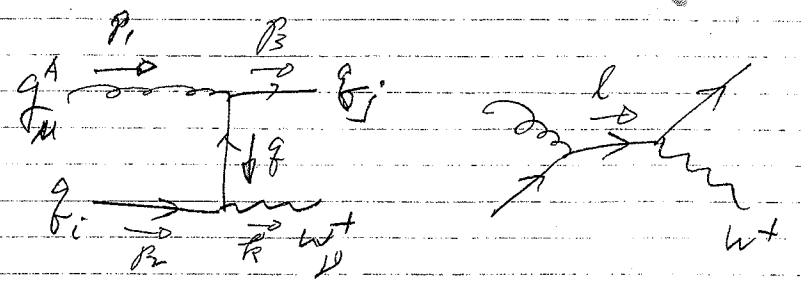


3 $g q_i \rightarrow W^+ q_j$



$i\mathcal{M} = -i \frac{g}{2\sqrt{2}} (-i g_s) \epsilon_{\mu}^A (p_1) \epsilon_{\nu}^B (p_2)$

$\bar{u}(p_2) \left[\gamma^{\mu} T^A \frac{i}{p_3 - p_1} \gamma^{\nu} (1 - \gamma_5) + \gamma^{\nu} (1 - \gamma_5) \frac{i}{p_1 + p_2} \gamma^{\mu} T^A \right] u(p_1)$

Squaring and summing over spins and colors, we obtain

$|\mathcal{M}|^2 = \frac{g^2}{8} g_s^2 \text{Tr}(T^A T^A) \cdot (u)^2$

$\left\{ \begin{aligned} & \text{Tr} \left(\frac{1}{p_3} \gamma^{\mu} (p_3 - p_1) \gamma^{\nu} (1 - \gamma_5) p_2 \gamma_{\nu} (1 - \gamma_5) (p_3 - p_1) \gamma_{\mu} \right) \frac{1}{t^2} \\ & + \text{Tr} \left(\frac{1}{p_3} \gamma^{\nu} (1 - \gamma_5) (p_1 + p_2) \gamma^{\mu} p_2 \gamma_{\mu} (p_1 + p_2) \gamma_{\nu} (1 - \gamma_5) \right) \frac{1}{s^2} \\ & + \text{Tr} \left(\frac{1}{p_3} \gamma^{\mu} (p_3 - p_1) \gamma^{\nu} (1 - \gamma_5) p_2 \gamma_{\mu} (p_1 + p_2) \gamma_{\nu} (1 - \gamma_5) \right) \frac{2}{st} \end{aligned} \right\}$

with $\begin{aligned} \hat{s} &= (p_1 + p_2)^2 = 2 p_1 \cdot p_2 \\ \hat{t} &= (p_1 - p_3)^2 = -2 p_1 \cdot p_3 \\ \hat{u} &= (p_2 - p_3)^2 = -2 p_2 \cdot p_3 \end{aligned}$

$\hat{s} + \hat{t} + \hat{u} = M^2$

1) We shall do the γ -matrices in $N = 4 - 2\epsilon$ dimensions,

$\{\gamma^{\mu}, \gamma^{\nu}\} = 2 g^{\mu\nu}$, $g^{\mu\nu} = \text{diag}(1, -1, \dots, -1)$

$g^{\mu\nu} g_{\mu\nu} = N$

$\gamma^{\mu} \gamma_{\mu} = N$

$\gamma^{\mu} \not{a} \gamma_{\mu} = -2(1 - \epsilon) \not{a}$

$\gamma^{\mu} \not{a} \not{b} \gamma_{\mu} = 4 a \cdot b - 2 \epsilon \not{a} \not{b}$

$\text{Tr}(\not{a} \not{b}) = 4 a \cdot b$

$\gamma^{\mu} \not{a} \not{b} \not{c} \gamma_{\mu} = -2 \not{c} \not{b} \not{a} + 2 \epsilon \not{a} \not{b} \not{c}$

$\text{Tr}(\not{a} \not{b} \not{c} \not{d}) = 4 (a \cdot b c \cdot d + a \cdot d b \cdot c - a \cdot c b \cdot d)$

The Dirac matrices have dimension $2^{N/2}$ for N even
 $2^{(N-1)/2}$ for N odd

However we will use $\text{Tr } \mathbb{1} = 4$

Also $\{\gamma_\mu, \gamma_5\} = 0$, $\gamma_5^2 = 1$
 $\text{Tr}(\gamma_5 \text{dirac}) = 0$

2) The first trace gives $-16(1-\epsilon)^2 \hat{s} \hat{t}$
 The 2nd may be obtained from the 1st by $p_2 \leftrightarrow -p_3$, or $s \leftrightarrow t$
 Hence, it gives $-16(1-\epsilon)^2 \hat{t} \hat{s}$
 The 3rd trace gives $16(1-\epsilon) (-\hat{u} M^2 + \epsilon \hat{s} \hat{t})$

3) Therefore

$$|\mathcal{M}|^2 = 2g^2 g_s^2 (\bar{u})^\epsilon \text{Tr}(T^A T^A) (1-\epsilon) \left[(1-\epsilon) \left(\frac{\hat{s}}{-\hat{t}} + \frac{-\hat{t}}{\hat{s}} \right) - 2 \frac{\hat{u} M^2}{\hat{s} \hat{t}} + 2\epsilon \right]$$

Now $\text{Tr}(T^A T^A) = C_A C_F = 3 \times \frac{4}{3} = 4$ (in QCD)

4) There are $(N-2)$ transverse spatial dimensions in N dimensions,
 so the gluon has $(N-2) = 2(1-\epsilon)$ spin components.
 The quark has only 2 spin components, since we have
 chosen to use $\text{Tr}(\mathbb{1}) = 4$ in calculating Dirac matrices.
 Thus the spin- and color-averaged

$$|\overline{\mathcal{M}}|^2 = \left[\frac{1}{2(1-\epsilon)}, \frac{1}{2} \right] \left[\frac{1}{8}, \frac{1}{3}, C_F \right] |\mathcal{M}|^2$$

$$= \frac{1}{4} \frac{1}{6} 2g^2 g_s^2 (\bar{u})^\epsilon \left[(1-\epsilon) \left(\frac{\hat{s}}{-\hat{t}} + \frac{-\hat{t}}{\hat{s}} \right) - 2 \frac{\hat{u} M^2}{\hat{s} \hat{t}} + 2\epsilon \right]$$

↓ ↓ ↓ ↓ ↓

Note If $f(x)$ depends only on $r = \sqrt{x_1^2 + \dots + x_n^2}$, then

$$\int d^n x f(x) = \int_0^\infty f(r) \cdot r^{n-1} dr \underbrace{\int_0^\pi \sin^{n-2} \theta d\theta \int_0^\pi \dots \int_0^\pi \sin \theta_2 d\theta_2 \int_0^{2\pi} d\theta_1}_{d\Omega}$$

Using $\int_0^\pi \sin^m \theta d\theta = \sqrt{\pi} \frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m+2}{2})}$, $d\Omega$

get $\int d^n \Omega = \sqrt{\pi} \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})} \sqrt{\pi} \frac{\Gamma(\frac{n-2}{2})}{\Gamma(\frac{n-1}{2})} \dots \sqrt{\pi} \frac{\Gamma(\frac{2}{2})}{\Gamma(\frac{3}{2})} \cdot (2\pi)$

$$= (\sqrt{\pi})^{n-2} \cdot \frac{\Gamma(1)}{\Gamma(\frac{n}{2})} \cdot (2\pi)$$

$$= \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$$

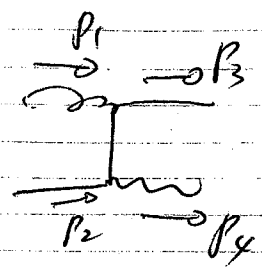
$$\int d^N p \delta^+(p^2) = \int d p_0 \delta^+(p_0^2 - |p|^2) \int d^{N-1} p$$

$$= \frac{1}{2p_0} \int_0^\infty d|p| \cdot |p|^{N-1} \cdot \int_0^\pi \sin^{N-2} \theta d\theta \int_0^\pi \dots \int_0^\pi \sin \theta_3 d\theta_3 \dots \int d\theta_1$$

($\because p_0 = |p|$) $\int d^{N-2} \Omega$

$$= \frac{1}{2} \int_0^\infty d|p| \cdot |p|^{N-3} \int_0^\pi \sin^{N-3} \theta d\theta \cdot \left[\int d^{N-2} \Omega \right]$$

5) N -dimensional phase space gives



$$P.S. = \int \frac{d^{N-1} p_3}{(2\pi)^{N-1} 2E_3} \frac{d^{N-1} p_4}{(2\pi)^{N-1} 2E_4} (2\pi)^N \delta^{(N)}(p_1 + p_2 - p_3 - p_4)$$

$$= \int \frac{d^N p_3}{(2\pi)^{N-1}} \int \frac{d^N p_4}{(2\pi)^{N-1}} (2\pi)^N \delta^{(N)}(p_1 + p_2 - p_3 - p_4) \delta^+(p_3^2) \delta^+(p_4^2 - M^2)$$

with $\delta^+(p_4^2 - M^2) = \delta(p_4^2 - M^2) \theta(E_4)$

In (q, q_i) c.m.s with (q_j) moving along the $(N-1)^{th}$ direction, we may write

$$p_3 = (|p_3|, \dots, |p_3| \cos \theta)$$

where the dots indicate $(N-2)$ unspecified momenta.

First, we use $\delta^{(N)}$ to integrate out $\int d^N p_4$, then perform the $(N-2)$ angular integration for $\int d^N p_3$.

For brevity, let's denote $\mathcal{P} \equiv p_3$, then

$$\int d^N p = \int d|p| |p|^{N-1} \int_{\theta=0}^{\pi} \sin^{N-2} \theta d\theta \int_{\phi=0}^{2\pi} d\phi \int \dots \int 1 \cdot d\theta_i \checkmark$$

with $0 \leq \theta_i \leq \pi$, except $0 \leq \theta_1 \leq 2\pi$.

$$\int d^N p \delta(p^2) = \int_0^\infty d|p| |p|^{N-3} \left[\int_0^\pi \sin^{N-3} \theta d\theta \right] \left[\frac{1}{2} \int d\Omega_{N-2} \right]$$

$N=4-2\epsilon$

$$= \left(\int_0^\infty d|p| |p|^{1-2\epsilon} \right) \left[\int_{-1}^1 d(\cos \theta) (1 - \cos^2 \theta)^{-\epsilon} \right] \left[\frac{1}{2} \frac{2\pi^{\frac{N-2}{2}}}{\Gamma(\frac{N-2}{2})} \right]$$

$$= \int_0^\infty d|p| |p|^{1-2\epsilon} \int_{-1}^1 d(\cos \theta) (1 - \cos^2 \theta)^{-\epsilon} \frac{\pi^{1-\epsilon}}{\Gamma(1-\epsilon)}$$

$$\int_0^\pi \sin^{N-3} \theta d\theta = - \int_0^\pi \sin^{N-4} \theta d \cos \theta = + \int_{-1}^1 (1 - \cos^2 \theta)^{\frac{N-4}{2}} d \cos \theta = \int_{-1}^1 (1 - \cos^2 \theta)^{-\epsilon} d \cos \theta$$

$$\frac{\frac{1}{4\pi} \frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)}}{(2\pi)^{2N-2} \cdot (2\pi)^n} = \frac{1}{4\pi} \frac{1}{\Gamma(1-\epsilon)} \frac{(4\pi)^{\epsilon} (2\pi)^{n-2}}{\pi^{-\epsilon} (2\pi)^{2\epsilon} (2\pi)^{2-2\epsilon}}$$

$n = 4 - 2\epsilon$
 $n - 2 = 2 - 2\epsilon$

$$= \frac{1-\epsilon}{\pi} \frac{1}{\Gamma\left(\frac{n-2}{2}\right)} = \boxed{\frac{\pi^{\frac{n-2}{2}}}{\Gamma\left(\frac{n-2}{2}\right)}}$$

Hence

$$P.S. = \frac{(2\pi)^N}{(2\pi)^{N-1} (2\pi)^{N-1}} \frac{\pi^{1-\epsilon}}{\Gamma(1-\epsilon)} \int_0^\infty d|p_3| |p_3|^{1-2\epsilon} \int_1^1 d(\cos\theta) (1-\cos^2\theta)^{-\epsilon}$$

$$M_3 \rightarrow 0 \quad \delta\left(\frac{\hat{s}}{s} - M^2 - 2\sqrt{\frac{\hat{s}}{s}} |p_3|\right)$$

$$\left(\text{Since } |p_3| = E_3 = \frac{\hat{s} - M^2}{2\sqrt{\hat{s}}}, \Rightarrow \frac{\hat{s}}{s} = 2\sqrt{\frac{\hat{s}}{s}} |p_3| + M^2 \right)$$

Furthermore,

$$\int_0^\infty d|p_3| |p_3|^{1-2\epsilon} \delta\left(\frac{\hat{s}}{s} - M^2 - 2\sqrt{\frac{\hat{s}}{s}} |p_3|\right) \Rightarrow |p_3| = \frac{\hat{s} - M^2}{2\sqrt{\hat{s}}}$$

$$= \frac{1}{2\sqrt{\frac{\hat{s}}{s}}} \left(\frac{\hat{s} - M^2}{2\sqrt{\hat{s}}}\right)^{1-2\epsilon} = \frac{1}{2^{2-2\epsilon}} \left(\frac{\hat{s} - M^2}{\hat{s}}\right)^{1-2\epsilon} \left(\frac{1}{\sqrt{\hat{s}}}\right)^{2\epsilon}$$

Since

$$\frac{\pi^{1-\epsilon}}{(2\pi)^{1-2\epsilon}} = 2^{-2+2\epsilon} \pi^{-1+\epsilon} = (4\pi)^{-1+\epsilon}$$

So

$$P.S. = \left[\frac{1}{4\pi} (4\pi)^{\epsilon} \frac{1}{\Gamma(1-\epsilon)} \right] \left[\frac{1}{4^{1-\epsilon}} \left(1 - \frac{M^2}{\hat{s}}\right)^{1-2\epsilon} \left(\frac{1}{\hat{s}}\right)^{\epsilon} \right]$$

$$\cdot \left[\int_{-1}^1 d(\cos\theta) (1-\cos^2\theta)^{-\epsilon} \right]$$

$$\text{Let } y = \frac{1}{2}(1 + \cos\theta), \text{ then } dy = \frac{1}{2} d(\cos\theta)$$

$$\cos\theta = 2y - 1$$

$$1 - \cos^2\theta = 1 - (2y-1)^2 = (1-(2y-1))(1+(2y-1)) = 2y(2-2y) = 4y(1-y)$$

then

$$\left[\int_{-1}^1 d(\cos\theta) (1-\cos^2\theta)^{-\epsilon} \right] = \left[2 \int_0^1 dy \cdot (4)^{-\epsilon} \cdot (y(1-y))^{-\epsilon} \right]$$

Therefore

$$P.S. = \left[\frac{1}{4\pi} (4\pi)^{\epsilon} \frac{1}{\Gamma(1-\epsilon)} \right] \left[\frac{1}{4} \frac{1}{(4)^{\epsilon}} \left(1 - \frac{M^2}{\hat{s}}\right)^{1-2\epsilon} \left(\frac{1}{\hat{s}}\right)^{\epsilon} \right]$$

$$\cdot \left[2 \int_0^1 dy (y(1-y))^{-\epsilon} \right]$$

$$= \frac{1}{8\pi} \left(\frac{4\pi}{\hat{s}}\right)^{\epsilon} \frac{1}{\Gamma(1-\epsilon)} \left(1 - \frac{M^2}{\hat{s}}\right)^{1-2\epsilon} \int_0^1 dy (y(1-y))^{-\epsilon}$$

$$6) \hat{t} = -2 p_1 \cdot \beta_3 = -2 \frac{\sqrt{s}}{2} \frac{\sqrt{s}}{2} \left(1 - \frac{M^2}{s}\right) (1 - \cos\theta)$$

$$= -\frac{\hat{s}}{s} \left(1 - \frac{M^2}{s}\right) (1 - y)$$

$$u = -2 p_1 \cdot \beta_3 = -\frac{\hat{s}}{s} \left(1 - \frac{M^2}{s}\right) y$$

(since $\hat{s} + \hat{t} + \hat{u} = M^2$)

7) Therefore, the cross section for $g q_i \rightarrow W^+ q_j$ is

$$\hat{\sigma} = \frac{1}{2s} \frac{1}{4} \frac{1}{6} \frac{1}{2} \frac{1}{s} \frac{1}{s} (P.S.) \rightarrow \text{where?}$$

$$= \left(\frac{1}{s}\right)^2 \left[\frac{1}{4} \cdot \frac{1}{6} \cdot 2 g^2 g^2 (M^2)^\epsilon \right] \left[\frac{1}{8\pi} \frac{1}{P(1-\epsilon)} \left(\frac{4\pi}{M^2}\right)^\epsilon \left(\frac{M^2}{s}\right)^\epsilon \left(1 - \frac{M^2}{s}\right)^{1-2\epsilon} \right]$$

$$\cdot \left\{ \int_0^1 dy y^{-\epsilon} (1-y)^{-\epsilon} \left[(1-\epsilon) \left(\frac{1}{\left(1 - \frac{M^2}{s}\right)(1-y)} + \left(1 - \frac{M^2}{s}\right)(1-y) \right) - 2 \left(\frac{M^2}{s} \right) \frac{y}{1-y} + 2\epsilon \right] \right\}$$

Note: The collinear singularity resides in the y integral; in the limit of $\epsilon \rightarrow 0$.

Use $\int_0^1 dy y^\alpha (1-y)^\beta = \frac{\Gamma(1+\alpha) \Gamma(1+\beta)}{\Gamma(2+\alpha+\beta)}$

drop it

$$\Gamma(-\epsilon) \sim \frac{-1}{\epsilon}$$

$$\Gamma(x+1) = x \Gamma(x), \quad \Gamma(n) = (n-1)! \quad n = \text{integer} \geq 1$$

The above y integral gives

$$\left\{ \right\} = \frac{-1}{\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left(\frac{1-\epsilon}{1-\frac{\hat{s}}{s}} - 2 \frac{\hat{s}}{s} (1+\epsilon) \right) + \frac{1}{2} \left(1 - \frac{\hat{s}}{s}\right)$$

$$\frac{\hat{s}}{s} = \frac{M^2}{s}$$

$$\text{Use } \frac{-1}{\epsilon} \left(\frac{4\pi M^2}{M^2}\right)^\epsilon \frac{\hat{s}^\epsilon}{s} (1-\frac{\hat{s}}{s})^{-2\epsilon} = \frac{-1}{\epsilon} \left[1 + \epsilon \left(\ln \frac{4\pi M^2}{M^2} \frac{\hat{s}}{s} \frac{1}{\left(1-\frac{\hat{s}}{s}\right)^2} + \dots \right) \right]$$

$$= \frac{-1}{\epsilon} + \left(\ln \frac{M^2}{4\pi M^2} \frac{\left(1-\frac{\hat{s}}{s}\right)^2}{\frac{\hat{s}}{s}} \right)$$

In the limit of $\epsilon \rightarrow 0$.

$$\hat{\sigma}(q_i \rightarrow W^+ q_j) = \frac{\pi}{12} \alpha_s \frac{\alpha}{\sum_W^2} \left(\frac{1}{\hat{s}} \right).$$

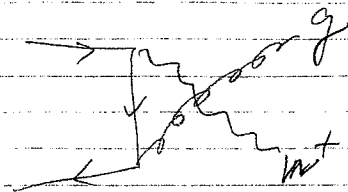
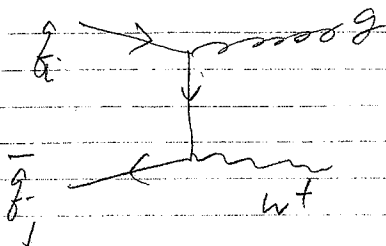
$$\cdot \left\{ \left(\frac{\hat{s}^2}{\tau} + (1 - \frac{\hat{s}}{\tau})^2 \right) \left(\frac{-1}{\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} + \ln \frac{M^2 (1-\tau)^2}{4\pi \mu^2 \tau} \right) + \frac{3}{2} \frac{\Gamma}{\tau} - \frac{3}{2} \frac{\hat{s}^2}{\tau} \right\}$$

Note ($\ln M^2$) comes from the decomposition of

$$\left(\frac{\hat{s}}{\tau} \equiv \frac{M^2}{\hat{s}} \right)$$

$$\left(\frac{4\pi M^2}{\hat{s}} \right)^\epsilon = \left(\frac{4\pi M^2}{M^2} \right)^\epsilon \left(\frac{M^2}{\hat{s}} \right)^\epsilon$$

4 $g_i \bar{g}_j \rightarrow W^+ g$



1) In the previous section, we did

$$g(p_1) + \bar{g}_i(p_2) \rightarrow \bar{g}_j(p_3) + W^+(p_4)$$

now, it is

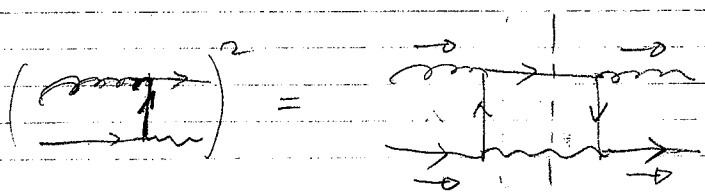
$$\bar{g}_i(p_1') + \bar{g}_j(p_2') \rightarrow g(p_3') + W^+(p_4')$$

Therefore, we can relabel the momenta

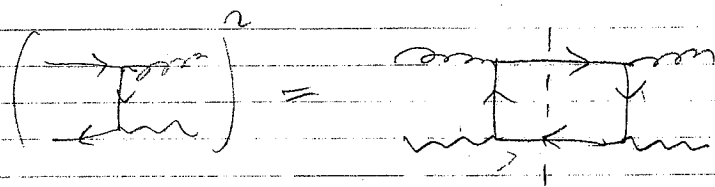
$$\left. \begin{array}{l} p_1 \rightarrow -p_3' \\ p_2 \rightarrow p_1' \\ p_3 \rightarrow -p_2' \\ p_4 \rightarrow p_4' \end{array} \right\} \Rightarrow \left. \begin{array}{l} s \rightarrow t \\ t \rightarrow u \\ u \rightarrow s \end{array} \right\}$$

to get this $18M^2$ from previous one.

But, there is an overall minus sign should be included, because



$$z^{2\epsilon} (1-z)^{-1-2\epsilon}$$



(This fermion loop gives a relative minus sign to the above one.)

2) Use the relation

$$z \equiv \frac{M^2}{s} \Rightarrow \hat{z}$$

$$\int_0^1 z^\epsilon (1-z)^{-1-2\epsilon} = \frac{1}{\epsilon} \delta(1-z) + \frac{1}{(1-z)_+} - \epsilon \left(\frac{\ln(1-z)}{1-z} + \frac{\ln z}{-1-z} \right) + \mathcal{O}(\epsilon^2)$$

$$\int_0^1 z^\epsilon (1-z)^{-\epsilon} = 1 + \epsilon \ln \frac{z}{1-z} + \mathcal{O}(\epsilon^2)$$

with $\int_0^1 dz \left(g(z)_+ h(z) \right) \equiv \int_0^1 dz g(z) (h(z) - h(1))$, which is finite at $z=1$.

get

$$\sigma(\bar{q}q \rightarrow W^+g) = \frac{\pi^2 \alpha_s^2}{3} \frac{4}{2\pi} \left(\frac{4\pi M^2}{M^2} \right)^\epsilon \frac{P(1-\epsilon)}{P(1-2\epsilon)} \frac{1}{5}$$

$$\left\{ \frac{2}{\epsilon^2} \delta(1-\hat{z}) - \frac{2}{\epsilon} \frac{1+\hat{z}^2}{(1-\hat{z})_+} + 4(1+\hat{z}^2) \left(\frac{\ln(1-\hat{z})}{1-\hat{z}} \right)_+ - 2 \frac{1+\hat{z}^2}{1-\hat{z}} \ln \hat{z} \right\}$$

Note: in this case, the spin average factor is

$$\left(\frac{1}{2} \cdot \frac{1}{2} \right),$$

which should be compared with the $(\bar{q}q \rightarrow Wg)$ case, there it is

$$\left(\frac{1}{2} \cdot \frac{1}{N-2} \right) = \left(\frac{1}{2} \frac{1}{2(1-\epsilon)} \right), \quad (N=4-2\epsilon)$$

Also, the color factor is

$$\left(\frac{1}{3} \cdot \frac{1}{3} \right), \quad \text{instead of } \left(\frac{1}{3} \cdot \frac{1}{8} \right) \text{ as in } (\bar{q}q \rightarrow W^+g)$$

$$\frac{-2}{\epsilon} \frac{1+\hat{z}^2}{(1-\hat{z})_+} = \frac{-2}{\epsilon} \left[\left(\frac{1+\hat{z}^2}{1-\hat{z}} \right)_+ - \frac{3}{2} \delta(1-\hat{z}) \right]$$

$$= \underbrace{\frac{-2}{\epsilon} \left[\left(\frac{1+\hat{z}^2}{1-\hat{z}} \right)_+ \right]}_{\text{collinear}} + \underbrace{\frac{3}{\epsilon} \delta(1-\hat{z})}_{\text{soft}}$$

$$\frac{1+\hat{z}^2}{(1-\hat{z})_+} = \left(\frac{1+\hat{z}^2}{1-\hat{z}} \right)_+ - \frac{3}{2} \delta(1-\hat{z})$$

$$\frac{1+\hat{z}^2}{(1-\hat{z})_+} = \left(\frac{1+\hat{z}^2}{1-\hat{z}} \right)_+ + \frac{3}{2} \delta(1-\hat{z})$$

$\tau = \frac{M^2}{s}$
 $\hat{z} = \frac{1-\tau}{2}$
 $\hat{z} \rightarrow 0$
 $\hat{z} = \frac{1-\tau}{2}$
~~soft~~
 this is soft