varepsilon = \varepsilon_{IR} = n-4 \qquad \text{e.g.} \quad \int \frac{d^n p}{(p^2)^2}$$

* The hard part has

$$\alpha_s(\mu), \quad \frac{Q^2}{\mu^2}, \quad \varepsilon = \frac{2}{n-4}$$

dependence.

- $F_2(Q^2) = \hat{f} \otimes \hat{\sigma}(\alpha_s(\mu), \frac{Q^2}{\mu^2}, \varepsilon)$

Contain UV and IR div.

$= \hat{f} \otimes \hat{\sigma}(\alpha_s(R^2), \frac{Q^2}{\mu^2}, \frac{R^2}{\mu^2}, \varepsilon)$

R^2 renorm. scale

Only Collinear. div.

$R \frac{d\alpha_s(R^2)}{dR} = \beta(\alpha_s(R^2))$

$= \hat{f} \otimes \Gamma(\alpha_s(R^2), \frac{M^2}{\mu^2}, \frac{M^2}{R^2}, \varepsilon) \otimes C(\alpha_s(R^2), \frac{Q^2}{M^2}, \frac{M^2}{R^2})$

process independent

The residue term including pole $\frac{1}{\varepsilon}$

is A-P spliting fun.

Wilson Coeff.

process dependent

M : mass fact. scale.

$= \underbrace{f(\alpha_s)}$ $\otimes$ $C(\alpha_s)$

Usually, choose scale

$R = M = Q$

POF at LL.

$Q\frac{df_i}{dQ} = - P_{ij} \otimes f_j$

A-P spliting

- Logs due to Collinear Singularity

$$-\left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon}\cdot\frac{1}{\varepsilon}\,P_{qq}$$

$$= -\left(\frac{4\pi\mu^2}{M^2}\,\frac{M^2}{Q^2}\right)^{\varepsilon}\frac{1}{\varepsilon}\,P_{qq}$$

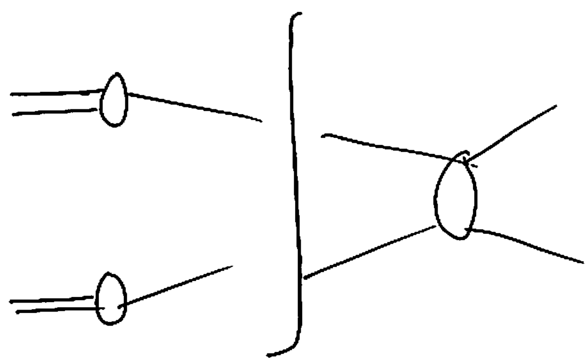
$$= -\left(\frac{4\pi\mu^2}{M^2}\right)^{\varepsilon}\cdot\frac{1}{\varepsilon}\,P_{qq} - \ln\frac{M^2}{Q^2}\cdot P_{qq}$$

$$= \underbrace{-\left(\frac{4\pi\mu^2}{M^2}\right)^{\varepsilon}\frac{1}{\varepsilon}\,P_{qq}}_{\text{part in } f} + \underbrace{\ln\frac{Q^2}{M^2}\cdot P_{qq}}_{\text{part in } C}$$

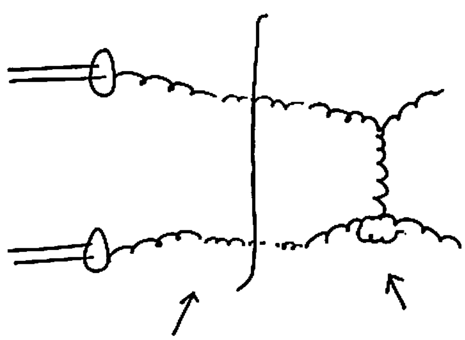
finite & process dependent

- Scale choice

— Hadron collider : 2-jet events.



e.g.



M factorization scale

R renormalization scale

$P_{gg}\, \alpha_s \ln\left(\frac{M^2}{Q^2}\right)$ from factorization

$\beta_1 \alpha_s \ln\left(\frac{R^2}{Q^2}\right)$ from running of α_s

— If calculate to all order in α_s, the σ should be indep. of R & M.

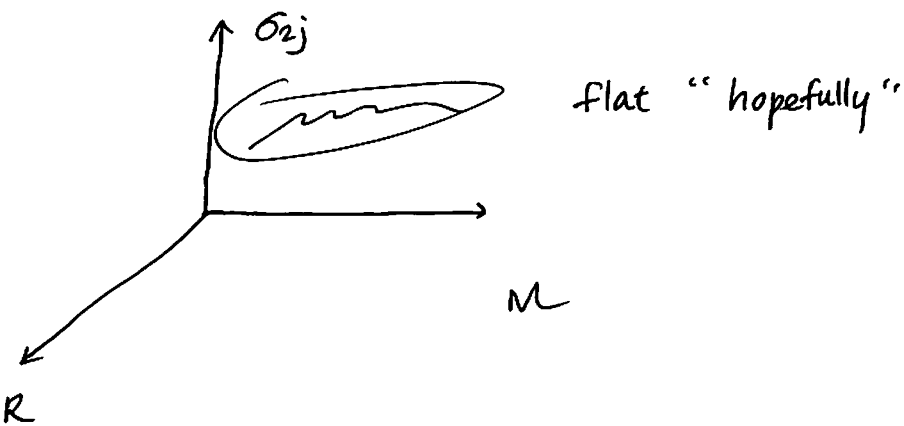
— If finite order calculation, we choose

$R = Q$ & $M = Q$ to "eliminate" possible large logs.

$\beta_1 \alpha_s \ln\left(\frac{R^2}{Q^2}\right)$ from running of α_s

$P_{qq}\, \alpha_s \ln\left(\frac{M^2}{Q^2}\right)$ from factorization

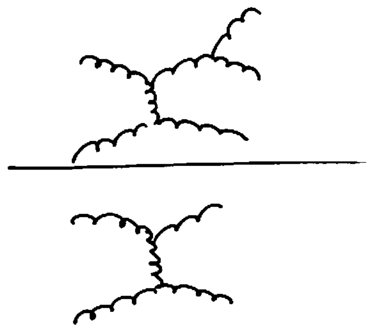
— In practice, we vary R & M to test (or estimate) higher order corrections.



— Define

$$K \equiv \frac{\text{event rate for } \geq 3 \text{ jets}}{\text{event rate for } \geq 2 \text{ jet}}$$

$\sim$



$\sim \alpha_s$

K has to be renormalization Group Invariant.

$$\Rightarrow \quad \mu \frac{dK}{d\mu} = 0$$

— We only have $\sigma_{\geq 3jet}$ at tree level (i.e. no logs)

We should only use $\sigma_{\geq 2jet}$ at tree-level.

$$\frac{\sigma_{\geq 3jets}^{(\alpha_s(K))}}{\sigma_{\geq 2jet}^{(\alpha_s(K))}} = (\cdots)\ \alpha_s(K)$$

$\Rightarrow$ Use LL <u>resutes</u>

α_s, PDF, amplitude