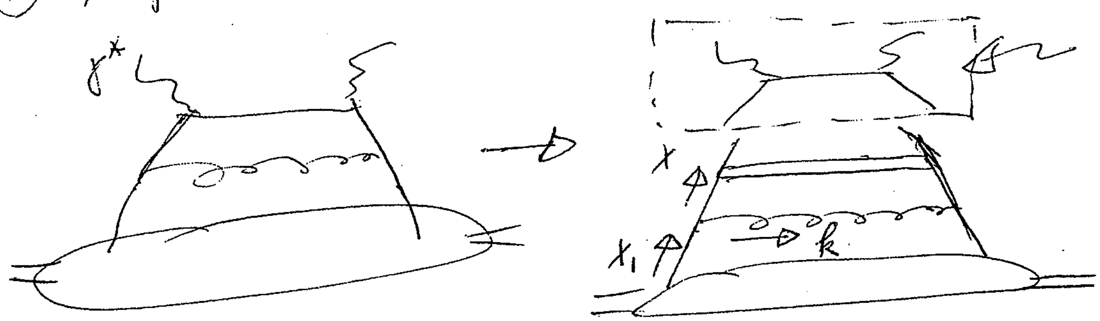
Parton distribution function (PDF)

At the scale Q^2, the number of quarks seen by the detector (virtual photon) with momentum fraction x to $x+dx$ is $q_v(x,Q^2)\,dx$.
(valence quark)

When the scale changes from Q^2 to $Q^2+\Delta Q^2$, the number of quarks seen by the detector changes to $q_v(x,Q^2+\Delta Q^2)\,dx$.

The difference comes from two sources:

(1) A positive contribution (in axial gauge)

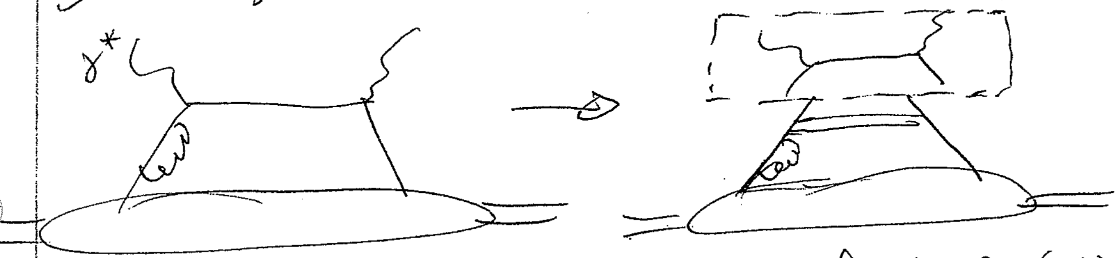


Wilson coeff

$x_1 > x$

$$\Rightarrow \frac{d\,q_v(x,Q^2)}{d\ln Q^2} = \frac{\alpha_s}{2\pi}\int_x^1 dx_1\,\frac{1}{x_1}\,C_F\left(\frac{1+\left(\frac{x}{x_1}\right)^2}{1-\left(\frac{x}{x_1}\right)}\right) q_v(x_1,Q^2)$$

(2) A negative contribution (in axial gauge),



$$\Rightarrow \frac{d\,q_v(x,Q^2)}{d\ln Q^2} = \frac{-\alpha_s}{2\pi}\int dz\,C_F\left(\frac{1+z^2}{1-z}\right) q_v(x,Q^2)$$

22-141 50 SHEETS
22-142 100 SHEETS
22-144 200 SHEETS
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The combination of ① and ② gives

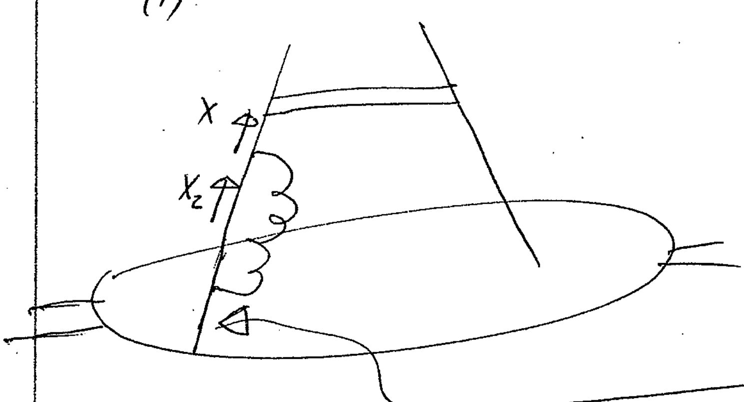
$$\frac{d q_v(x,Q^2)}{d\ln Q^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z}\, \underbrace{C_F\left(\frac{1+z^2}{1-z}\right)_+}_{\text{This is the splitting kernal of } P_{q\to q}(z)} q_v\left(\frac{x}{z}, Q^2\right)$$

$\left(C_F = \frac{4}{3} \text{ in QCD.}\right)$

where we define

$$\int_x^1 dz \left(\frac{f(z)}{1-z}\right)_+ h(z) \equiv \int_0^1 dz\, \theta(z-x)\frac{f(z)}{1-z}\, h(z) - \int_0^1 dz\, f(z)\cdot h(1)$$

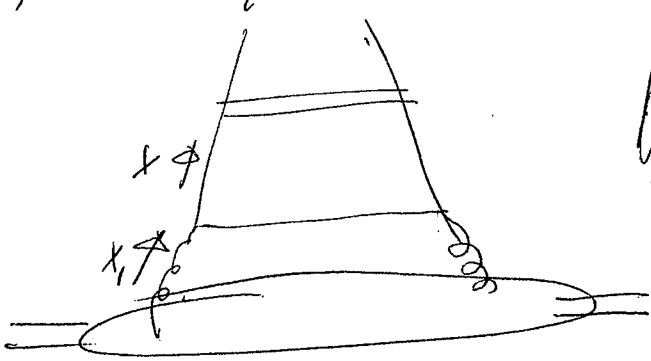
<u>Note</u> (i) The negative contribution in ② can be seen as



$x_z < x$

It comes from the <u>parton</u> after gluon radiation.

(ii) For sea-quark, we have to add the contribution from



$$P_{g\to q}(z) = \frac{z^2+(1-z)^2}{2}$$

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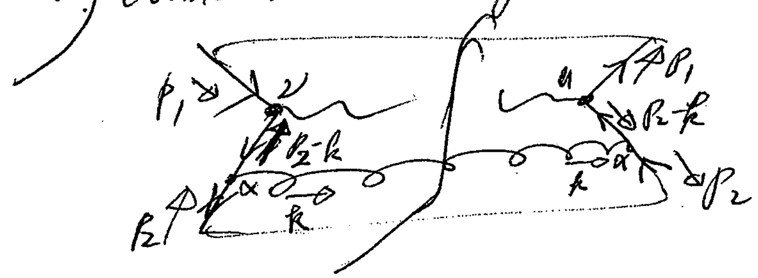
How to understand the "cut vertex"?

1) Consider the diagram



$$(A) = -\mathrm{Tr}\left\{ \not p_1 \Gamma^\mu \underbrace{\frac{(\not k - \not p_2)}{(-2p_2\cdot k)} \gamma_\alpha \not p_2 \gamma^\alpha \frac{(\not k-\not p_2)}{(-2p_2\cdot k)}} \Gamma^\nu \right\}$$

with

$|k^+| \gg |k^-|, |k_T|$, $\qquad p_2^\mu = (p_2^+, 0, \underline{0})$

$(k \parallel p_2)$

2) We want to rewrite (A) as

$$\mathrm{Tr}\{\not p_1 \Gamma^\mu \underbrace{\not p_2}_{(p_2^+\gamma^-)} \Gamma^\nu\} \cdot \mathrm{Tr}\underbrace{\{\quad ? \quad\}}_{(\gamma^+)\left(\frac{k^+\gamma^-}{p_2^+}\right)}$$

Find this!

Recall that

$$(\not k-\not p_2)(-2\not p_2)(\not k - \not p_2) = (-2)\not k\not p_2\not k = -2\not k(-\not k\not p_2 + 2(p_2\cdot k))$$
$$= 2\not k(-2p_2\cdot k)$$

$\not k\not k = k^2 = 0$
$\not p_2\not p_2 = p_2^2 = 0$

Since $\not k = \frac{k^+}{p_2^+} p_2^+\gamma^- = \left(\frac{p_1\cdot k}{p_1\cdot p_2}\right)\not p_2$,

so,

$$(A) = -\mathrm{Tr}\left\{\not p_1 \Gamma^\mu \left(\frac{2\, p_1\cdot k}{p_1\cdot p_2}\right)\frac{\not p_2}{(-2p_2\cdot k)}\Gamma^\nu\right\}$$

$$= \mathrm{Tr}\{\not p_1 \Gamma^\mu \not p_2 \Gamma^\nu\}\left(\frac{p_1\cdot k}{(p_1\cdot p_2)(p_2\cdot k)}\right)$$

22-141 50 SHEETS
22-142 100 SHEETS
22-144 200 SHEETS
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3) Observing that

$$\mathrm{Tr}\{\gamma^+ \not{k}\} = \mathrm{Tr}\{\gamma^+ k^+ \gamma^-\} = k^+ \,\mathrm{Tr}\{\gamma^+\gamma^-\},$$

and $\mathrm{Tr}\{\gamma^\mu \gamma^\nu\} = 4 g^{\mu\nu}$,

$$\mathrm{Tr}\{\gamma^+\gamma^-\} = \frac{1}{2}\left(\mathrm{Tr}\{\gamma^0\gamma^0\} - \mathrm{Tr}\{\gamma^3\gamma^3\}\right) = \frac{1}{2}(4)(1-(-1)) = 4$$

$$\Rightarrow -\frac{1}{p_2^+}\mathrm{Tr}\left\{\gamma^+ \frac{(\not{k}-\not{p}_2)}{(-2p_2\cdot k)}\gamma_\alpha \not{p}_2 \gamma^\alpha \frac{(\not{k}-\not{p}_2)}{(-2p_2\cdot k)}\right\}$$

$$= -\frac{1}{p_2^+}\mathrm{Tr}\left\{\gamma^+ \frac{2\not{k}(-2p_2\cdot k)}{(-2p_2\cdot k)(-2p_2\cdot k)}\right\}$$

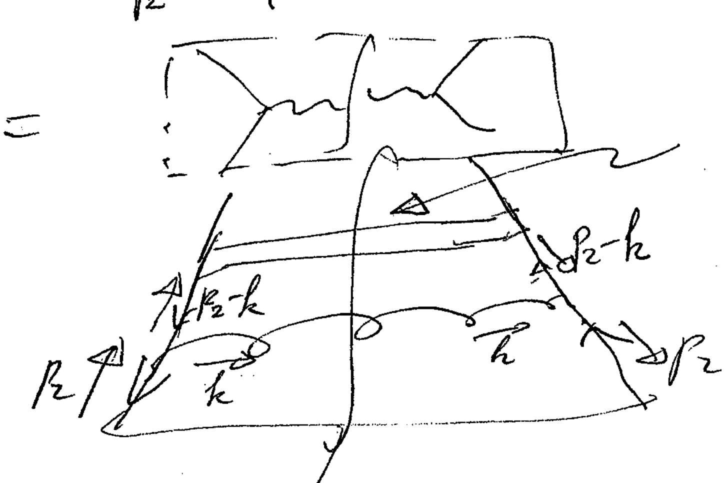
$$= +\frac{1}{p_2^+}\frac{1}{(p_2\cdot k)}\mathrm{Tr}\{\gamma^+\not{k}\}$$

$$= +\frac{1}{p_2^+}\frac{k^+}{(p_2\cdot k)}\mathrm{Tr}\{\gamma^+\gamma^-\} = \frac{4k^+}{(p_2\cdot k)p_2^+} = \frac{4(p_1\cdot k)}{(p_2\cdot k)(p_1\cdot p_2)}$$

Hence,

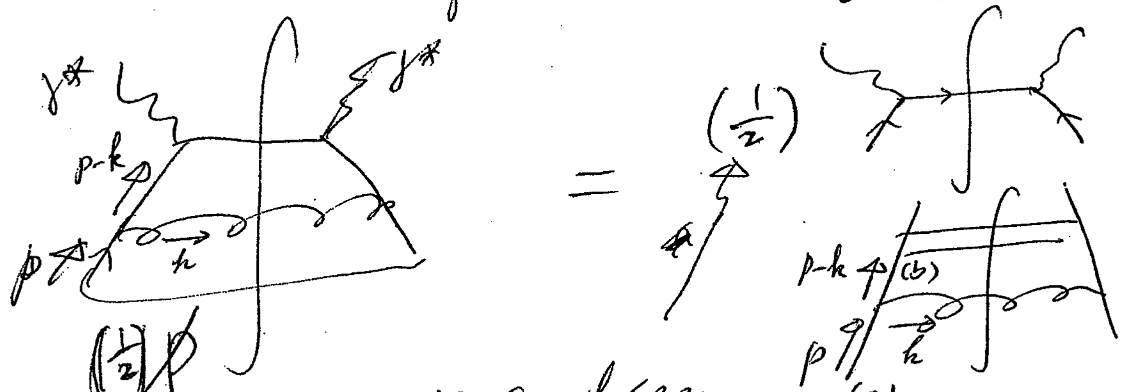
$$A = \mathrm{Tr}\{\not{p}_1 \bar{\Gamma}^\mu \not{p}_2 \Gamma^\nu\}\cdot$$

$$\frac{1}{4p_2^+}\mathrm{Tr}\left\{\gamma^+ \frac{(\not{k}-\not{p}_2)}{(-2p_2\cdot k)}\gamma_\alpha \not{p}_2 \gamma_\beta \frac{(\not{k}-\not{p}_2)}{(-2p_2\cdot k)}\right\}\cdot(-g^{\alpha\beta})$$



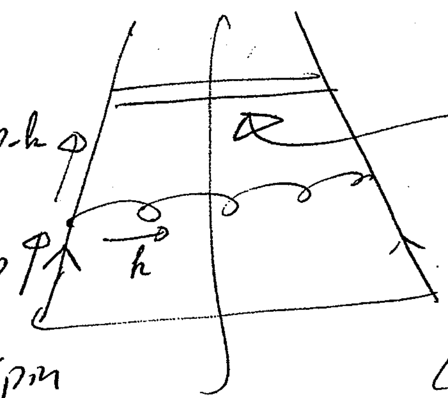
$$= \quad \frac{\gamma^+}{4p_2^+}\underbrace{\delta\left(z - \frac{(p_2^+-k^+)}{p_2^+}\right)}_{\text{indicating the collinear approximation.}}$$

4) Consider DIS (deep inelastic scattering) process



average over quark spin

Hence, the probability of finding quark (b) carrying the momentum fraction $x \equiv \left(\frac{p^+ - k^+}{p^+}\right)$ inside the quark (a) (after averaging over the spin of quark (a)) is

$\left(\frac{1}{2}\right) \cdot$  $\frac{\gamma^+}{2p^+}\,\delta\!\left(x - \frac{p^+ - k^+}{p^+}\right)$

average over quark (a) spin

$p^\mu \equiv (p^+, 0, \underline{0})$

This is the definition of "cut vertex" for defining parton distribution function (PDF).

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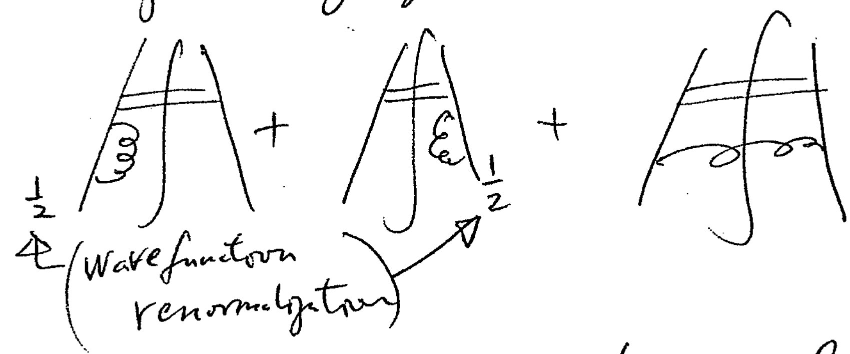
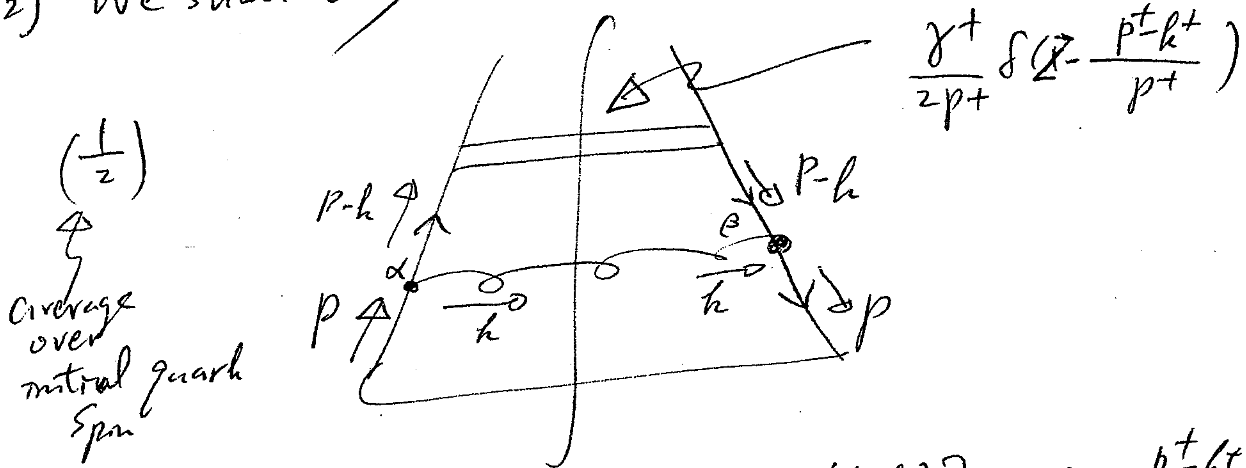
5) Calculate parton splitting kernel from the definition of PDF.

(1) In light-cone gauge, it includes



(2) We shall only calculate the real emission diagram



$$(B) \equiv \mathrm{Tr}\left\{\left(\frac{\gamma^+}{2p^+}\right)\cdot\frac{(\not p-\not k)}{(p-k)^2}\cdot\gamma^\alpha\left(\frac{\not p}{2}\right)\gamma^\beta\frac{(\not p-\not k)}{(p-k)^2}\right\}\cdot\delta\left(x-\frac{p^+-k^+}{p^+}\right)$$

$$\cdot\left(-g_{\alpha\beta}+\frac{n_\alpha k_\beta+k_\alpha n_\beta}{(n\cdot k)}\right)\cdot\left(\text{color factor}\right)\left(g_s^2\right),$$

where we choose $n^\mu=(0,1,\underline{0})$ for $p^\mu=(1,0,\underline{0})$, and $k^\mu \parallel p^\mu$.

The (color factor) is $\frac{1}{3}\cdot\left(\frac{8}{2}\right)=\frac{4}{3}=C_F$, and

~~the~~ $g_s^2=4\pi\alpha_s$.

(3) The cut-line through the gluon line gives the phase space factor

$$\int \frac{d^n k}{(2\pi)^n}(2\pi)\delta^+(k^2) = \int \frac{dk^+ dk^- d^{n-2}k_T}{(2\pi)^n}(2\pi)\delta^+(2k^+k^- - k_T^2)$$

Using the $\delta(2k^+k^- - k_T^2)$ function ~~to~~ integrate out $\int dk^-$, we have

$$\int \frac{dk^+ d^{n-2}k_T}{(2\pi)^{n-1}}\cdot\left(\frac{1}{2k^+}\right), \quad \left(\delta^+(2k^+k^- - k_T^2) = \frac{1}{2k^+}\delta\left(k^- - \frac{k_T^2}{2k^+}\right)\right)$$

where $\int d^{n-2}k_T = \int d\Omega \int k_T^{(n-2-1)} dk_T$

$$= \frac{2\pi^{\left(\frac{n-2}{2}\right)}}{\Gamma\left(\frac{n-2}{2}\right)}\int \frac{dk_T^2}{2}\left(k_T^2\right)^{\frac{n-4}{2}}$$

Or,

$$\int \frac{d^n k}{(2\pi)^n}(2\pi)\delta^+(k^2) = \int \frac{dk^+}{2k^+(2\pi)^{n-1}}\,\frac{\pi^{\left(\frac{n-2}{2}\right)}}{\Gamma\left(\frac{n-2}{2}\right)}\cdot\int dk_T^2\cdot\left(k_T^2\right)^{\frac{n-4}{2}}$$

(4) A straight-forward calculation shows, without including $(C_F \cdot g_s^2)$,

$$B = (2)\cdot\left\{\frac{2(p\cdot n)}{(p\cdot k)(n\cdot k)}\left(1 - \frac{n\cdot k}{(p\cdot n)}\right) + \frac{n\cdot k}{(p\cdot k)(p\cdot n)}\right\}$$

Using the definition of z from $\delta\left(z - \frac{p^+ - k^+}{p^+}\right)$, we obtain $k^+ = (1-z)p^+$

$$dk^+ = -p^+ dz$$

$$\left(\begin{array}{l} p^+ - k^+ = z p^+ \\ p^+ \end{array} \longrightarrow k^+ = (1-z)p^+\right)$$

Hence,

$$B = (2)\left\{ \frac{2p^+}{k^+}\left(1-\frac{k^+}{p^+}\right) + \frac{k^+}{p^+} \right\}\frac{1}{(p\cdot k)}$$

$$= (2)\left\{ \frac{2}{1-z}\left(1-(1-z)\right) + (1-z) \right\}\frac{1}{(p\cdot k)}$$

$$= (2)\left(\frac{1+z^2}{1-z}\right)\frac{1}{(p\cdot k)}$$

where

$$\frac{1}{p\cdot k} = \frac{1}{p^+k^-} = \frac{1}{p^+\left(\frac{k_T^2}{2k^+}\right)} = \frac{z(1-z)}{k_T^2}$$

Let $n = 4-2\epsilon$, then

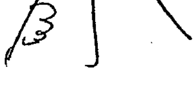
$$\frac{\alpha_s}{2\pi}\int\frac{dk^+}{2k^+}\int d^2k_T\,(k_T^2)^{1-\epsilon}\cdot\left(\frac{1}{p\cdot k}\right)\cdot\left(\frac{1+z^2}{1-z}\right)C_F\cdot\delta\left(z-\frac{p^+-k^+}{p^+}\right)$$

$$=\frac{\alpha_s}{2\pi}\int\frac{dk_T^2}{(k_T^2)^{1-\epsilon}}\cdot\int\frac{dk^+(1-z)}{k^+}\left(\frac{1+z^2}{1-z}\right)\cdot C_F\cdot\delta\left(z-\frac{p^+-k^+}{p^+}\right)$$

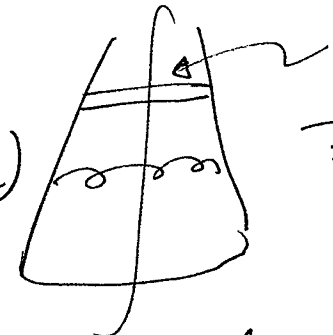
$$=\frac{\alpha_s}{2\pi}\int\frac{dk_T^2}{(k_T^2)^{1-\epsilon}}\cdot\int dz\left(\frac{1+z^2}{1-z}\right)C_F\cdot\delta\left(z-\frac{p^+-k^+}{p^+}\right)$$

$$=\frac{\alpha_s}{2\pi}\int\frac{dk_T^2}{(k_T^2)^{1-\epsilon}}\underbrace{\left(\frac{1+z^2}{1-z}\right)C_F}_{\text{splitting kernal}\ \left(\text{from real diagram only}\right)}.$$

Note ① In 't Hooft-Feynman gauge, B is obtained from summing

② Adding virtual diagram, , will change $\left(\frac{1+z^2}{1-z}\right)$ to $\left(\frac{1+z^2}{1-z}\right)_+$.

We have shown that

$\left(\frac{1}{2}\right)$  $\dfrac{\gamma^+}{2p^+}\,\delta\!\left(z-\dfrac{p^+-k^+}{p^+}\right)$

$$= \int \frac{d^n k}{(2\pi)^n}(2\pi)\,\delta^+(k^2)\left\{\frac{1}{(p\cdot k)}\left(\frac{1+z^2}{1-z}\right)\right\}\cdot\frac{\alpha_s(\mu_R)}{2\pi}C_F\cdot\delta\!\left(z-\frac{p^+-k^+}{p^+}\right) \quad (1)$$

where μ_R is the renormalization scale for defining $\alpha_s(\mu)$.

Recall that

$$d^n k = dk^+ dk^- d^{n-2}\underline{k}_T$$

and

$$d^{n-2}\underline{k}_T = \int d\Omega^{n-2}\int k_T^{(n-2)-1}\,dk_T$$

$$= \frac{2\pi^{\left(\frac{n-2}{2}\right)}}{\Gamma\left(\frac{n-2}{2}\right)}\int k_T^{n-3}\,dk_T \qquad dk_T^2 = 2k_T\,dk_T$$

$$= \frac{2\pi^{\left(\frac{n-2}{2}\right)}}{\Gamma\left(\frac{n-2}{2}\right)}\cdot\int\frac{dk_T^2}{2}\left(k_T^2\right)^{\frac{n-4}{2}}$$

Thus,

$$\int\frac{d^n k}{(2\pi)^n}(2\pi)\,\delta^+(k^2) = \int\frac{dk^+}{2k^+(2\pi)^{n-1}}\cdot\frac{\pi^{\left(\frac{n-2}{2}\right)}}{\Gamma\left(\frac{n-2}{2}\right)}\int dk_T^2\left(k_T^2\right)^{\frac{n-4}{2}}$$

Let's introduce the factorization scale μ_F to define the perturbative PDF so that

$$\int\frac{dk_T^2}{\left(k_T^2\right)^{\frac{4-n}{2}}} \longrightarrow \int_0^{\mu_F}\frac{dk_T^2}{\left(k_T^2\right)^{\frac{4-n}{2}}}$$

The integral for defining the perturbative PDF, ~~is~~ i.e. Eq. ①, is

$$\frac{\alpha_s(\mu_R)}{2\pi}\int \frac{dk^+}{2k^+}\int dk_T^2 \left(k_T^2\right)^{\frac{n-4}{2}} \cdot \left(\frac{1}{p\cdot k}\right)\cdot\left(\frac{1+z^2}{1-z}\right) C_F \cdot \delta\left(z-\frac{p^+-k^+}{p^+}\right)$$

$$= \frac{\alpha_s(\mu_R)}{2\pi}\cdot\int \frac{dk_T^2\,(k_T^2)^{\frac{n-4}{2}}}{(k_T^2)}\cdot\int\frac{dk^+}{k^+}\frac{(1-z)(1+z^2)}{1-z} C_F \cdot\delta\left(z-\frac{p^+-k^+}{p^+}\right)$$

$$\left(\frac{1}{p\cdot k} = \frac{1}{p^+k^-} = \frac{1}{p^+\left(\frac{k_T^2}{2k^+}\right)} = \frac{2(1-z)}{k_T^2}\right)$$

$$= \underbrace{\frac{\alpha_s(\mu_R)}{2\pi}\int_0^{\mu_F}\frac{dk_T^2\,(k_T^2)^{\frac{n-4}{2}}}{k_T^2}}\cdot\left(\frac{1+z^2}{1-z}\right) C_F .$$

$$\left(\int\delta\left(z-1+\frac{k^+}{p^+}\right)\frac{dk^+}{k^+} = \frac{p^+}{k^+} = \frac{1}{1-z}\right)$$

This integral has infrared singularity (collinear singularity) as $k_T \to 0$

Because of the infrared singularity, we should use

$$n \equiv 4 + 2\epsilon_{IR}$$

so that

$$\int_0^{\mu_F}\frac{dk_T^2}{(k_T^2)^{1-\epsilon_{IR}}} = \frac{1}{\epsilon_{IR}}\left(k_T^2\right)^{\epsilon_{IR}}\Bigg|_0^{\mu_F} = \frac{1}{\epsilon_{IR}}\left(\mu_F^2\right)^{\epsilon_{IR}}$$

$$\Rightarrow ① = \frac{\alpha_s(\mu_R)}{2\pi}\frac{1}{\epsilon_{IR}}\left(\mu_F^2\right)^{\epsilon_{IR}}\cdot\left(\frac{1+z^2}{1-z}\right) C_F .$$

Now, let's rewrite $g_s \to g_s \mu_R^{\epsilon}$, and $\epsilon \equiv -\epsilon_{IR} \equiv \frac{4-n}{2}$

The complete expression of ① should be

$$\frac{\alpha_s(\mu_R)\cdot(\mu_R^2)^{\epsilon}}{2\pi} \frac{1}{-\epsilon} (\mu_F^2)^{-\epsilon} \left(\frac{1+z^2}{1-z}\right) C_F$$

When μ_F is chosen to be equal to μ_R, then

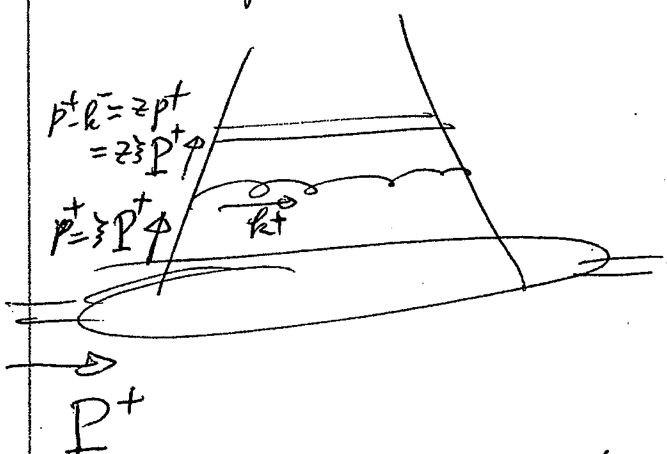
$$(\mu_R^2)^{\epsilon}\cdot(\mu_F^2)^{-\epsilon} = \left(\frac{\mu_R^2}{\mu_F^2}\right)^{\epsilon} = 1^{\epsilon} = 1,$$

and

$$① \longrightarrow \frac{\alpha_s(\mu)}{2\pi} \frac{-1}{\epsilon} \left(\frac{1+z^2}{1-z}\right) C_F \quad \text{for } \mu_R = \mu_F \equiv \mu.$$

(5) The above result is the probability for finding a quark (b) inside a quark (a).

Factorization theorem assures that the same form also applies to hadron.



In parton model, we define

$$x = \frac{p^+ - k^+}{P^+} = \frac{z p^+}{P^+} = z\xi$$

$$\Rightarrow \quad z = \frac{x}{\xi}$$

Hence, the probability of finding a quark (b) carrying the momentum fraction x inside the hadron (P^+) is

$$\frac{\alpha_s}{2\pi} \int \frac{dk_T^2}{(k_T^2)^{1-\epsilon}} \int \frac{d\xi}{\xi} \left(\frac{1 + \left(\frac{x}{\xi}\right)^2}{1 - \left(\frac{x}{\xi}\right)} \right)_+ \cdot C_F \cdot f_{(a)/P}\left(\xi\right)$$

$$\equiv f_{(b)/P}(x)$$

$$\left(\int^{Q^2} \frac{dk_T^2}{k_T^2} \to \ln Q^2 \right)$$

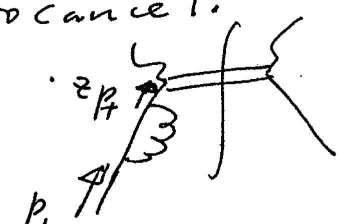
$\Rightarrow$ The evolution in scale Q is

$$\frac{d f_{b/P}(x, Q^2)}{d \ln Q^2} = \frac{\alpha_s}{2\pi} \int \frac{d\xi}{\xi} \underbrace{\left(\frac{1 + \left(\frac{x}{\xi}\right)^2}{1 - \left(\frac{x}{\xi}\right)} \right)_+ \cdot C_F}_{\text{splitting kernel}} \cdot f_{(a)/P}\left(\xi, Q^2\right)$$

From $\left(\frac{1+z^2}{1-z}\right)$ to $\left(\frac{1+z^2}{1-z}\right)_+$

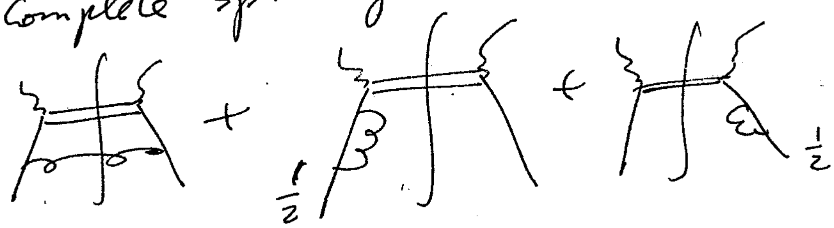
1) The function $\left(\frac{1+z^2}{1-z}\right)$ is divergent at $z=1$, which corresponds to $k^+\equiv(1-z)p^+\to 0$, i.e. soft gluon contribution.

Based on KLN theorem, all the soft-singularities have to cancel.

$\Rightarrow$ 

must contribute to $\delta(1-z)$

$\Rightarrow$ The complete splitting kernal.



must take the form

$$C_F\left(\frac{1+z^2}{1-z}\right) + a\cdot\delta(1-z)$$

a ← yet to be determined.

2) The physical interpretation of splitting functions

(1) Since QCD Lagrangian conserves
① fermion flavor
and ② momentum,

$\Rightarrow$ ①: $\displaystyle\int_0^1 dx\left[q(x,Q^2)-\bar{q}(x,Q^2)\right] = V_q$ (~~No.~~ Number of valence quark is conserved.)

Valence value for the flavor q in the target, e.g. $V_q = 2, 1, 0 \ldots$ for the $u, d, s \ldots$ flavors inside a <u>proton</u>.

②: $\displaystyle\int_0^1 dx\, x\left[\sum_{\text{flavors}}\left(q(x,Q^2)+\bar{q}(x,Q^2)\right) + G(x,Q^2)\right] = 1$

(momentum conservation)

~~Note~~

(2) By definition of the splitting kernals:

① $\Rightarrow$ $\displaystyle\int_0^1 dx\, P_{q\leftarrow q}(x) = 0$

② $\Rightarrow$ $\displaystyle\int_0^1 dx\, x\cdot\left[P_{q\leftarrow q}(x) + P_{G\leftarrow q}(x)\right] = 0$

③ $\Rightarrow$ $\displaystyle\int_0^1 dx\cdot x\left[2f\, P_{q\leftarrow G}(x) + P_{G\leftarrow G}(x)\right] = 0$

Note D ... (αs) and the zeroth order term $\phi_{i/k} = \delta_{ik}\,\delta(1-x)$

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3) Define a "+" description, so that

$$\int_0^1 dx\, \underset{\text{a test function}}{\phi(x)} \underset{\text{a distribution}}{\left[F(x)\right]_+} \equiv \int_0^1 dx \left(\phi(x) - \phi(1)\right) F(x)$$

Hence $\int_0^1 dx \left[F(x)\right]_+ = 0$

This suggests that we should have, ~~in the end~~ after summing the real and virtual contributions,

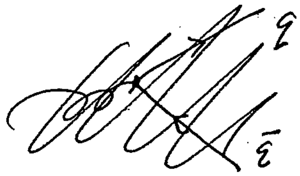
$$P^{(1)}_{q \leftarrow q}(z) = C_F \left[\frac{1+z^2}{1-z}\right]_+$$

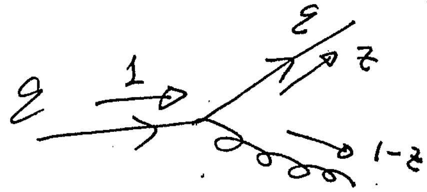
$$= C_F \left\{ \frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z) \right\}$$

finite, when integrated with a test function

Because $\int dx\, P^{(1)}_{q \leftarrow q}(x) = 0$.

4) From the momentum conservation at the QCD vertex





we can ~~easily~~ see that

$$P^{(1)}_{q\leftarrow q}(z) = P^{(1)}_{G\leftarrow q}(1-z) \qquad \text{for} \quad z<1$$

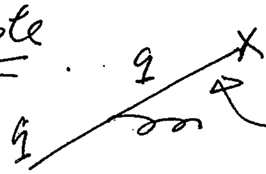
Switching momentum to get the $P_{q\leftarrow g}$

Hence
$$P^{(1)}_{G\leftarrow q}(z) = C_F \frac{1+(1-z)^2}{1-(1-z)}$$

$$= C_F \frac{1+(1-z)^2}{z}$$

(Since $P^{(1)}_{G\leftarrow q}(z)$ doesn't have a soft singularity at $z=1$, it does not ~~need~~ acquire a "+" description.)

<u>Note</u>



Can have soft & Collinear singularities

Can only have Collinear singularity
(i.e. no soft singularity)

5) For completeness, we summarize the result for a $SU(N)$ non-abelian gauge theory with each flavor of quarks in the N-dimensional (fundamental) representation (with t^a the generators).

$$\Rightarrow \quad \mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab} \qquad (a, b, c = 1, 2, 3 \cdots N^2-1)$$

$$[t^a, t^b] = i f_{abc} t^c$$

The Casimir factors

$$C_A \equiv C_2(A) \quad \leftarrow \text{adjoint representation}$$
$$= N$$

$$\delta_{ab} C_2(A) = \sum_{c,d} f_{acd} f_{bcd} \qquad (a, c, d = 1, 2, \cdots N^2-1)$$

$$C_F \equiv C_2(R_q) \quad \leftarrow \text{reducible representation containing } n_f \text{ flavor of quarks}$$
$$= \frac{N^2-1}{2N}$$

$$\delta_{ij} C_2(R_q) = \sum_a (t^a t^a)_{ij}$$

$$T \equiv T(R_q)$$
$$= \frac{n_f}{2}$$

$$\delta_{ab} T(R_q) = \sum \mathrm{Tr}(t^a t^b)$$

sum over quark flavors

(In QCD, $N=3$, and

$$C_A = 3, \quad C_F = \frac{4}{3}, \quad T = \frac{n_f}{2}$$)

In summary

$$P^{(1)}_{q\leftarrow G}(z) = \frac{T}{n_f}\left(z^2+(1-z)^2\right)$$

$$= \frac{1}{2}\left(z^2+(1-z)^2\right),$$

$$\left(T=\frac{n_f}{2}\right)$$

$$P^{(1)}_{G\leftarrow G}(z) = 2C_A\left[\frac{z}{(1-z)_+}+\frac{1-z}{z}+z(1-z)\right]+\left(\frac{11}{6}C_A-\frac{2T}{3}\right)\delta(1-z)$$

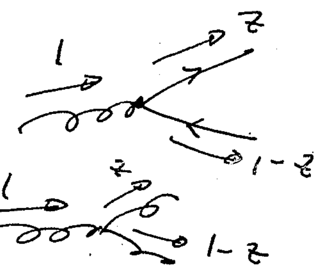
$$= 2C_A\left[\frac{z}{(1-z)_+}+\frac{1-z}{z}+z(1-z)\right]+\left(\frac{11}{6}C_A-\frac{n_f}{3}\right)\delta(1-z)$$

Note:

$$P^{(1)}_{q\leftarrow G}(z) = P^{(1)}_{q\leftarrow G}(1-z)$$

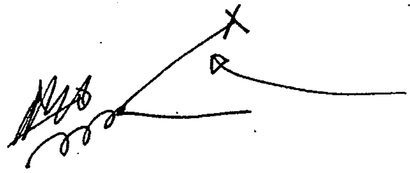
&

$$P^{(1)}_{G\leftarrow G}(z) = P^{(1)}_{G\leftarrow G}(1-z)$$

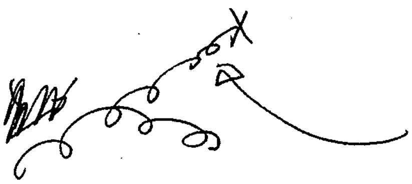


can be derived from the momentum conservation of the QCD vertex.

Note



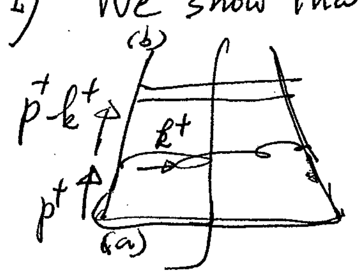
no soft singularity, cf. $P^{(1)}_{q\leftarrow G}$,
only collinear singularity



It has both
soft & collinear singularities,
(i.e. as $z\to 1$
or $z\to 0$)

From parton PDF to hadron PDF

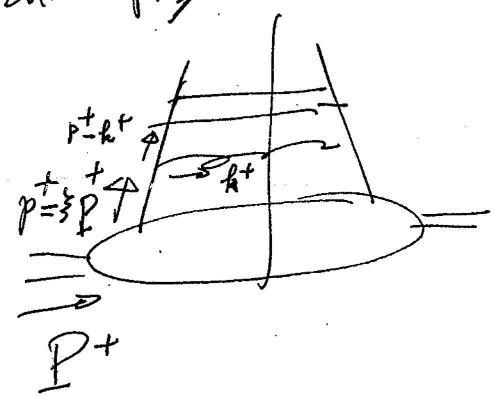
1) We show that



gives $\displaystyle \frac{\alpha_s}{2\pi}\int\frac{dk_T^2}{(k_T^2)^{1-\epsilon}}\int_0^1 dz\left(\frac{1+z^2}{1-z}\right)C_F\,\delta\!\left(z-\frac{p^+-k^+}{p^+}\right)$

which is the probability of finding a quark (b) inside a quark (a).

Factorization theorem assures that the same form also applies to hadron.



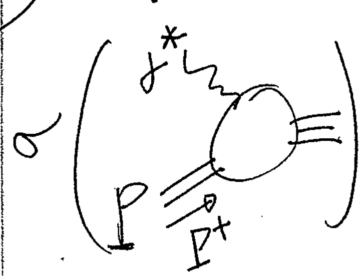
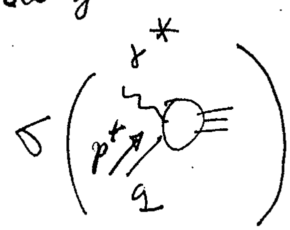
In parton model, we define

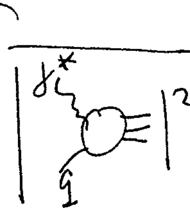
$$x \equiv \frac{p^+-k^+}{P^+} = \frac{z\,p^+}{P^+} = z\,\xi$$

where $\displaystyle \xi \equiv \frac{p^+}{P^+}$

$$\Rightarrow \quad z = \frac{x}{\xi}.$$

2) Consider the scattering, from Factorization theorem, (parton model)

$\sigma\Big($  $\Big) = \displaystyle\int d\xi\,\phi_{q/P}(\xi)\cdot\sigma\Big($  $\Big)_{p^+=\xi P^+}$

$= \displaystyle\int d\xi\,\phi_{q/P}(\xi)\,\frac{1}{2\hat{s}}\int\Big|$  $\Big|^2 d\Phi$ ← phase space

$(\hat{s}=\xi s)$ $\quad = \displaystyle\frac{1}{2S}\int\frac{d\xi}{\xi}\,\phi_{q/P}(\xi)\int\Big|$  $\Big|^2 d\Phi$

3) Therefore, the probability of finding a quark (b) ~~(b) carri~~ with fraction x of the hadron momentum (P^+) is

$$\cancel{\frac{\alpha_s}{2\pi}\int\frac{dk_T^2}{(k_T^2)^{1-\epsilon}}}\int\frac{d\xi}{\xi}\, q_{(a)/P}(\xi)\left\{\frac{\alpha_s}{2\pi}\int\frac{dk_T^2}{(k_T^2)^{1-\epsilon}}\cdot\int dz\,\delta\!\left(z-\frac{x}{\xi}\right)C_F\left(\frac{1+z^2}{1-z}\right)\right\}$$

$$=\frac{\alpha_s}{2\pi}\int\frac{dk_T^2}{(k_T^2)^{1-\epsilon}}\int\frac{d\xi}{\xi}\,C_F\left(\frac{1+\left(\frac{x}{\xi}\right)}{1-\left(\frac{x}{\xi}\right)}\right)\cdot q_{(a)/P}(\xi)$$

$$\equiv q_{(b)/P}(x)$$

$\Rightarrow$ The evolution in scale Q is $\quad\left(\int^{Q^2}\frac{dk_T^2}{k_T^2}\to\ln Q^2\right)$

$$\frac{d\,q(x,Q^2)}{d\ln Q^2}=\frac{\alpha_s}{2\pi}\int\frac{d\xi}{\xi}\underbrace{\left(\frac{1+\left(\frac{x}{\xi}\right)^2}{1-\left(\frac{x}{\xi}\right)}\right)_+}\cdot C_F\cdot q(\xi,Q^2)$$

splitting kernal (after including virtual correction)

22-141 50 SHEETS
22-142 100 SHEETS
22-144 200 SHEETS