PKU-HUST Lectures 2020-21

Lecture II

- Review of { Goals / Lecture 1

- Time delays and Limit Cycles
  (CTRC - Multi-predator) $\Leftrightarrow$ Memory Effn.
- Variability - Statistical Approach I

- From Time → Space-Time : Fronts
  Unstable / 2nd Order → Fisher Eqn.
  Bistable / 1st Order → Fitzhugh - Nagumo Eqn.
  → next

→ will extend into Lect. III.

i) Review

- Goals: Topics in Nonlinear Dynamics for MFE.

N.B.: Mix

- topics with relevance already established (i.e. predator-prey)

- topics of possible future relevance - good to learn! ?
  - source of new ideas

may be relevant

⇒ Ecological Niches → multi-scale

- Last time:

→ DW - ZF

Prey - Predator

Key physics: Reynolds work:

$$\int \langle \tilde{v}_r \tilde{v}_\perp \rangle \langle V_\perp \rangle$$

phase $\langle k_x k_y \rangle \to \langle V_\perp \rangle'$

- H-Theorem: For:

$$\frac{dH}{dt} = H\,F(H,P)$$

predator

$$\frac{dP}{dt} = P\,G(H,P)$$

prey

→ 4 conditions

→ system will have either

- stable fixed pt.
- stable LCO

or

$\boxed{\text{LCO} - \text{unstable fixed PT.}}$

- Simple DW-ZF Pred-Prey

→ 2 non-trivial Fixed Pts.

No Flow

$$\mathcal{E} = \gamma_0/\alpha_1$$

$$\boxed{u = 0}$$

Flow

$$\mathcal{E} = u/\alpha_2$$

$$u = \left(\gamma_0 - \frac{\alpha_1}{\alpha_2}u\right)/\alpha_2$$

$u = v^2$

~ B > C condition: $\boxed{\gamma_0 - \frac{\alpha_1}{\alpha_2}u > 0}$

also $u > 0$

~ modes: $\gamma = -\gamma_0$

No flow

$\gamma = -(u - \alpha_2\gamma_0/\alpha_1)$ → soft

2nd Order
Phase Transition

$\gamma_u \to 0$, $T_{trans} \to \infty$ at threshold

so slaving E to flow:

$$\boxed{\frac{1}{2}\frac{du}{dt} = \left(\frac{\alpha_2}{\alpha_1}\gamma_0 - u\right)u - \frac{\alpha_2^3}{\alpha_1}u^2}$$

+ T_{eff}
(shear)

and driven TDGL $\Rightarrow$ Bias

H_0 in ferromagnet

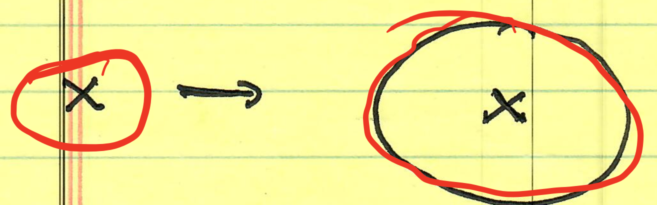
$\Rightarrow$ LCO's......

Minorsky → feedback & control

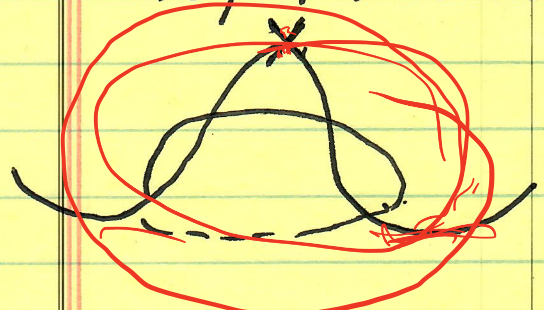
(ii) LCO → Overshoot

→ What? (in Pred-Prey context)

- attractor in 2-species system is a closed curve



- stable, not neutral.



Mexican Hat

- appears in K-Thm by:
  - system satisfies theorem
  - yet fixed point is linearly unstable.

- usually revealed by x, y → r, θ coordinates (see standard books)

Drazin

→ Why? Limit Cycles ? May
(More interesting)

"In the real world, the growth rate of a species' population will often not respond immediately to changes in its own population, or that of an interacting species, but rather will do so after a time lag."

— R. May

LCO's – Time Lags

~ Inspirational version (for students):

"Alternatively, it can be that the time delay in a resource limitation effect is essentially a natality lag: one has to natal

wait until the next generation is confronted with the havoc its predecessors wrought."

— R. May
(broadly applicable)

So

time delay ⟷ memory kernel

- time delays are destabilizing, leading to over-shoot ⟵ LCO

- $\frac{dN}{dt} = r N$ — exponential (Malthusian) growth

R. Malthus

$$\frac{dN}{dt} = \left(r - \frac{N}{K}\right) N$$ — Logistic Eqn.

(~~Eqn~~ eqn — Pred-Prey)

random #

$\gamma = r \rightarrow r - \frac{N(t)}{K}$ saturation level

$K \rightarrow$ carrying capacity

$x_{n+1} = \alpha x_n (1 - x_n)$ Feigenbaum

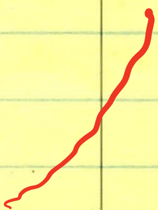
$\Rightarrow N(t)$ feeds back on $\gamma(t)$.

$\rightarrow$ with time delay:

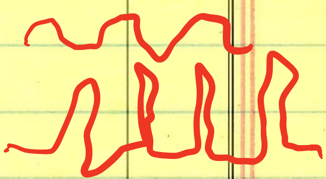
$\gamma(t) \rightarrow r - \frac{N(t - \tau)}{K}$ → time delay

Business cycle

$$\frac{dN}{dt} = r\,N\left(1 - \frac{N(t-T)}{K}\right)$$



→ $N(t-T)$ feeds back on $N(t)$ growth. If $N(t)$ growing fast, $N(t-T) \ll N(t)$



⇒ no saturation ↔ $\boxed{\text{instability}}$

n.b. $T = 0$ → Logistic saturation for $N = K$. → carrying capacity

→ stability HW

Then "how fast is 'fast'?"

Clearly, comparison is r vs. $1/T$

r ↓ growth rate

$1/T$ ↓ rate defined by time delay

Can see: $r \gg 1/T$

⇒ $Tr \gg 1$ → $\boxed{\text{unstable}}$ (long)

$r \ll 1/T$ (short)

⇒ $Tr \ll 1$ → stable

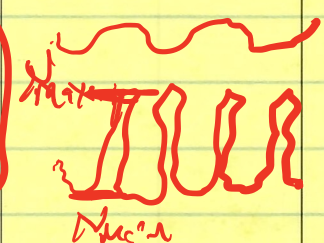
More systematically,

$$\frac{dN(t)}{dt} = r\,N(t)\left[1 - N(t-T)/K\right]$$

$\hat{\tau} = rT \qquad \hat{t} = rt$

$x = N/K$

$$\Rightarrow \boxed{\frac{dx(\hat{t})}{d\hat{t}} = x(\hat{t})\left[1 - x(\hat{t}-\hat{\tau})\right]}$$



- LCO for $\hat{\tau} > 1/2\pi \rightarrow rT > 1/2\pi$

  $\hat{\tau} < 1/2\pi \rightarrow rT < 1/2\pi$

- see table → cycle amplitude very sensitive to time delay
- see pic → cycle approaches burst for longer delay.
- In general, more general

ecology → time to mature   Nicholson's Blowflies

Memory function / Response fctn.

$$N(t - \underset{T,\ \text{distrib.}}{T}) \;\; \cancel{N(t)} \;\rightarrow\; \int_{-\infty}^{t} N(t')\,\underbrace{Q(t-t')}_{\text{Memory function / Kernel}}$$

May T 4.1
99.

MODELS WITH FEW SPECIES

TABLE 4.1. Properties of limit cycle solutions of equation (4.9)

time delay $\tau = rT$	cycle amplitude $N(\max)/N(\min)$	Cycle period
1.57	1.00	—
1.6	2.56	4.03T
1.7	5.76	4.09T
1.8	11.6	4.18T
1.9	22.2	4.29T
2.0	42.3	4.40T
2.1	84.1	4.54T
2.2	178	4.71T
2.3	408	4.90T
2.4	1,040	5.11T
2.5	2,930 3×10^3	5.36T

realistically, this time delay will depend n
ation at some particular instant in past
on an average over past populations
$T)/K$ but rather on the weighted average

... functions of τ (Martin, private communication).

Hutchinson's model has a time delay of exactly T for the vegetation or whatnot to respond. More generally, and

Amplitud evolution for different time delays

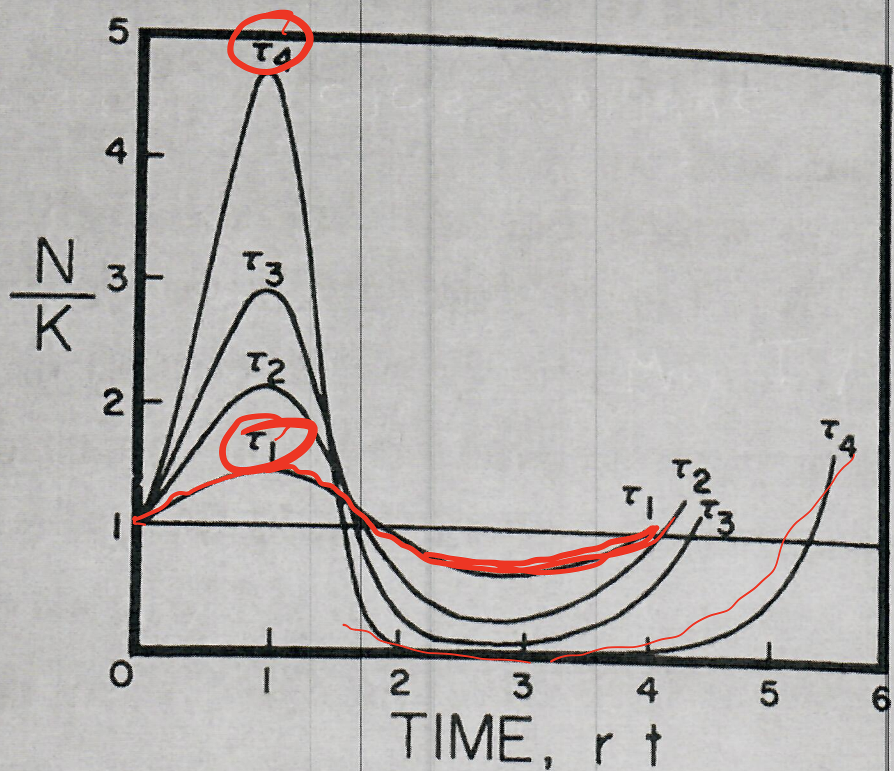


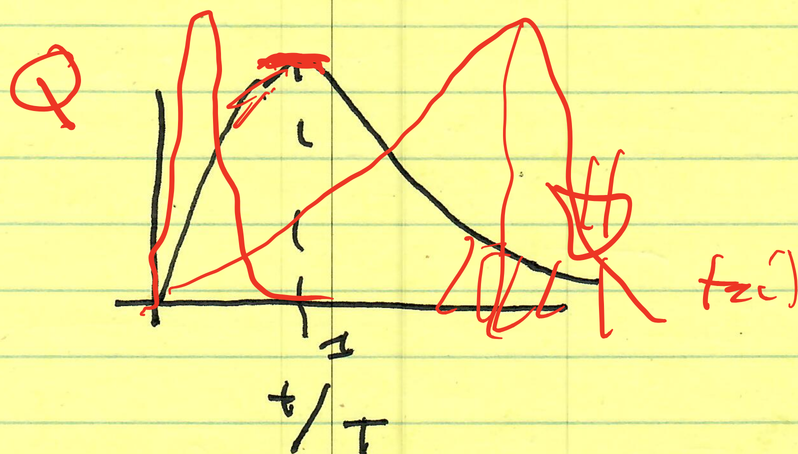
FIGURE 4.5. The oscillations undergone in one complete cycle by the population $N(t)$ whose dynamics obey the time-delayed logistic equation (4.9) with too long a time delay. For $rT = \tau < 1/2\pi$, there is a stable equilibrium point at $N/K = 1$ (the horizontal line). For $\tau > 1/2\pi$, the final periodic solutions are shown for various values of τ, as indicated: $\tau_1 = 1.6$; $\tau_2 = 1.75$; $\tau_3 = 2$; $\tau_4 = 2.5$ (after Jones, 1962b).

→ i.e. feedback on γ results from weighted average over past populations

→ Q(t−t') is weighting factor → growth rate response.

effective time delay.

i.e.



Q(t−t') → δ(t−T−t') recovers single time delay form

⇓

(self) memory kernel.

structure of self-memory kernel L(t).

→ general form of Logistic Eqn.:

$$\frac{dN(t)}{dt} = r\,N(t)\left[1 - \int_{-\infty}^{t} \frac{N(t')}{K}\, Q(t-t')\, dt'\right]$$

integral

~ transducence

f(t)

→ structure of memory kernel determines stability (fixed pt.) or instability (LCO)

i.e. t s/t $Q = Q_{max}$

$\equiv t_p$ → kernel peak

then: $t_p r$ $\begin{matrix} > \\ < \end{matrix}$ 1 $\begin{matrix} \to \text{unstable} \\ \to \text{stable.} \end{matrix}$

Some furthur questions:

- Control theory perspective?
- Classic example? – Nicholson's Blowflies
- What of prey + predator vs. resource limited prey?
- Physics of memory kernel?

⟺ relevance to plasma problems.

(i.) Control Theory

- $r \to r_0 (1 - N/K)$

  ~ stabilizing negative feedback

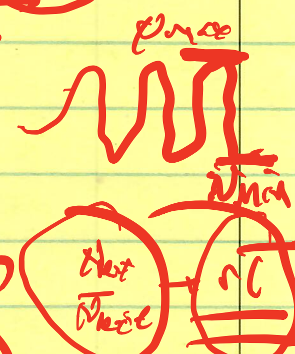
- Control theory ⇒ if negative feedback on time delay T_d s/t

  $T_d >$ natural system period

  ⇒ instability.

  ($T_d > \tau$)

period



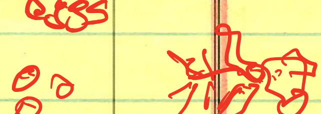
(ii.) Example – Nicholson's Blowflies

- experiments on Lucilia cuprina (blowfly)

Many generations

- Data
  - ⇒ growth rate r
  - ⇒ K set by food supply
  - ⇒ T time delay on resource limitation, as maturation time for blowfly larvae.

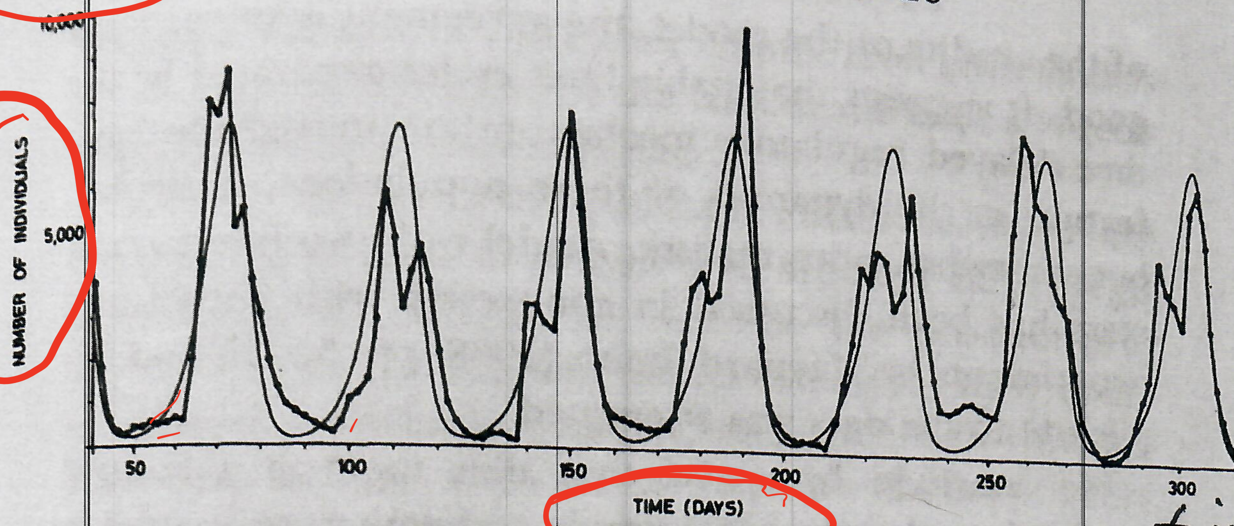
eggs



~ See figure.

126.

Flies



time

FIGURE 4.8. From the one-parameter family of limit cycles generated by the time-delayed logistic equation (4.9) (see Figure 4.5 and Table 4.1), we display that which best fits the oscillations in Nicholson's blowfly populations. The experimental data are from Nicholson (1954); the theoretical curve, with $rT = 2.1$, is in good agreement considering the crudity of the model.

In fitting this model to the data, we have only the dimensionless parameter rT at our disposal; K is absorbed in setting the scale of the y-axis for the population, and r in scaling the x-axis for the time (cf. equation (4.11)). This single parameter rT is completely determined by Nicholson's data (Figure 4.8) because it depends sensitively on the ratio between maximum and minimum values of the oscillating population (cf. Table 4.1). We estimate $rT \sim 2.1$. Thence, again consulting Table 4.1, the appropriate theoretical period may be compared with the observed experimental oscillation period, to conclude that the time delay T is roughly 9 days. In fact, these blowfly larvae take around 11 days to become adult (Nicholson, 1957, Figure 6). The theory also predicts that if the amount of ground liver be doubled, that is if K be doubled, then everything should be exactly as before, except that the

B.

So

— N_{max} / N_{min} → $r T$ (table)
obs. population

$rT \sim 2.1$

— again, oscillation period → $\boxed{\text{delay}}$, via table.
observe

Recall delay observable → hatching time

$T_{delay} \sim \boxed{9 \text{ days}}$

but 11 days to mature (not bad).

Nicholson's Blowflies
→ dimensionless scaling
of Logistic Eqn.

Logistics

iii.) What of $\boxed{\text{Predator - Prey}}$?

Consider:

Rabbits

$$\boxed{\begin{aligned} \frac{dH}{dt} &= r H(t)\left[1 - \frac{H(t-T)}{K}\right] - \alpha H(t) P(t) \\ \frac{dP}{dt} &= -b P(t) + \beta P(t) H(t) \end{aligned}}$$

Large Cats

if $\alpha = \beta$, $T = 0$, identical to:

$$\begin{bmatrix} \dfrac{\partial E}{\partial t} = \gamma_0 E(t) - \beta E(t)^2 - \alpha E(t) U(t) \\ \dfrac{\partial U}{\partial t} = -\mu U(t) + \alpha E(t) U(t) \end{bmatrix}$$

$\rightsquigarrow$ DW - ZF system

$\rightsquigarrow$ Vegetat~~[illegible]~~ion - Herbivore - Carnivore
(Cow) (Large Cat)

Vegetation $\Rightarrow$ $r \to r\left[1 - \frac{H}{K}\right]$ resource limitation, due grazing

Now, critical comparison is between

- Time delay
- natural time scale of system

Comparison clear if:

$H_{\text{fixed pt}} \ll K$ ← Prey F.P

self-note: $H_s \sim K$, ~~$H_{\text{fixed}} \ll K$~~

i.e.

$b/\beta < K$ → H_{fixed} for $\frac{dP}{dt} = 0$ less than carrying capacity

$D\mu/d_H <$ ~~α~~ α/β

⇒ resource limitation irrelevant

$$\frac{dH}{dt} = rH - \alpha H P$$

$$\frac{dP}{dt} = -bP + \beta P H$$

Lotka-Volterra

HW - show

⇒ $\boxed{\omega \sim (rb)^{1/2}}$ → Lotka-Volterra Oscillation ~~[illegible]~~ frequency.

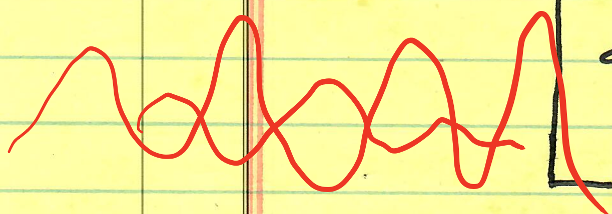
natural rate for predator-prey self-regulation

(not LCO!)

Then stability → damped oscillation *

unstability → LCO

* damped → [pred-prey / Lotka-Volterra oscillation
+
damping due resource limitation]



HW → work out implications DW-ZF.

i.e. $\omega T \gg 1$ → LCO, unstable.

$\omega T \ll 1$ → stable

? $\omega T \longleftrightarrow$ L mode
I phase

iv.) $\boxed{\text{Memory Kernel}}$.

↳ aside]

→ Consider related problem of what is a heat bath

→ $\boxed{\text{dynamical perspective.}}$

Local / Nonlocal in time

Pt. Memory Kernel

→ $\boxed{\text{generalized Langevin Eqn.}}$

friction thermal fluct's

c.f.
$$\boxed{\frac{\partial}{\partial t} V + \frac{\beta}{M} V = \tilde{f}}$$
FDT

and resource limitation / self-saturation

has similar structure: (NL friction).

self-saturation

$$\frac{\partial \mathcal{E}}{\partial t} + \beta\, \mathcal{E}^2 = r\mathcal{E} - \alpha \mathcal{E} U$$
+ Noise

No Memory

$$\beta \int Q(t-t')\, \mathcal{E}(t')\, \mathcal{E}(t')$$

Mech. Heat Bath

+ Noise

Physics controlling?

Memory Kernel

all NL interaction models

c(ii.) A Dynamical Model of a Heat Bath, and How Interact with ?

(Zwanzig '73)

What does Heat Bath Mean ?

How dynamically describe arbitrary motion in a heat bath ?

System: q, p System

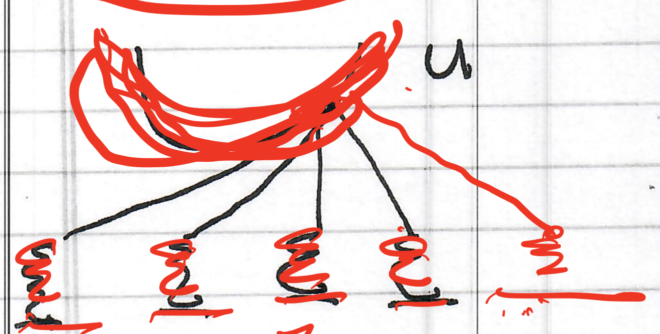
$$H_S = \frac{p^2}{2m} + U(x)$$ — system.

Bath: collection of h.o.'s coupled to motion

$$H_B = \sum_j \left(\frac{p_j^2}{2} + \frac{1}{2}\omega_j^2 \left(q_j - \frac{\gamma_j}{\omega_j^2} x \right)^2 \right)$$

collection of oscillators

coupling to system



freq., dist., N

R. Zwanzig

Write EOMs:

$$\frac{dx}{dt} = p$$

$$\frac{dp}{dt} = -U'(x) + \sum_j \gamma_j \left(q_j - \frac{\gamma_j}{\omega_j^2} x \right)$$

elim in terms p & x

and

$$\frac{dq_j}{dt} = p_j$$

$$\frac{d}{dt} p_j = -\omega_j^2 q_j + \gamma_j x$$

Ultimately: – seek Langevin Eqn. (generalized)

– express q_j in terms p, x

i.e. bath coords irrelevant.

If x known: Formally,

i.c.'s

$$q_j(t) = q_j(0)\cos\omega_j t + p_j(0)\frac{\sin\omega_j t}{\omega_j} + \gamma_j\int_0^t ds\, x(s)\,\frac{\sin\omega_j(t-s)}{\omega_j}$$

Seeking "Langevin Eqn.", I.B.P. $\Rightarrow$

(RHS dp/dt)

i.c.

$$q_j(t) - \frac{\gamma_j}{\omega_j^2}x(t) = \left(q_j(0) - \frac{\gamma_j}{\omega_j^2}x(0)\right)\cos(\omega_j t)$$

i.c.

$$+ p_j(0)\frac{\sin\omega_j t}{\omega_j}$$

$\Downarrow$ RHS of p eqn

$$-\gamma_j\int_0^t ds\,\frac{p(s)}{M}\,\frac{\cos\omega_j(t-s)}{\omega_j^2}$$

So plugging into dp/dt eqn.:

"noise"

$$\boxed{\frac{dp(t)}{dt} = -U'(x(t)) - \int_0^t ds\, K(s)\,\frac{p(t-s)}{M} + \underbrace{F_p(t)}}$$

(Sort of) Langevin Eqn.! Interaction kernel!

Non-Markovian

$$\frac{\partial v}{\partial t} + \frac{\beta}{\kappa}v = \ldots\ldots$$

effective damping / drag

Here:

$$\boxed{K(s) = \sum_j \frac{\gamma_j^2}{\omega_j^2}\cos(\omega_j t)} \rightarrow \left\{ \begin{array}{l} \text{kernel set} \\ \text{by bath} \\ \text{properties.} \end{array} \right.$$

Noise $F_p(t)$ from i.c.'s:

$$\boxed{\begin{aligned} F_p(t) = {} & \sum_j \gamma_j\, p_j(0)\, \frac{\sin \omega_j t}{\omega_j} \\ & + \sum_j \gamma_j \left(q_j(0) - \frac{\gamma_j}{\omega_j^2} x(0) \right) \cos(\omega_j t) \end{aligned}}$$

$K(s) \rightarrow$ memory function

i.e. contrast:

B. M. $\quad \dfrac{dv}{dt} = -\dfrac{\gamma}{m} v + \tilde{f}/m$

no memory — local in time

simple $\Rightarrow$ "Markovian"

Here:

$$\frac{dp}{dt} = -U'(x(t)) - \int_0^t ds\, K(s)\, \frac{p(t-s)}{m} + F_p(t)$$

→ Memory ←

general ⇒ "Drag" depends on time history → "Non-Markovian"

Now, in this model, can adjust Memory function .... via bath h.o. distribution

$$\boxed{K(t) = \sum_j \frac{\gamma_j^2}{\omega_j^2} \cos(\omega_j t)}$$

a simple example of memory fctn..

then if:

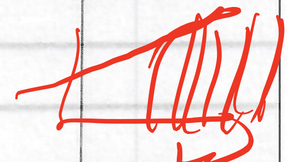
- continuous spectrum.
- $\sum_j \to \int d\omega\, g(\omega)$
  density states
- $\gamma \to \gamma(\omega)$
  coupling dependence.

$$K(t) = \sum_j \gamma_j^2 \cos \omega_j t$$

$$\rightarrow \int_0 d\omega \, g(\omega) \, \gamma(\omega)^2 \cos \omega t$$

$g(\omega) \equiv$ density of states

$g(\omega) \sim \omega^2$



Now if:

- $g(\omega) \sim \omega^2$ (wts high ω)
- $\gamma(\omega) \sim$ const

[high freq. wiggles → kicks]

$$K(t) = \gamma^2 \delta(t) \quad \delta(t-\tau)$$

→ localized kernel

no history

and

$$\frac{dp}{dt} = -U'(X(t)) - \int_0^t \gamma^2 \delta(s) \frac{p(t-s)}{m} + F_p(t)$$

$$\downarrow$$

$$\frac{\gamma^2}{m} p(t)$$

Markovian dynamics
→ bath at high freq. only

Markovian Langevin eqn. structure.

V4.

1". – density of states → $\gamma(\omega)$ – γ const } determine memory kernel.

→ spectral distribution of scatterers!

What about noise?

$$F_p(t) = \sum_j \gamma_j \, p_j(0) \frac{\sin \omega_j t}{\omega}$$

$$+ \sum_j \gamma_j \left(q_j(0) - \frac{\gamma_j}{\omega_j^2} x(0) \right) (\cos \omega_j t)$$

$p_j(0)$, $q_j(0)$ distributed according:

$$f_{eq} \cong \exp[-H_B/T]$$

so

$$\left\langle \left(q_j(0) - \frac{\gamma_j}{\omega_j^2} x(0) \right)^2 \right\rangle = \frac{T}{\omega_j^2} \quad \text{etc.}$$

$$\langle p_j(0)^2 \rangle = T$$

Can deduce $\langle F_p^2 \rangle$.

$\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right)$ overshoot

25.

stability ⟷ structure of Q ← Kubo

F D T.: Memory Kernel → damping

$\boxed{\langle F_p(t) F_p(t') \rangle = T\, k(t-t')}$

(recall $\langle \tilde{f}(t) \tilde{f}(t') \rangle = \gamma T\, \delta(t-t')$).

Next: Fokker-Planck Theory!

→ T vs LCO

stability

stable → no LCO

$rT \ll 1$

FP

unstable → LCO

$rT \gg 1$

Kernel

Physics

Memory

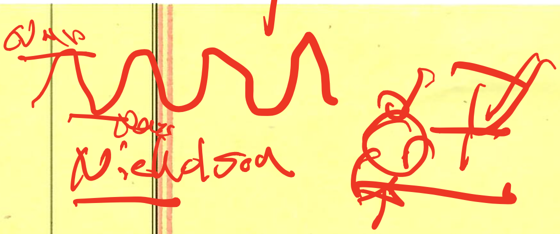
Limit Cycle

structure distribution of Memory

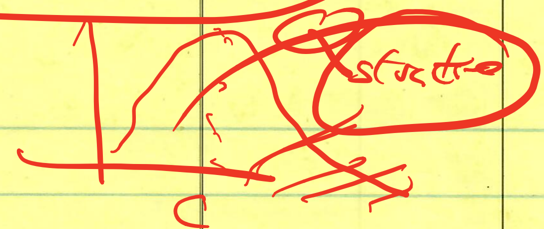
QLT+

Time Delay → not easy

$r\left[1 - N(t-T)\right] N \rightarrow r\left[1 - \int Q(t-t')\, N(t')\right] N(t)$



response for turbulence energy



→ Upshot: Memory Kernel

Prey → Fletn → $\mathcal{E}$

instab $\alpha \to 1 - \mathcal{E}^2$

$\ddot{y} + (1 - y^2)\dot{y} + \omega^2 y = 0$ Van-der-Pol

$\ddot{y} + \omega \dot{y} + \omega^2 y = 0$

Any modal interaction theory, etc.

$\Rightarrow$

$$\partial_t \mathcal{E}_{k,\omega} + d_{k,\omega} \mathcal{E}_{k,\omega} = \ldots \qquad \sim \mathcal{E}^2 \qquad \mathcal{E}^2$$

coupling coeff

$$d_{k,\omega} \cong \sum_{k',\omega'} CC(k, k') \, R_{\substack{k+k' \\ \omega+\omega'}} \mathcal{E}_{k',\omega'}$$

resonance

Non-Markovian $\quad k' \lesssim k, \; \omega \quad \omega'$

- has form of memory-weighted resource limitation effect.

- Non-Markovian structure ⟷ ~ time delay

→ Spectral / coupling structure ⇒ strength of resource limitation.

→ effective time delay.

spectral dist of memory function eff. Time delay → $Q(t-t')$ → spectral distributio

$P(t-\boxed{t'})$

What if K-carrying capacity — varies?

Stochastically

e.i.e.)

– Multiplicative Noise, (Simple)
('Noisey' Coefficients)

Consider Logistic Eqn → Populations

$$\frac{dN}{dt} = N(k - N)$$

N → Malthusian growth (exponential)

$(k-N)$ → saturation by competition $\sim N^2$

$N \equiv$ # of population

$$x_{n+1} = \alpha x_n (1 - x_n)$$

Logistic Map

$N=0$, $N=k$ are fixed pts

Now, could consider variability in k, and treat as stochastic variable

$$\frac{dN}{dt} = N\left(k_0 + \tilde{\gamma}(t) - N\right) + \tilde{\kappa}(t)$$

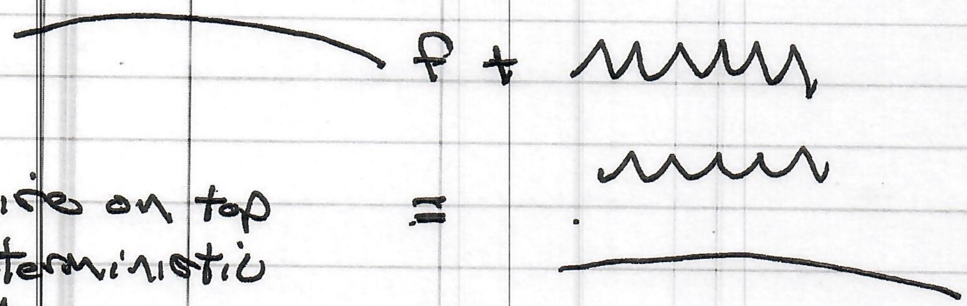
$\tilde{\gamma}(t)$ → variability in resources

⇒ multiplicative noise
→ rate

$\tilde{\kappa}(t)$ → external input variability

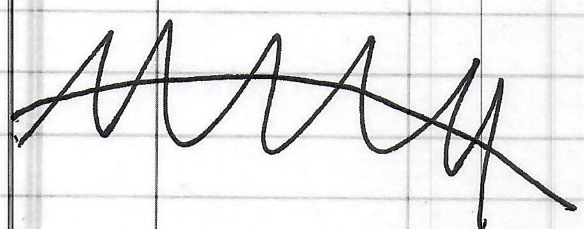
⇒ additive noise.

i.e. additive:



noise on top deterministic base

Multiplicative:



multiplies by fast, random quantity

$f(N, t)$ → population pdf
↓
→ Fokker–Planck Equation for $f(N)$

→ here $\langle \tilde{\gamma}(t)\, \tilde{\gamma}(t') \rangle = |\tilde{\gamma}_0|^2 \tau_{ac}\, \delta(t-t')$
(simple case).

Delta correlated for simplicity.

N.B. This is a "textbook model".

→ additive, as usual

$$\langle \tilde{\alpha} \tilde{\gamma} \rangle \doteq 0$$

Then : $$\frac{dN}{dt} = N\left(\gamma_0 + \tilde{\gamma}(t) - N\right) + \tilde{\alpha}$$

&

$$\frac{\partial}{\partial t} f(N,t) = -\frac{\partial}{\partial N}\left[\left(\gamma_0 N - N^2\right) f(N,t) - \frac{\partial}{\partial N}\left(D\, f(N)\right)\right]$$

For D:

$$\langle \Delta N \, \Delta N \rangle = \int^t dt' \int^t dt'' \, \langle \tilde{\gamma}(t') \tilde{\gamma}(t'') \rangle \, N^2 \; !$$

$$+ \int^t dt' \int^t dt'' \, \langle \alpha(t') \alpha(t'') \rangle$$

$$= |\gamma_0|^2 \tau_{ac} \, N^2 \, t + |\tilde{\alpha}_0|^2 \tau_{ac}$$

Nonlinearity in D

→ one trademark feature of multiplicative noise

→ Note: $N \to 0 \Rightarrow D \to 0$

Rate variation ⇒ Pdf spread in proportion to population.

→ Additive correction significant at low N.

Now, ignoring additive correction,

$$\boxed{\partial_t f(N) = -\frac{\partial}{\partial N}\left\{ (k_0 N - N^2) f(N,t) - \frac{\partial}{\partial N}\left(\frac{|\tilde{\gamma}_0|^2}{2} \tau_{ac} N^2 f(N,t) \right) \right\}}$$

is Fokker-Planck Equation

and stationarity:

$$N(k_0 - N) f(N) = \frac{\partial}{\partial N}\left(\frac{|\tilde{\gamma}_0|^2}{2} \tau_{ac} N^2 f(N) \right)$$

so
Norm

$$\boxed{f(N) = c \cdot n^{[2(k_0/\sigma^2)-2]}\, e^{-2N/\sigma^2}}$$

$\sigma^2 = |\tilde{\gamma}_0|^2 \tau_{ac}$

Power

exponential tail

Need $k_0^2 > (\sigma^2/2)^2 \Leftrightarrow f > 1/n$ $\quad n \to 0$

i.e. $\boxed{k_0 > \frac{|\tilde{\gamma}_0|^2 \tau_{ac}}{2}}$

to avoid log. singularity $\int f(N)\, dN$.

Physics of $k_0 > \frac{|\tilde{\gamma}_0|^2 \tau_{ac}}{2}$?

Growth big enough to avoid extinction due flctns

Convenient to linearize around fixed point:

Validity ?

$$\frac{dN}{dt} = (k + \tilde{\gamma} - N) N$$

$$N = k_0 + \tilde{n}$$

$$\frac{d\tilde{n}}{dt} = (k_0 + \tilde{n})(\cancel{k_0} + \tilde{\gamma} - \cancel{k_0} - \tilde{n})$$

$$\simeq k_0 \tilde{\gamma} - k_0 n + O(n^2)$$

so

$$\partial_t f(n) = -\frac{\partial}{\partial n}\left[-k_0 n f(n) - \frac{\partial}{\partial n}\left(\frac{k_0^2 \gamma_0^2 \tau_{ac}}{2} f(n) \right) \right]$$

to

linearize abt fixed pt.

$$= -\frac{\partial}{\partial n}\left[-k_0 n f(n) - \frac{\partial}{\partial n}\left(\frac{k_0^2}{2} \Gamma^2 f(n) \right) \right]$$

$\Rightarrow$ zero flux / stationarity:

$$\boxed{f(n) = \underbrace{c}_{\text{const}} \exp\left[-n^2 / k_0 \Gamma^2 \right]}$$

Valid for: $\left\langle \left(\tilde{n} / N_0 \right)^2 \right\rangle = \left\langle \left(\tilde{n} / k_0 \right)^2 \right\rangle < 1$

Now $\langle \tilde{n}^2 \rangle = \frac{\Gamma^2 k_0}{2}$

so $\left\langle \left(\tilde{n} / k_0 \right)^2 \right\rangle < 1 \Rightarrow \boxed{\frac{\Gamma^2}{2 k_0} < 1}$

→ again ; $\underbrace{\sigma^2 < 2k_0}$

i.e. fluctuations small compared to logistic growth.

N.B. :

- can determine time evolution
- can get moments
- spatio-temporal dynamics.