PKU - HUST Lectures 2020

Lecture III : $\boxed{\text{Space - Time Evolution of Transitions}}$

- Comment on 'Time Delay' → end

- From time → space - time

  ⇒ $\boxed{\text{Transition Fronts}}$ (c.f. $\boxed{\text{Murray}}$) J.D. Murray

  → Fisher Equation (unstable) epidemics

  → Fitzhugh - Nagumo Equation (bistable)

  → 2 basic prototypes

→ Overview of Fronts

(a)

– Recall Predator-Prey, with fast fluctuations slaved

→ reduced model reduced to: Logistic

$(T - T_c)$ $\propto [T]$

$$\boxed{\partial_t u = (\gamma - \mu c_1)\, u - c_2 u^2}$$

(a la' TDGL)

2 non-trivial fixed-pts:

[transition as soft mode]

$\gamma < \mu c_1 \rightarrow u = 0$

$\gamma > \mu c_1 \rightarrow u = (\gamma - \mu c_1)/c_2$

But model is layer-averaged:

What if relax → spatial evolution?

⇒ i.e. viscous diffusion
viscosity

- neoclassical
- turbulent

X

$\nearrow_j$ NL

then: $\rightarrow$ $\quad u = v^2$

$$\boxed{\partial_t u - \partial_x D \partial_x u = (\gamma - u c_1) u - c_2 u^2}$$

transition evolves in space, time

$\Rightarrow$ 'bare-bones' prototype: $\quad D \equiv$ const

Pearl-Ahey Logistic + space

$$\boxed{\partial_t f - D \partial_x^2 f = \gamma f - c f^2}$$

$\rightarrow$ Fisher Eqn. $\quad$ Fisher Eqn. KPP (1937)

= Logistic + Diffusion

(N.B. Also can have Fisher – Burgers)

$\rightarrow$ useful for epidemics, 'slow' combustion, populations as well as plasmas

see W. Van Saarloos review

$\rightarrow$ basic point:

describe evolution of two domains

Fronts → $L \to$ lt...

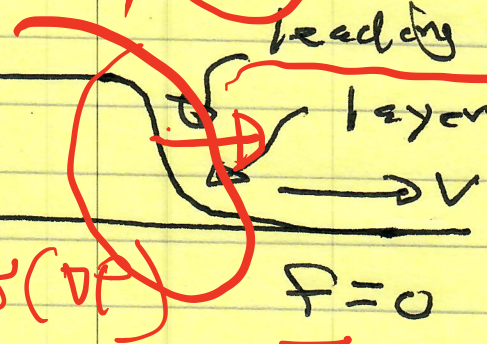
Barrier build-up γ/L

superćritical bifurcation front

X

4.

leading edge



$f = \gamma/\sigma$

$f = \gamma/\sigma$ stable

layer → thickness

$\to V$

$\gamma \to \gamma(\nabla P)$

$f = 0$

$f = 0$ unstable

i.e. advance of 1 domain into another

P.D. et. al. '75
Gürcan, P.D '04 → spreading
Garbet, et. al, '07

Murray
Von Saarloos

+ many others.

E_r

see:

E_r' E_r'

K. Ida mid 2000

L. Schmitz → APS 2020 ✱

⇒ Analysis for Fisher.

Fisher Fronts → Basic Paradigm

5.

.) Pattern Formation Paradigms I : Epidemic Front Propagation – Fisher Equation

i.) Motivation

→ basic paradigm is nonlinear dynamics: {logistic problem}

map : $x_{n+1} = a\,x_n(1-x_n)$
growth ↑ saturation ↑

continuous system : $\frac{dx}{dt} = \gamma_0 x - \alpha x^2$

Fixed pts : $\begin{cases} x_0 = 0 \\ x_0 = \gamma_0/\alpha \end{cases}$

stability: $\begin{cases} -i\omega = \gamma_0 - 2\alpha x_0 \\ \gamma = \gamma_0 - 2\alpha x_0 \end{cases}$

∴ Transition : $x_0 = 0$ (unstable) to $x_0 = \gamma_0/\alpha$ (stable) {i.e. growth to saturation of population}

→ in spatio-temporal generalization, allow diffusive dispersal of population ρ :

1D

$$\boxed{\frac{\partial \rho}{\partial t} - D\frac{\partial^2 \rho}{\partial x^2} = \gamma_0 \rho - \alpha \rho^2}$$ {Fisher Equation}

note: TDGL : $\frac{\partial \eta}{\partial t} - D\frac{\partial^2 \eta}{\partial x^2} = \underbrace{a(T)}_{T-T_c}\,\eta - b\,\eta^3$

[illegible]

- seek solutions of form:

$$\boxed{\rho = \rho(x - ct)}$$

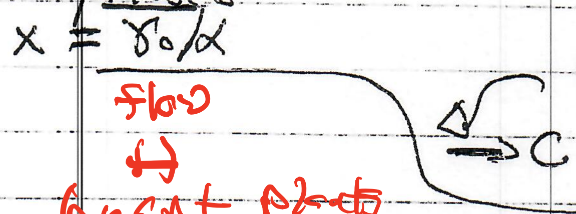
speed

i.e. [propagating solution of nonlinear equation

expect:

- propagation driven by $x_0 : 0 \to \gamma_0/\alpha$ transition instability

- solution to have form of front → domain wall separating regions of two phases



front / transition layer / domain wall

i.e. ① similar phase transitions
② combustion etc.

fires

- seek:

① → structure of solution
*② → propagation speed c → { what sets? physics of Eqn.
③ → stability of front

ii) Formulating problem:

if Fisher eqn:

$$\frac{\partial \rho}{\partial t} = k\rho(1-\rho) + D\frac{\partial^2 \rho}{\partial x^2}$$

[redacted] 7

8

$$t^* = kt \qquad \text{and omitting } * \Rightarrow$$
$$x^* = x(k/D)^{1/2}$$

$$\frac{\partial P}{\partial t} = P(1-P) + \frac{\partial^2 P}{\partial x^2}$$

$P = P(x - ct) \Rightarrow$ -1

-0

$$\left.\begin{array}{l} \boxed{P'' + cP' + P(1-P) = 0} \\ P(-\infty) = 1, \quad P(\infty) = 0 \end{array}\right\}$$

Now, can analyze via # of strategies:
convert to

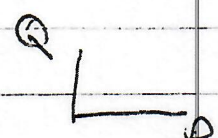
① dynamical system -2

$$\left\{\begin{array}{l} Q = P' \\ Q' = -cQ - P(1-P) \end{array}\right.$$

$\Rightarrow$

$$P' = Q$$
$$Q' = -cQ - P(1-P)$$

and



$$\left.\frac{dQ}{dP} = \frac{-cQ - P(1-P)}{Q}\right\} \text{ phase plane trajectories.}$$

Observe similarity:

$\rho(x-ct)$

- Fisher Eqn. (generalized) and 1D mechanics

$U = \gamma\rho^2/2 - c\rho$

$$-D\frac{\partial^2\rho}{\partial x^2} - c\frac{\partial\rho}{\partial x} = -\frac{\partial U(\rho)}{\partial\rho}$$

inertia — friction — force

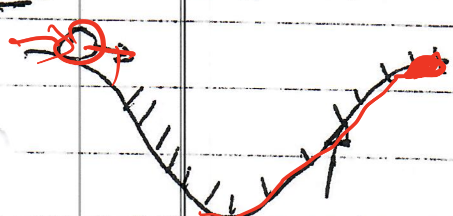
$$m\ddot{x} + \gamma\dot{x} = -\frac{\partial U(x)}{\partial x}$$

$\gamma \to \rho$

→ c to stabilize transition in moving frame

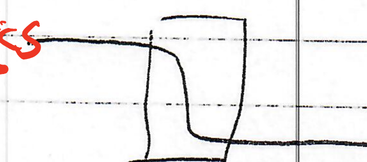
→ γ drag to balance force

i.e. ball motion



arrival here depends on i.c.'s due friction

C → dynamics



kink motion →

Front → { marginally stable in co-moving frame.

∴ Can expect:

→ sensitivity of trajectory to initial condition (i.e. push at $t=0$ to arrive at X?)

→ condition for propagation (over / under damping)

Now, trajectories have two critical points:

$$\rho = 0, \quad Q = 0$$
$$\rho = 1 \quad Q = 0 \qquad 2 \text{ F.P}$$

Can linearize about these:

$$-\gamma \tilde{\rho} = \tilde{Q}$$

$$-\gamma \tilde{Q} = -c\tilde{Q} - \tilde{\rho} + 2\rho_0 \tilde{\rho}$$

For $(0,0)$:
$$-\gamma \tilde{\rho} = \tilde{Q}$$
$$-\gamma \tilde{Q} = -c\tilde{Q} - \tilde{\rho}$$

$$0 = \begin{vmatrix} -\gamma, & -1 \\ 1, & -\gamma - c \end{vmatrix} \Rightarrow \begin{array}{l} \gamma(\gamma + c) + 1 = 0 \\ \gamma^2 + c\gamma + 1 = 0 \end{array}$$

$$\gamma = -\frac{c}{2} \pm \frac{1}{2}\left(c^2 - 4\right)^{1/2} \longrightarrow \left[\begin{array}{l} c \geq c_{min} = 2 \text{ for} \\ \text{non-negative definite} \\ \rho \text{ (avoid oscillation)} \end{array} \right.$$

population should ~~not~~ oscillate

For $(0,1)$:

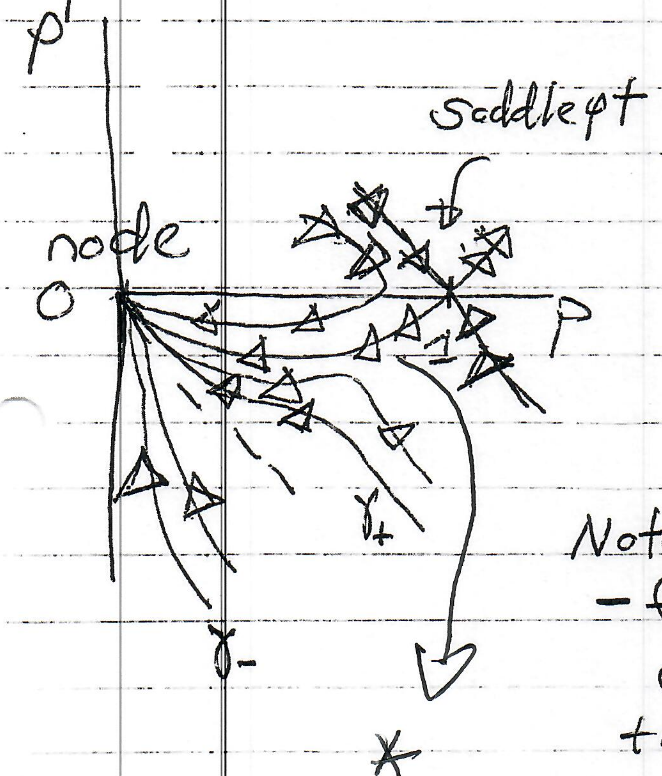
$$0 = \begin{vmatrix} -\gamma, & -1 \\ -1, & -\gamma - c \end{vmatrix} \Rightarrow \gamma(\gamma + c) - 1 = 0$$

$$\gamma = -\frac{c}{2} \pm \frac{1}{2}\left(c^2 + 4\right)^{1/2}$$

Thus, $(0,0)$: stable node for $c^2 > 4$
stable $\left\{ \begin{array}{l} \text{focus} \\ \text{spiral} \end{array} \right.$ for $c^2 < 4$

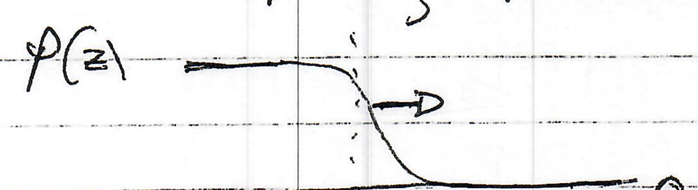
$(0,1)$: saddle point

$\Rightarrow$ phase plane trajectories:



Clearly, $\exists$ a phase space trajectory from $(1,0) \to (0,0)$ which
i.) falls in $\rho > 0$
ii.) $\rho' < 0$ (front)
for all wave speeds $c > 2$
$\Rightarrow$ front solution

Note:
- formally, travelling wave solutions exist for $c < c_{min} = 2$, but these are unphysical, as ρ oscillates ($\rho < 0$?)
- $c > c_{min}$ solution has $\rho > 0$, $\rho' < 0$ $\Rightarrow$ front



(physical solution)

$\therefore$ analysis establishes minimum speed for propagating front solution $c_{min} = 2(KD)^{1/2}$

leading edge analysis — Key

- consider edge of evolving wave propagating from $-\infty$ to $+\infty$

$\rho(x,0) \sim A e^{-\alpha x}$ $\quad x \to \infty \quad$ $e^{-\alpha x}$

- linearizing Fisher Eqn (about unstable fixed point): Reaction-Diffusion

$$\frac{\partial}{\partial t}\rho = \rho + \frac{\partial^2 \rho}{\partial x^2}$$

exponential

$\rho = A e^{-\alpha(x-ct)}$ (propagating leading edge)

$$\boxed{\alpha c = 1 + \alpha^2} \Rightarrow \alpha^2 - \alpha c + 1 = 0$$

$$\boxed{\alpha = \frac{c}{2} \pm \frac{1}{2}(c^2 - 4)^{1/2}}$$

c_{min}

$\omega(k) = 0 \quad k(\omega)$

$f(\omega, k) = 0$

$F(c, \alpha)$

∴ consistency with leading edge hypothesis structure forces $c > c_{min} = 2 \Rightarrow 2(kD)^{1/2}$

$c_{min} = 2 \to c = 2(kD)^{1/2} \quad 2(rD)^{1/2}$

Key Point:
- in fixed frame, instability occurs at each point, as ρ transitions $0 \to 1$

A
- c_{min} specifies minimal speed such that marginal

L2.

Observe:

→ $$\boxed{C_{min} = 2(kD)^{1/2}}$$

$$\Delta x = (D/k)^{1/2}$$

↓ kink width

[can sharpen kink via $D\downarrow$ or $k\uparrow$ (increase rate local instability)]

→ observe that with diffusion: Δx, c_{min} emerge from marginal stability analysis

react- diff

$$\gamma = k - k^2 D \qquad \gamma = 0 \qquad \gamma \sim \gamma P + D\nabla^2 P$$

$$\gamma = 0 \Rightarrow k \sim 1/\Delta x \sim \left(\frac{k}{D}\right)^{1/2}$$

→ $C_{min} \sim (kD)^{1/2}$ but diffusion → D/L^2

local transition instability → k

$$\frac{1}{\tau} \sim \frac{c}{L} \sim \left(\frac{kD}{L^2}\right)^{1/2}$$

$$\boxed{1/\tau_{trans}} \sim \left(\frac{D}{L^2}k\right)^{1/2}$$

→ [geometric mean of {diffusion, transition (reaction)} time scale]

γ

i.e. {propagation is synergism of local transition instability with diffusive coupling (spatially)}

↓

important physics!

Why C_{min}?

stability maintained $\left(\partial/\partial t\,(\sim\gamma) \to -c\,\frac{\partial}{\partial x} \right)$.

- leading edge analysis illustrates wave speed dependence on conditions at $x = \pm\infty$.

→ Kolmogorov

Note: KPP $\boxed{\text{proved}}$ that if:

a.) $P(x,0)$ has compact support

b.) $P(x,0) = P_0(x) > 0$

$$P_0(x) = \begin{cases} 1, & x \le x_1 \\ 0, & x \ge x_2 \end{cases} \qquad x_1 < x_2$$

c.) $P_0(x)$ continuous $x_1 < x < x_2$

(i.e. kink structure), then:

→ $\boxed{P(x,t) \text{ evolves to } P(x - C_{min}\,t)}$

C_{min} → leading edge analysis

Key issue: minimum speed is one selected

i.e. Counter-intuitive point is that pattern/front in Fisher Equation which is selected is one with minimum speed (marginal stability!?)

$c = 2(\gamma D)^{1/2}$

iii.) Front Stability

→ clearly physically interesting solution should be stable

what learned from stability

→ while wave-front unstable to far-field perturbations, KPP thm. suggests insensitivity to i.e. perturbations with compact support)

Predator

$P \to U$

slow

$\therefore$ natural to investigate stability

For stability $\boxed{P = P(x-ct, t)}$

2 time scales

front prop. time dependence

instability

2 time dep

$\kappa = \gamma = 1$

$\theta = 1$

$$\frac{\partial P}{\partial t} = P(1-P) + c\frac{\partial P}{\partial x} + \frac{\partial^2 P}{\partial x^2}$$

$$\boxed{P = P_0(\underbrace{x-ct}_{z}) + \epsilon \tilde{P}(z, t)}$$

$(z \equiv x - ct)$

$$\therefore \frac{\partial \tilde{P}}{\partial t} = \tilde{P} - 2P_0(z)\tilde{P} + c\frac{\partial \tilde{P}}{\partial z} + \frac{\partial^2 \tilde{P}}{\partial z^2}$$

$$\Rightarrow \frac{\partial \tilde{P}}{\partial t} = (1 - 2P_0(z))\tilde{P} + c\frac{\partial \tilde{P}}{\partial z} + \frac{\partial^2 \tilde{P}}{\partial z^2}$$

Now $\tilde{P} = \tilde{P}(z)e^{-\gamma t}$

$$\Rightarrow \left\{\frac{\partial^2 \tilde{P}}{\partial z^2} + c\frac{\partial \tilde{P}}{\partial z} + (1 + \gamma - 2P_0(z))\tilde{P} = 0\right\}$$

eigenmode equation

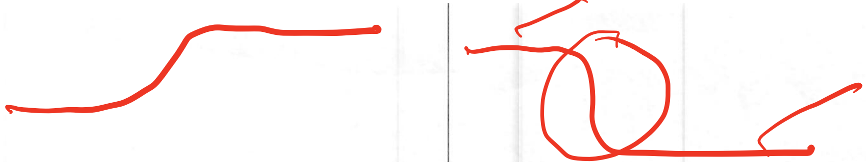
$\gamma > 0 \to$ stable

for $\gamma = 0$ have: $\boxed{\tilde{P}'' + c\tilde{P}' + (1 - 2P_0(z))\tilde{P} = 0}$

observe:

$P(x-ct)$

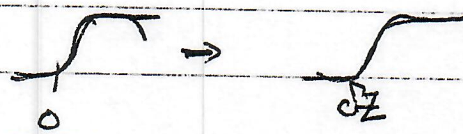
$P_0(z) \to P_0(z + \delta z)$



$$0 = \frac{\partial^2 P}{\partial Z^2} + c\,\frac{\partial P}{\partial Z} + P(1-P)$$

$P = P_0(Z)$ is solution. Now, consider infinitesimal shift of solution:

$P_0(Z + \delta Z)$



$\Rightarrow$

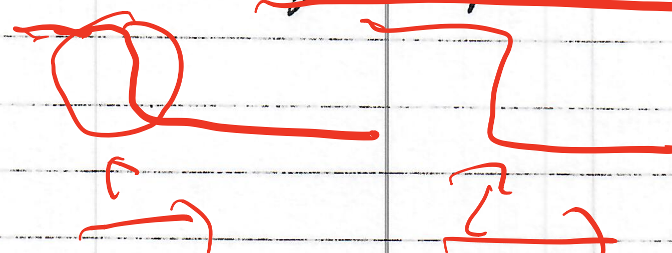
$$0 = \frac{\partial^2}{\partial Z^2}\left(P_0(Z) + \delta Z\,\frac{dP_0(Z)}{dZ}\right) + c\,\frac{\partial}{\partial Z}\left(P_0(Z) + \delta Z\,\frac{dP_0}{dZ}\right)$$

~~base~~

$$+ P_0(1-P_0) + \left(\frac{dP_0}{dZ} - 2P_0(Z)\,\frac{dP_0}{dZ}\right) + O(\delta Z^2)$$

$$= (P_0')'' + c\,(P_0')' + (1 - 2P_0(Z))\,P_0' \qquad \text{eigenmode at } \gamma = 0.$$

$\therefore$ $\gamma = 0$ is "translation mode" $\Rightarrow$ related to translational invariance of system / momentum conservation of kink.



$\Rightarrow$ for stability, need:

$\gamma > 0 \Rightarrow \lim_{t\to\infty} \tilde{P} = 0$

$\gamma = 0 \Rightarrow \lim_{t\to\infty} \tilde{P} = \frac{dP_0}{dZ}$ (front translation)

travel speed $(D/\gamma)^{1/2}$

$\gamma = \gamma$ γ const. $\gamma(x)$

d gap-jumping

Now, substituting $\boxed{\tilde{\rho} \to \delta\rho\, e^{-cz/2}}$ $\Rightarrow$

S.F

$$\delta\rho'' + \left(\gamma - \left(2\rho_0(z) + \frac{c^2}{4} - 1\right)\right)\delta\rho = 0$$

$\delta\rho(\pm L) = 0$

$\gamma > 0$ $e^{-\gamma t}$

$$\Rightarrow \gamma = \frac{1}{\left(\int \delta\rho^2 dz\right)}\left[\int dz \left(\left(\frac{c^2}{4} - 1 + 2\rho_0(z)\right)\delta\rho^2 + \frac{1}{2}(\delta\rho')^2\right)\right]$$

some L

$$\boxed{\gamma \geq 0 \Leftrightarrow c^2 \geq c^2_{min} = 4}$$

– leading edge

– stab.

i.e. Cmin emerges from stability analysis for front.

– dyn. syst

iv.) ~~Asymptotic Analysis of Nonlinear Problem~~

~~→ would be re-assuring to demonstrate viability of (leading edge) analysis – i.e. obtain analytic form for nonlinear front~~

~~→ proceed via singular perturbation theory approach~~

c ?

stab.

Cmin. → Marginal stability in co-moving frame.

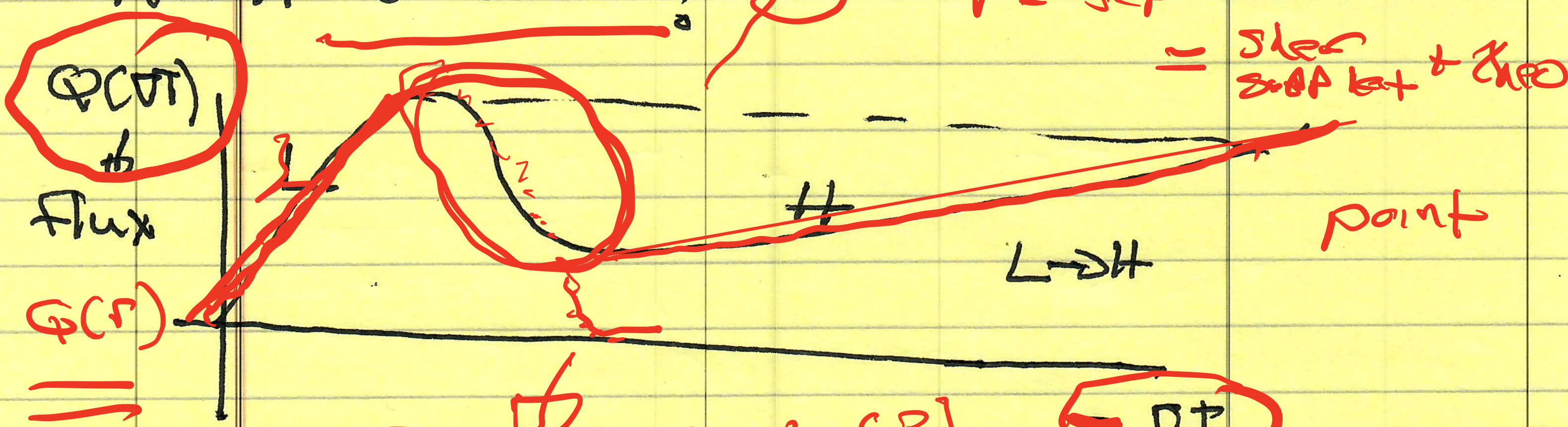
→ l_{t} → $c = c_{marg}$.

II.) Bistability ?

S ~

- Recall S-curve!

$r = r_{sep}$



Itohs (?)

c.f. Hinton '91

exp. A. Hubbard early 2000

K. Ida 2010, 2003

Cat a point!

⇓

how develop in space-time

Flux landscape

L → Q vs $-\nabla T$ → steep
→ large χ_T

H → Q vs $-\nabla T$ → shallow
→ small χ_{neo}.

N.B. → Fick → Bi-stable Fick

How? → shear suppression

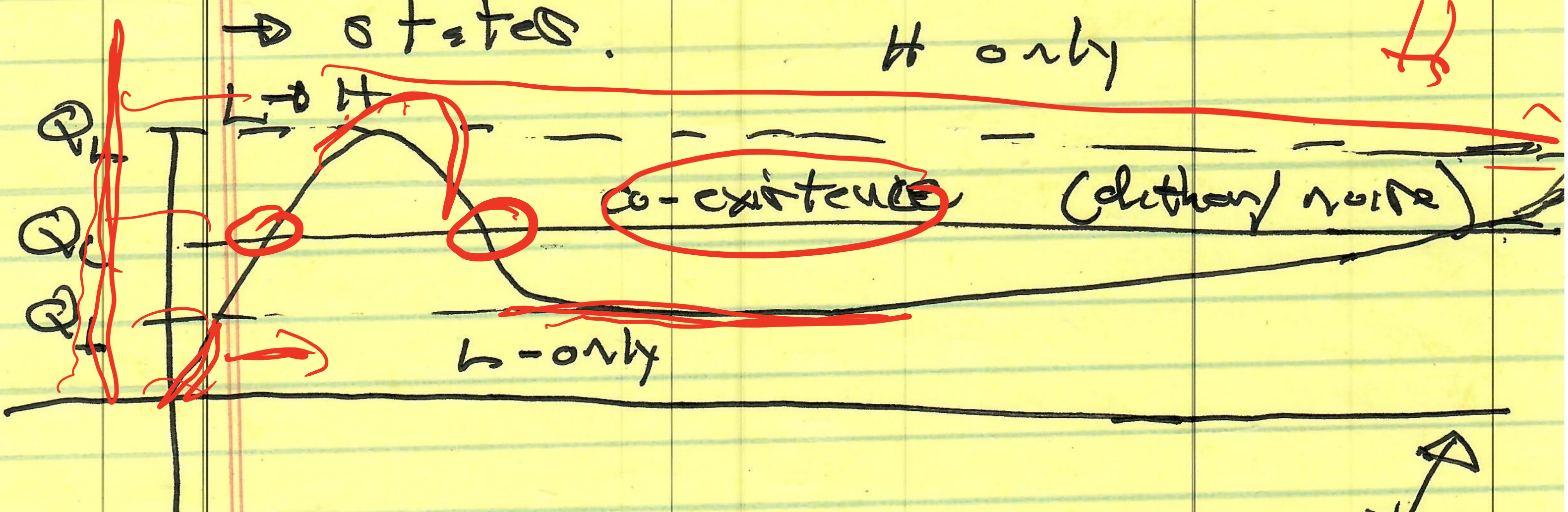
$$Q = -\frac{\chi_T}{1+\alpha(V_E'^2/\gamma^2)}\nabla T - \chi_{neo}\nabla T$$

$V_E' \to T$ with radial force balance,
Radial F.B. rolls over at large ∇T

$P_{in} \to Q_0$

Q_0 ↔ power input

→ states.



other physics
may enter here
⇒ MHD stability (ELM)

Fisher → 1 stable

2 stable states

Signature Feature : R_i – stability

Why?

$$\partial_t T = -\nabla Q + S_0$$

↳ heat source

$$Q_0 = Q(\nabla T)$$

roots

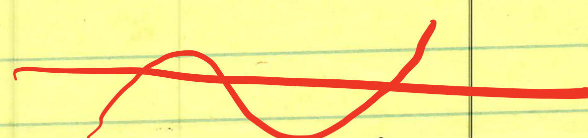
$$\partial_t \nabla T = -\nabla^2 Q$$

∇T_0 → roots

$$\boxed{\nabla T = (\nabla T)_0 + \delta \nabla T}$$

fixed pt root

consider perturbation



const   const

$$\partial_t\left(\cancel{\nabla T_0} + \delta \nabla T\right) = -\nabla^2\left(\cancel{Q(\nabla T_0)} + \frac{\delta Q}{\delta \nabla T}\,\delta(\nabla T)\right)$$

$$\boxed{\partial_t\,\delta(\nabla T) = -\nabla^2 \frac{\delta Q}{\delta \nabla T}\,\delta(\nabla T)}$$

$$= +\nabla^2 \frac{\delta Q}{\delta(-\nabla T)}\,\delta(\nabla T)$$

X

diffusion eq'n!

So observe:

$$\partial_t [\delta(\nabla T)] = \nabla^2 \left[\frac{\delta Q}{\delta(-\nabla T)} \right] [\delta(\nabla T)]$$

$\Updownarrow \Rightarrow$ perturbation domain

$$\partial_t f = \nabla^2 D_{eff} f$$

Diffusion Eqn.

$$D_{eff} = \delta Q / \delta(-\nabla T)$$

$\delta Q / \delta(-\nabla T) > 0 \rightarrow D_{eff} > 0$ stable

$\delta Q / \delta(-\nabla T) < 0 \rightarrow D_{eff} < 0$ unstable

S-curve $Q(\nabla T) \rightarrow$ slope $\rightarrow$ sign of D.

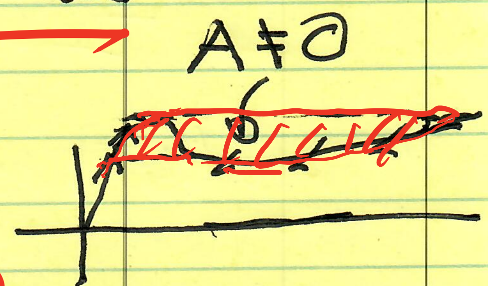
3 [redacted] roots; 2 stable, 1 unstable — bi-stable

1 root; 1 stable

X

Begs question

⇒ how understand co-existence of 2 stable roots?



⇒ thresholds, hysteresis

c.f. — Hinton '91
— Lebedev, P.D. '97 → fronts
" " et al '97 → flux landscape
Malkov, P.D. '07 → analysis

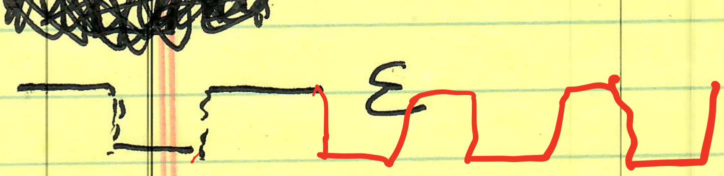
et seq.

⇒ Staircases (another discussion)

i.e. staircases = alternating sequence of bistable domains, connected by jumps.

bistability

Cahn-Hilliard Eqn



The proto-type:

$$\boxed{\frac{\partial u}{\partial t} = f(u) + D\,\partial_x^2 u}$$

Reaction - Diffusion Eqn.

$\rightarrow$ Fitzhugh - Nagumo Eqn. (FN)

- n.b.: $f(u) = A\,(u-u_1)(u_2-u)(u-u_3)$

$A\,(u-u_1)(u_2-u)(u-u_3)$

$A \to \gamma$

3 roots

reaction function

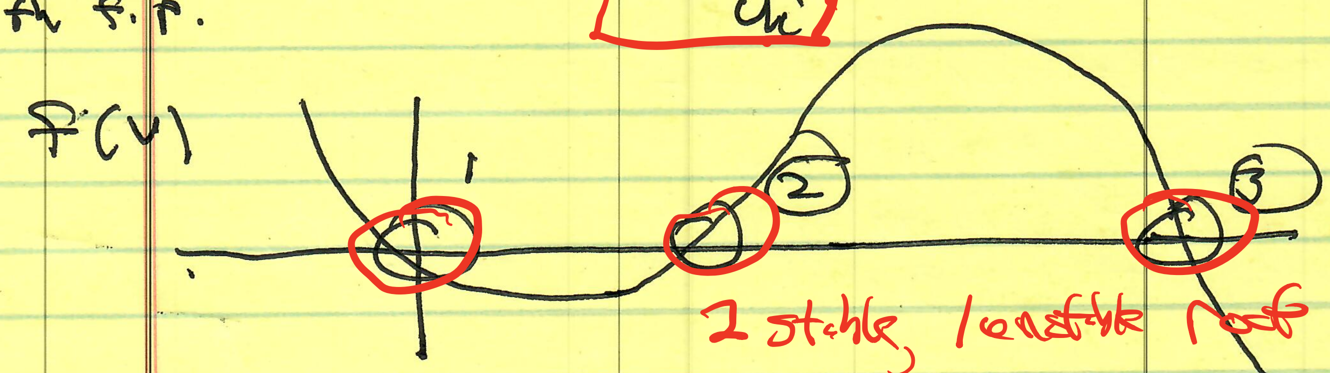
$A \sim \gamma \rightarrow$ rate

key pt: 3 fixed pts: u_1, u_2, u_3

stability: $\dfrac{\partial \tilde{u}}{\partial t} = \dfrac{\partial f}{\partial u}\Big|_{u_i} \tilde{u} + D\,\partial_x^2 \tilde{u}$

$\tilde{u} = u - u_i$

ith f.p.



2 stable, 1 unstable root

pt: $\dfrac{\partial f}{\partial u}\Big|_{u_2} > 0 \rightarrow$ instability

X

$\left.\frac{\partial f}{\partial u}\right|_{u_1, u_3} < 0 \rightarrow$ stable

→ WTFN? (General Culture)

- FN is 'toy' version of Hodgkin-Huxley model of neuron signals (muscle, etc ....) Nobel Prize

→ switch {on, off} ⟺ bistability

on-off switch

c.f. at least a look at Hodgkin-Huxley paper (1952) is highly recommended.

FN system:

$$\frac{\partial v}{\partial t} = \underset{\text{bi-stable}}{f(v)} - w + \overset{\text{current}}{I_a} + \frac{\partial^2 v}{\partial x^2}$$

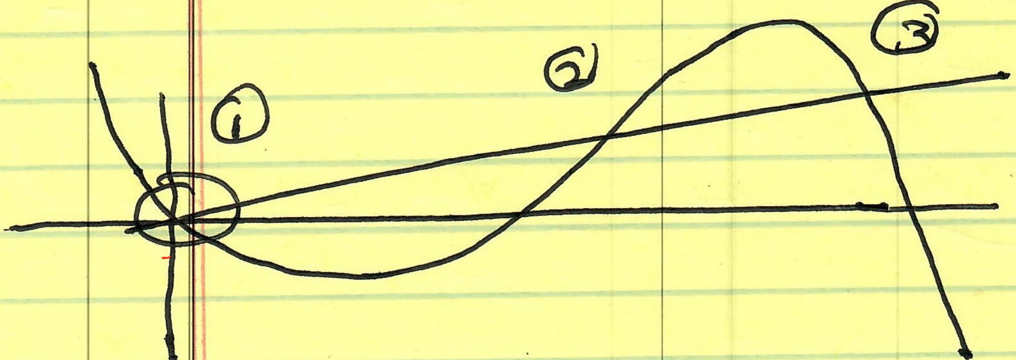
i.e. fast channel → sodium

$$\frac{\partial w}{\partial t} = bv - \gamma w$$

i.e. slow, as calcium

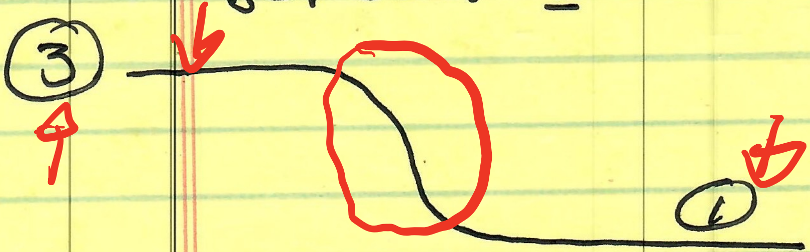
so, for fixed points:

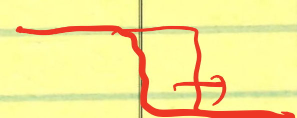
$$\left.\begin{aligned} w &= f(u) = w(u) \\ w &= bv/\gamma \end{aligned}\right\} \text{f.p.}$$



evident: (1), (3) stable $f' < 0$
(2) unstable $f' > 0$

$\Rightarrow$ front can link/allow transition between 2 stable fixed points.





$\Rightarrow$ unstable root 'powers' front motion (ala' Fisher).

can further simplify to FN equation:

$$\frac{\partial u}{\partial t} = D\,\partial_x^2 u + f(u)$$

$$f(u) = A\,(u-u_1)\,(u_2-u)\,(u-u_3)$$

⇒ minimal bistable model → switching of neurons
counterpart of Fisher

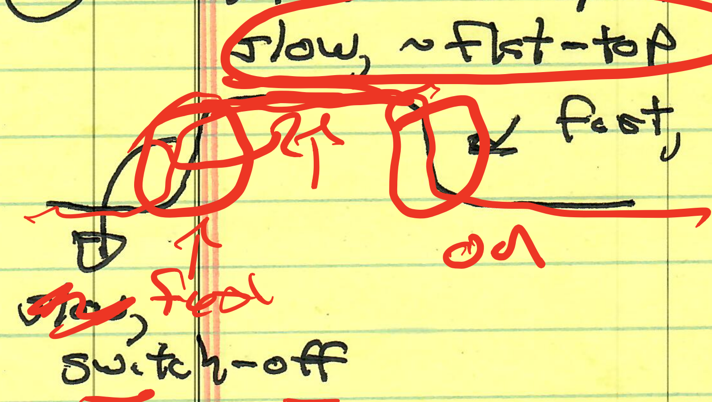
excitable media → threshold to activate

→ Note:

① One 'goal' of FN system is to describe pulses in excitable media

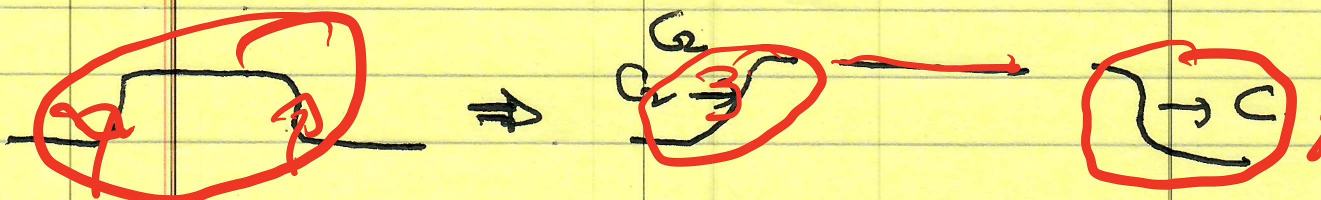
(MFE application?)

② What is a pulse?



X

Idea of FN: assemble 3 parts



to maintain coherence:

$$C_1 = C_3$$

~ key element of pulse is switch on-off front

~ switch-on starts from $U=0$ $\underline{\infty}$

$$\boxed{\partial_t u = D\,\partial_x^2 u + f(u)}$$

~ above.

③ F and FN:

Levi?

$$\partial_t u = D\,\partial_x^2 u + \gamma u(1-u)$$

↑ — reactions.

$$\partial_t u = D\,\partial x^2 u + f(u)$$

bistable

⇒ reaction - diffusion equation

i.e. nonlinear reaction and diffusion separate.

In MFE: (also) "reaction-diffusion"

but:

$$\partial_t T = -\underline{\nabla} \cdot \underline{Q}(\nabla T) + S_0$$

$$Q = \frac{-\chi_T}{1 + \alpha V_E'^2} \nabla T - \chi_{neo} \nabla T$$

NL of $\frac{\partial}{\partial}$ of S

↑ ⟹ "reaction"

⟹ "reaction is in the diffusion"

i.e. nonlinear diffusion.

- harder!

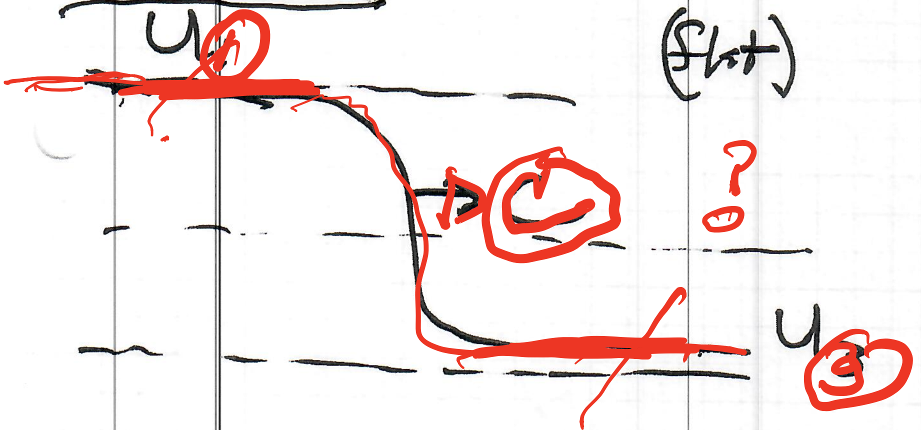
⟹ Cahn-Hilliard Equation (coming)

⟹ best understand FN first.

FN analysis →

Looking for { Fronts.
speed

As usual:



$c>0$

$c<0$

$c=0 \to$ co-existence

As usual, look for speed c via traveling wave:

$$u = u(x-ct)$$

$$\boxed{-cu' = Du'' + f(u)}$$

$$u'\left[-cu' = Du'' + f(u)\right]$$

so

$$-c\int_{x(u_3)}^{x(u_1)} u'^2\,dx = D\int_{x(u_3)}^{x(u_1)} u'u''\,dx + \int_{x(u_3)}^{x(u_1)} u' f(u)\,dx$$

$$\quad ① \qquad\qquad\qquad ② \qquad\qquad\qquad ③$$

$$② = \int_{x(u_3)}^{x(u_1)} u'u''\,dx = \int_{x(u_3)}^{x(u_1)} \frac{d}{dx}\left(u'^2/2\right)$$

$= 0$, as $u' = 0$ on both sides front.

② $\int_{x(u_3)}^{x(u_1)} dx\, u_x\, f(u) = \int_{x(u_3)}^{x(u_1)} du\, f(u)$

$= \int_{u_3}^{u_1} du\, f(u)$

③ $= \int_{u_3}^{u_1} du\, f(u)$

~~④ $= -c \int_{-\infty}^{+\infty} u'^2\, dx$~~

non zero

⑤ $\boxed{c = -\int_{u_3}^{u_1} f(u)\, du \Big/ \int_{-\infty}^{+\infty} u'^2\, dx}$ $\Rightarrow$ front speed

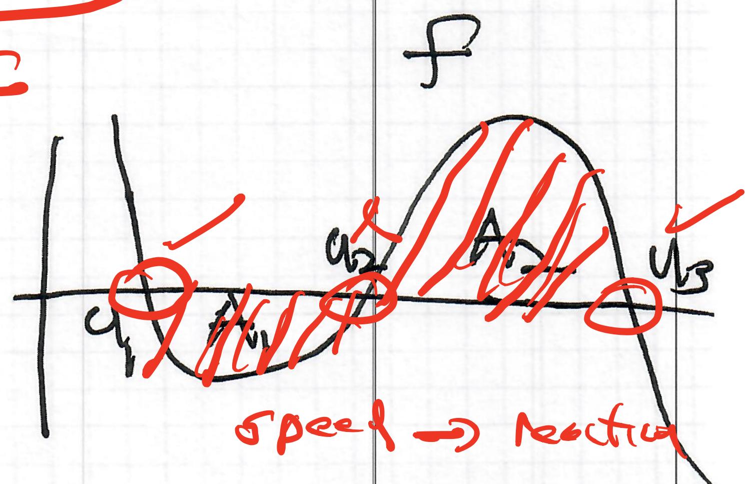
front speed.

>0

factor

$\boxed{c} \sim -\int_{u_3}^{u_1} f(u)\, du$

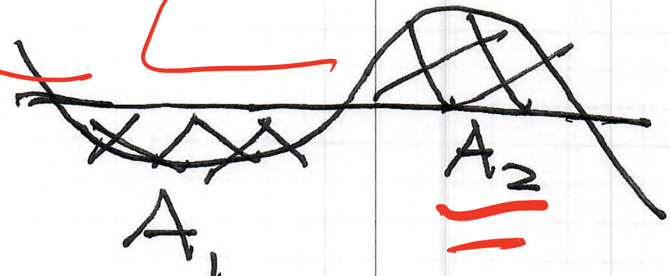
$= \int_{u_1}^{u_3} du\, f(u)$



speed → reaction

i.e. c linked to area under f → curve with zero crossings.

50 31.



$A_2 > A_1 \rightarrow$ $\boxed{c > 0}$ – front advances toward $x > 0$

$x_1 \rightarrow x_3$

$A_1 < A_2 \rightarrow c < 0$ – front advances toward $x < 0$

$x_1 \leftarrow x_3$

$A_1 = A_2 \rightarrow c = 0$ – front stationary – co-existance

→ Area rule is akin to Maxwell construction in thermodynamics. phase coexistence

Equal areas ⟺ co-existance of phases.

root – phenom

→ Bistable dynamics is much richer than simple "Pac-Man" dynamics of Fisher..

HW Hinton → roots (Hinton)

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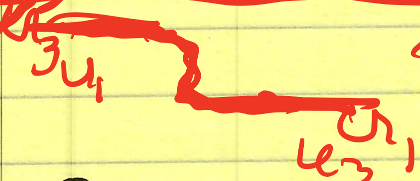
Calculate c ? → approximation ?

$$\frac{\partial u}{\partial t} = A(u-u_1)(u_2-u)(u-u_3) + D\frac{\partial^2 u}{\partial x^2}$$

$u = u(x-ct)$, $\quad u(-\infty) = u_3$, $\quad u(\infty) = u_1$ — asymptotic behavior

$$\boxed{L(u) = 0 = Du'' + cu' + A(u-u_1)(u_2-u)(u-u_3)}$$



simple solution:

- assume u satisfies

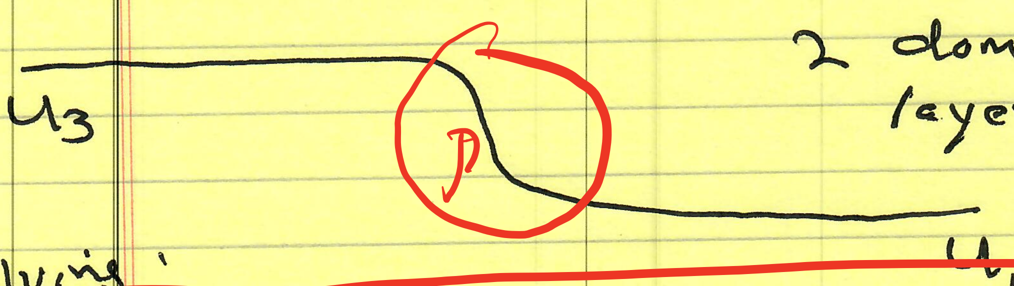
$$u' = a(u-u_1)(u-u_3)$$

→ trial, determine a

why?

$u' = 0$ at u_1, u_3

i.e. has form:



2 domains + layer.

solving:

$$\boxed{u(z) = \frac{u_3 + k\, u_1 \exp[a(u_3-u_1)z]}{1 + k \exp[a(u_3-u_1)z]}}$$

Plugging into $0 = L(u)$,

$$L(u) = D\left[\underline{a(u-u_1)(u-u_3)}\right]'$$

$$+ \, \underline{c\left[a(u-u_1)(u-u_3)\right]}$$

$$+ \, \underline{A(u-u_1)(u_2-u)(u-u_3)}$$

and repeat for u' terms in above.

$\times$

33.

$$L(u) = D\left(a(u-u_1)(u-u_3)\right)$$

$$+\left[c\, a(u-u_1)(u-u_3) + A(u-u_1)(u_2-u)(u-u_3)\right]$$

$$= Da\left[u'(u-u_3) + (u-u_1)u'\right]$$

$$+ \left[\text{above}\right]$$

and plug in u' again

$$= Da\left[a(u-u_1)(u-u_3)(u-u_3) + a(u-u_1)^2(u-u_3)\right]$$

$$+ \left[\quad\right] = 0$$

$A = 2Da^2$

re-grouping:

$$L(u) = (u-u_1)(u-u_3)\left[(2Da^2-A)u - (Da^2(u_1+u_3) - ca - Au_3)\right]$$

so

speed c

HW#2 → Translate FN to L→H problem

X

34.

so $L(u) = 0$

$\Rightarrow$ $2Da^2 = A$

$Da^2(u_1 + u_3) - Au_2 - ca = 0$

or $a = (A/2D)^{1/2}$.

speed!

and $\boxed{c = (AD/2)^{1/2}(u_1 - 2u_2 + u_3)}$

values of 3 roots

$A \Leftrightarrow \gamma$ $\Rightarrow$ $\boxed{c \sim (D\gamma)^{1/2}}$, as before

speed

and $\boxed{c \cong \sqrt{D\gamma}\, c(u_1, u_2, u_3)}$

param. in reaction.

$\boxed{u_2 = (u_1 + u_3)/2}$ → stationary front

$c = 0$

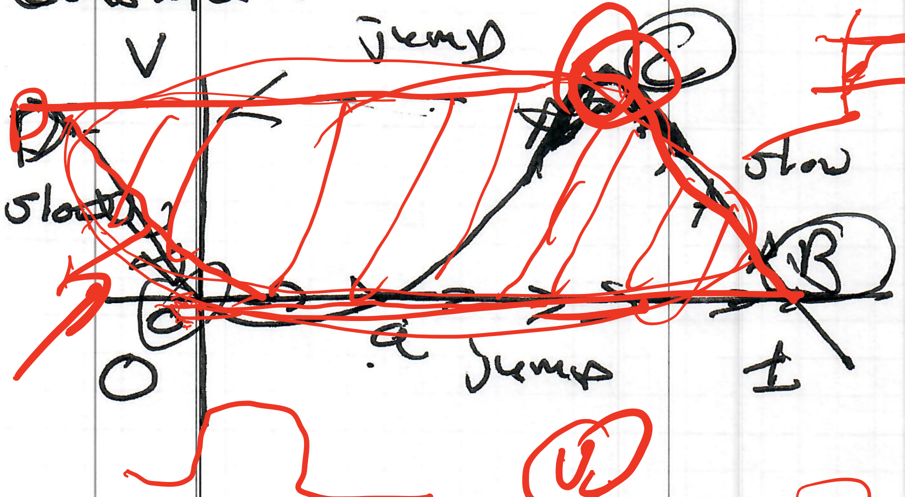
Thus, speed set by parameters in reaction function. Weighted balance between these is the key.

E' F' $u \sim E'$

Toward Pulses / Waves

35.

Consider:



σ B C D

Hysteresis loop

"Excitable media"

$0 < a < 1$

$$\partial_t u = f(u) - v - D\partial_x^2 u$$

$$\partial_t v = bu - \gamma v$$

for slow eqn:

$$V_t = \epsilon(Lu - Mv)$$

Treat v as fixed for fast bifurcations

v_c

if $v \to 0$ i.e.

$$\partial_t u = D u_{xx} + f(u)$$

$$f(u) = u(a-u)(u-1)$$

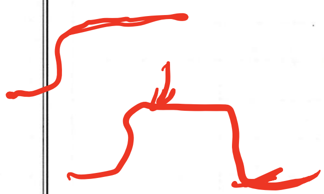
$\to$ leading edge set at $v = 0$

if V finite:

$$\boxed{u_t = D u_{xx} + f(u) - V_c}$$

↓

changes effective reaction function



so

$$c_- = \left(\frac{D}{2}\right)^{1/2}(u_c - 2u_p + u_0)$$

where u_c, u_p, u_0 roots of

$$f(u) = V_c$$

→ speed is function of amplitude of V

↓

like soliton

then pulse condition:

coherence

$$\boxed{C_+ = C_-(V_c)}$$

guarantees that pulse will not disperse → i.e. forward and backward transitions propagate together at same speed. This sets a critical amplitude V_c for

X

37.

pulse to be excited.

→ Slow evolution links the up, back transitions.

→ The FN model is basic simple model of pulse in excitable media.

→ so constructed from basic element of bistable front.

If time → spirals, etc. coming

In MFE — "Reaction is in the diffusion".

— More akin Cahn-Hilliard

coming

Staircase

Well separated

Ashourvan

2 step

turbulent QH

2-3 steps

u_1 → steel u_3

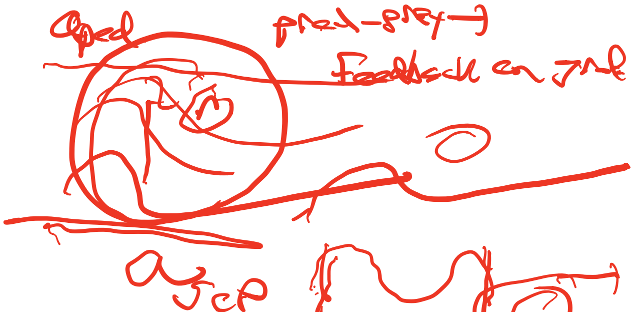
How build-up ?

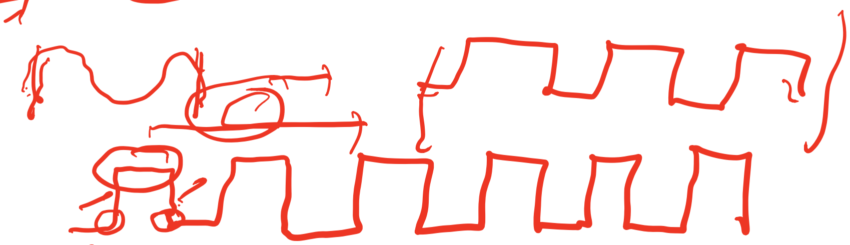
modulation + NL

bistable function

pred-prey →

Feedback on gate

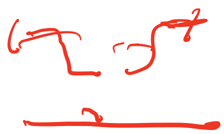




pulses

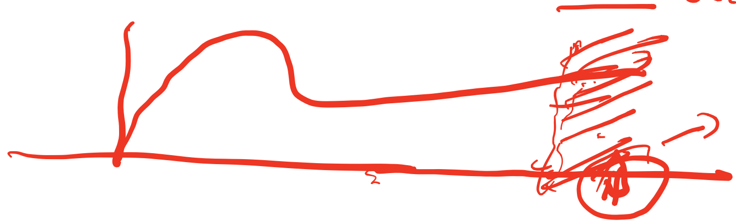
ε $C_- = C_+$ chain pulse → stationary SCs

$C_- = C_+ = 0$ →



~ lattice of (small) carriers

Ideal cell → ELA



→ cell

→ DC