Running Couplings in the SM

Grand Unified Theory

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Running Couplings in the SM

- Define

$$\alpha_i = \frac{g_i^2}{4\pi}, \tag{1}$$

with $i = 1, 2, 3$ for $U(1)_Y \times SU(2)_L \times SU(3)_C$.

- The one-loop renormalization group equations are

$$\frac{dg_i}{d\ln Q} = -b_i \frac{g_i^3}{(4\pi)^2}, \quad \text{or} \quad \frac{d\alpha_i}{d\ln Q} = -b_i \frac{\alpha_i^2}{2\pi}. \tag{2}$$

- In the SM,

$$b_3 = \frac{33}{3} - \frac{4}{3}n_g, \tag{3}$$

$$b_2 = \frac{22}{3} - \frac{4}{3}n_g - \frac{1}{6}n_h, \tag{4}$$

$$b_1 = -\frac{4}{3}n_g - \frac{1}{10}n_h, \tag{5}$$

where

$$n_g \;=\; \text{no. of generations} \;= 3 \quad \text{(in the SM)},$$

$$n_h \;=\; \text{no. of Higgs doublet} \;= 1 \text{ (in the SM)},$$

for $Q \gtrsim 1$ TeV.

- The solution of RG equation

$$\frac{1}{\alpha_i(Q)} = \frac{1}{\alpha_i(\Lambda)} - \frac{b_i}{2\pi} \ln\left(\frac{Q}{\Lambda}\right), \tag{6}$$

where Λ is some high energy scale.

- If $\alpha_1(\Lambda) = \alpha_2(\Lambda) = \alpha_3(\Lambda)$, *i.e.*, unification of gauge couplings, then

$$\frac{1}{\alpha_3(Q)} = (1+B)\frac{1}{\alpha_2(Q)} - B\frac{1}{\alpha_1(Q)}, \tag{7}$$

where $B = \frac{b_3 - b_2}{b_2 - b_1}$.

- The value of $\alpha_i(Q = M_Z)$ are

$$\frac{1}{\alpha_3(M_Z)} = 8.50 \pm 0.14, \quad (8)$$

$$\frac{1}{\alpha_2(M_Z)} = 29.57 \pm 0.02, \quad (9)$$

$$\frac{1}{\alpha_1(M_Z)} = 59.00 \pm 0.02. \quad (10)$$

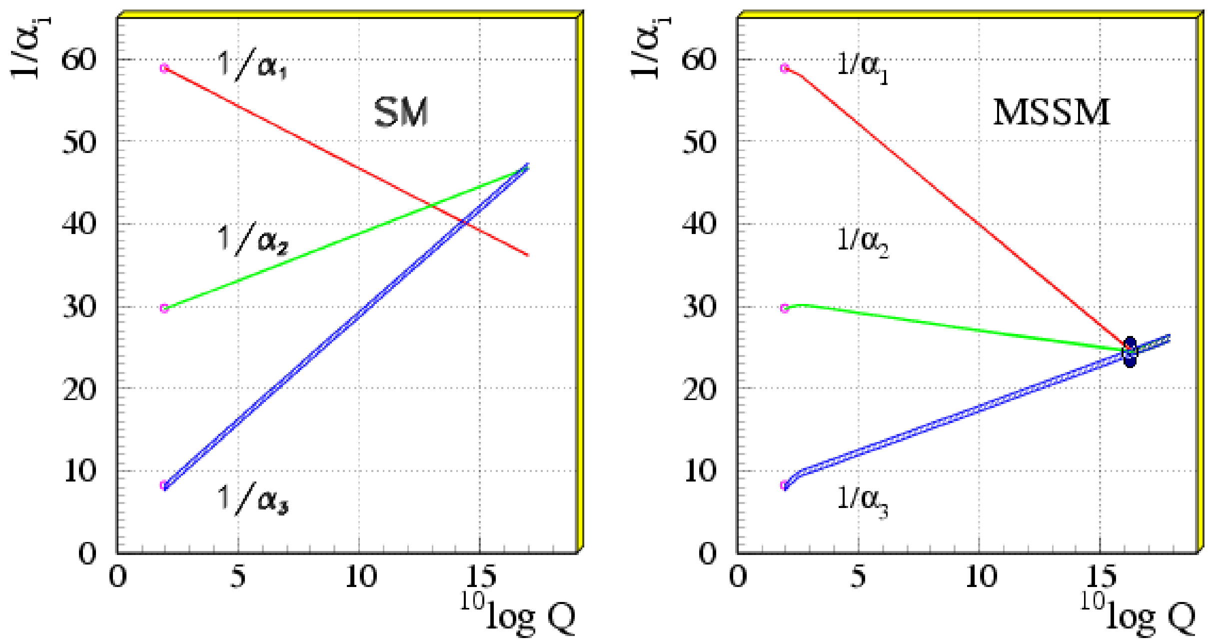
Thus,

$$B = 0.716 \pm 0.005 \pm 0.03 \quad \text{(from experimental data)}.$$

- This value does not agree with the SM prediction which gives

$$\begin{aligned} B_{SM} &\equiv \frac{b_3 - b_2}{b_2 - b_1} \\ &= \frac{\frac{33}{3} - \frac{22}{3} + \frac{1}{6}n_h}{\frac{22}{3} - \frac{2}{30}n_h} \\ &\simeq 0.53 \quad \text{for} \quad n_h = 1. \end{aligned} \quad (11)$$

To have $B = 0.716$, it needs $n_h = 6$.



- B_{SM} does not depends on the number of generators (n_g). It's because the quarks and leptons of each generation fill out multiplets of the SM or GUT gauge group.

- The SM can be embeded in the grand unified symmetry group $SU(5)$, where

$$SU(5) \supset SU(3)_C \times SU(2)_L \times U(1)_Y. \qquad (12)$$

- The SM generators (as 5×5 Hermitian matrices):

$$\begin{aligned}
SU(3)_C &: \begin{pmatrix} \lambda^a_{3\times3} & \\ & 0_{2\times2} \end{pmatrix}, \quad a = 1, 2 \cdots 8, \\
SU(2)_L &: \begin{pmatrix} 0_{3\times3} & \\ & \frac{\sigma^i}{2} \end{pmatrix}, \quad i = 1, 2, 3, \qquad (13) \\
U(1)_Y &: \sqrt{\frac{3}{5}} \begin{pmatrix} -\frac{1}{3}1_{3\times3} & \\ & \frac{1}{2}1_{2\times2} \end{pmatrix} \equiv \sqrt{\frac{3}{5}} Y,
\end{aligned}$$

where the convention of the normalization is such that $Tr(T^a T^b) = \frac{1}{2}\delta^{ab}$.

- The symmetry-breaking can be caused by the vacuum expectation value of a Higgs field in the adjoint representation of $SU(5)$,

$$\langle\Phi\rangle = v \begin{pmatrix} -\frac{1}{3}1_{3\times3} & \\ & \frac{1}{2}1_{2\times2} \end{pmatrix}, \tag{14}$$

which commutes with the SM generators.

- Since $\langle\Phi\rangle$ does not commute with the off-diagonal generators, it gives mass to those heavy gauge bosons and breaks $SU(5)$ to $SU(3)_C \times SU(2)_L \times U(1)_Y$.

- For every generation, the SM matter fields are the form of the followings:

$$\bar{5}: \begin{pmatrix} \bar{d} \\ \bar{d} \\ \bar{d} \\ e \\ \nu \end{pmatrix}_L \qquad 10: \begin{pmatrix} 0 & \bar{u} & \bar{u} & u & d \\ & 0 & \bar{u} & u & d \\ & & 0 & u & d \\ & & & 0 & \bar{e} \\ & & & & 0 \end{pmatrix}_L . \tag{15}$$

Here, there is no right-handed ν's.

- In the $SU(5)$ gauge theory,

$$g_3 = g_2 = g_1 = g_\Lambda, \tag{16}$$

where

$$g_3 = g_s, \qquad \text{strong interaction } SU(3)_C, \tag{17}$$
$$g_2 = g, \qquad \text{weak interaction } SU(2)_L, \tag{18}$$
$$g_1 = \sqrt{\frac{5}{3}} g', \qquad U(1)_Y. \tag{19}$$

Because the $SU(5)$ covariant derivative is

$$D_\mu = \partial_\mu - i g_\Lambda V_\mu^a T^a, \tag{20}$$

where g_Λ is the $SU(5)$ gauge coupling, and Λ is the unification scale.