

Lecture IV - Nonlinear Waves I

Next goal { avalanches
turbulence spreading }

Aval-Prey
LCOs ~~fronts~~
Transitions
→ ~~fronts~~
→ Fisher
→ FN

→ What's an avalanche?

pulse train

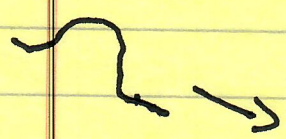
~ a pulse of a thermodynamic variable T, η , etc., or/and its flux, which traverses a distance L at a speed C s/t

$$C/L > \langle D \rangle / L^2$$

$$\langle D \rangle \sim D_{GB}$$

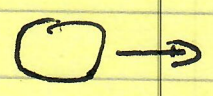
avg.

Can be a pulse train.



→ Pulse can be fluctuation intensity
→ spreading

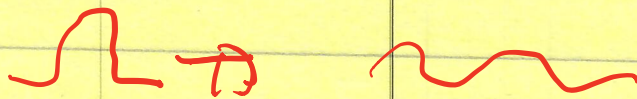
"spreading" in MFE



=
"Entrainment" in fluids

Structure of Turbulent Shear Flow
A.A. Townsend

→ Why Nonlinear waves & avalanches?



"The basic idea of wave propagation is that some recognizable feature of the disturbance moves with a finite velocity."

Landau & Lifshitz
Whitham

- G. B. Whitham

"Linear and Nonlinear Waves"

↑
highly recommended!

cf: Fronts!



[Nonlinear waves are underpinning for
concepts re: avalanches and
spreading. → Fundamental!]

Burgers - avalanches

So:



- basics: gas dynamics ("simple waves") and plasma solitons
→ collisionless shocks

⇒ shock, pulse formation

⇒ Jump conditions

⇒ steepening vs. dispersion →
Ion Acoustic shock

Critical Point:

⇒ Entropy production

then:

② - kinematic waves, traffic flow,
Flood waves, bore.

③ - Avalanche physics: Micro-Macro.

④ - Turbulence spreading and
entrainment physics.

Gas Dynamic Shocks / "Simple Waves"

3

1D Gas Dynamics

of {Lundau/Lifshitz
Fluids
Whitham}

cont.

$$\rightarrow \partial_t \rho + \partial_x (\rho v) = 0$$

(continuity)

$$\rightarrow \partial_t v + v \partial_x v + \frac{1}{\rho} \partial_x p = 0$$

mom.

(momentum balance)

$$p = p(\rho)$$

Entropy: - S initially homogeneous

$S = \text{const}$, till shock forms

pulse

$$p = p(\rho)$$

speed vs ρ rel.

1D

so

pulse follow rho.

flow

$$\frac{d\rho}{dt} = 0$$

along

$$\left(\frac{\partial x}{\partial t} \right)_p = \frac{d(\rho v)}{d\rho}$$

$$\partial_t \rho + \frac{d(\rho v)}{d\rho} \partial_x \rho = 0$$

$$\partial_t v + \left(v + \frac{1}{\rho} \frac{dp}{dv} \right) \partial_x v = 0$$

$$p = p(\rho)$$

$$v = v(\rho)$$

so

$$\left(\frac{\partial x}{\partial t} \right)_\rho = v + \rho \frac{dv}{d\rho}$$

$$\left(\frac{\partial x}{\partial t} \right)_\rho = \frac{d}{d\rho} (\rho v) = v + \rho \frac{dv}{d\rho}$$

n.b set in

$$\frac{d\rho}{dt} = 0$$

$$\frac{dv}{dt} = 0$$

characteristic for

characteristic
eqn. $\rightarrow$
all info

likewise:

$$\left(\frac{\partial x}{\partial t} \right)_v = v + \frac{1}{\rho} \frac{d\rho}{dv}$$



but

$$v = v(\rho)$$

$$\rho = \rho(\rho)$$

v is

$\Rightarrow$

$$\left(\frac{\partial x}{\partial t} \right)_\rho = \left(\frac{\partial x}{\partial t} \right)_v$$

1 pulse

$$\rho \frac{dv}{d\rho} + v = v + \frac{1}{\rho} \frac{d\rho}{dv}$$

$$d\rho = \rho^2 dv$$

$$\rho \frac{dv}{d\rho} = \frac{c_s^2}{\rho} \frac{d\rho}{dv}$$

$$(dv/d\rho)^2 = c_s^2/\rho^2$$

$$v = \pm \int \frac{c_s}{\rho} d\rho = \pm \int \frac{d\rho}{\rho c_s}$$

$\leftarrow$ velocity

solves for flow speed
in terms density

Note:

$$- V = \pm \int \frac{c_s}{\rho} d\rho$$

$$- \rho^{-\gamma} = \text{const.}$$

$$\gamma = 5/3$$

$$d\rho = c_s^2 d\rho = \alpha \gamma \rho^{\gamma-1} d\rho$$

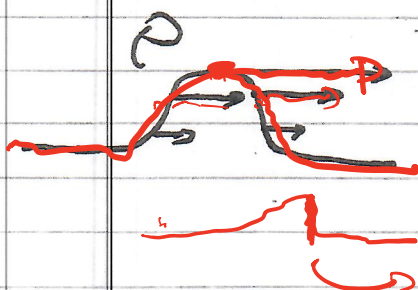
$$c_s^2 = \alpha \gamma \rho^{\gamma-1}$$

$$\alpha = \text{const}$$

$$V = \pm \int \frac{c_s^2}{\rho} d\rho = \pm \alpha \gamma \int \rho^{1/3} d\rho$$

$$V = \pm 3\sqrt{\alpha \gamma} \rho^{1/3}$$

- Flow speed increases with density



over-taken

- high density elements go faster

- then:

$$\frac{dx}{dt} = v + \frac{1}{\rho} \frac{d\rho}{dv}$$

$$v = \pm \int \frac{d\rho}{\rho c_s}$$

$$dv = \frac{d\rho}{\rho c_s}$$

$$\frac{dP}{dv} = \rho c_s$$

$$\rho = \rho(v)$$

$$\frac{\partial x}{\partial t + v} = v \pm c_s(v)$$

all cons - 1D - cont., EoM

$$x = t [v \pm c_s(v)] + f(v)$$

"simple waves" solution

all cons / nonlinear

Check: linearized limit

Non linear wave
→ no char.
ideal shock

$$x = t [\cancel{v} \pm c_s(v)] + \cancel{f(v)} + x_0$$

$$x = x_0 \pm c_s(v_0) t$$

→ Why "simple"?

No characteristic scale

→ alternate approach.

So if no characteristic scale
all quantities depend only on

$$\xi = x/t$$

Similarity soln.

$$f(x/t)$$

$$\rho = f(\xi)$$

$$\xi = x/t$$

$$F(x/t^2) \quad \underline{Z}$$

↳ Velocity formed by x, t in absence of c

so

$$\partial_x = 1/t \partial/\partial \xi$$

$$\partial_t = -\frac{\xi}{t} \frac{\partial}{\partial \xi}$$

$$\xi = \frac{x}{t}$$

$$\Rightarrow \partial_x \rho + \rho \partial_x v + v \partial_x \rho = 0$$

$$\partial_t v + v \partial_x v = -1/\rho \partial_x p$$

become:

$$-\frac{\xi}{t} \rho' + \frac{v}{t} \rho' + \frac{v}{t} \rho' = 0$$

$$1 = \partial/\partial \xi$$

$$-\frac{\xi}{t} v' + \frac{v}{t} v' = -\frac{c^2}{t} \frac{\rho'}{\rho}$$

so

$$(v - \xi) \rho' + \rho v' = 0$$

$$(v - \xi) v' = -c^2 \frac{\rho'}{\rho}$$

like linear dispersion theory, treat ξ as eigenvalue.

$$(v - \xi) \rho' + \rho v' = 0$$

$$-c^2 \frac{\rho'}{\rho} + (v - \xi) v' = 0$$

$$\frac{d\rho}{d\xi}$$

$$\frac{dv}{d\xi}$$

$$(v - \varepsilon)^2 = c_s^2$$

$$\Sigma = v \pm c_s$$

$$\left(\frac{x}{t}\right) = v \pm c_s$$

$$x = (v \pm c_s) t$$

From "eigenvector", relate v, ρ etc.

$$v - \varepsilon = -c_s$$

$$(v - \varepsilon) \rho' + \rho v' = 0$$

$$-c_s \rho' + \rho v'$$

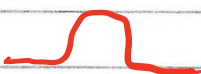
$$\rho dv = c_s d\rho$$

$$dv/d\rho = c_s/\rho$$

$$v = \int c_s d\rho/\rho = \int \underline{dp}/c_s \rho$$

- equivalent to previous, with $f(v) = 0$.

- corresponds to Kundu's version of simple wave!



$$x = \text{c.c.} + t(v \pm c_s)$$

Simple
wave

- can also write:

$$v = \int + (-dp dV)^{1/2}$$

$$d(1/\rho) = dV = -\frac{1}{\rho^2} d\rho, \quad dp = c^2 d\rho$$

$$\begin{aligned} v &= \int \left(\frac{1}{\rho^2} d\rho c^2 d\rho \right)^{1/2} \\ &= \int \frac{d\rho}{\rho} c \end{aligned} \quad \checkmark$$

→ Physics of Simple Wave:

no sub
on
adm.

- have established simple wave in gas dynamics, with key element of scale invariance

- now elucidate simple wave physics
⇒ shocks

Similarity Flow
 $f(v) = 0$

⇒ simple wave with

$$x = t [v \pm c(v)]$$

wave
front
refl.

Can write general solution,
for adiabatic process:

$$\rho \rho^{-\gamma} = \text{const}$$

$$T \rho^{-(\gamma-1)} = \text{const}$$

$$T = c_s^2 = \rho^{\gamma-1}$$

$$\frac{dP}{d\rho} = c_s^2 = T$$

$$P = \rho T$$

$$\rho^{-1} T^{1/\gamma-1} = \text{const}$$

as $c_s^2 \sim T$

$$\gamma - 1 = 2/3$$

$$\rho = \rho_0 \left(\frac{c}{c_0} \right)^{2/(\gamma-1)}$$

(eliminate
→ clarity
notation)

but $V = \pm \int c \frac{d\rho}{\rho}$

$$c = c_0$$

$$V = \pm \frac{2}{\gamma-1} \frac{(c - c_0)}{c_0}$$

so finally can write:

$$f(v)$$

11

$$c = c_0 \pm \frac{1}{2}(\gamma-1)v$$

$$\rho = \rho_0 \left(1 \pm \frac{(\gamma-1)v}{2c_0} \right)^{\frac{2}{\gamma-1}}$$

$$p = p_0 \left(1 \pm \frac{(\gamma-1)v}{2c_0} \right)^{\frac{2\gamma}{\gamma-1}}$$

so then:

$$x = t [v \pm c_0(v)] + f(v)$$

$$x = t \left[v \pm \left(c_0 \pm \frac{1}{2}(\gamma-1)v \right) \right] + f(v)$$

Now:

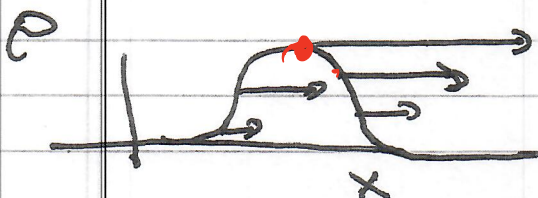
— point on wave profile moves at

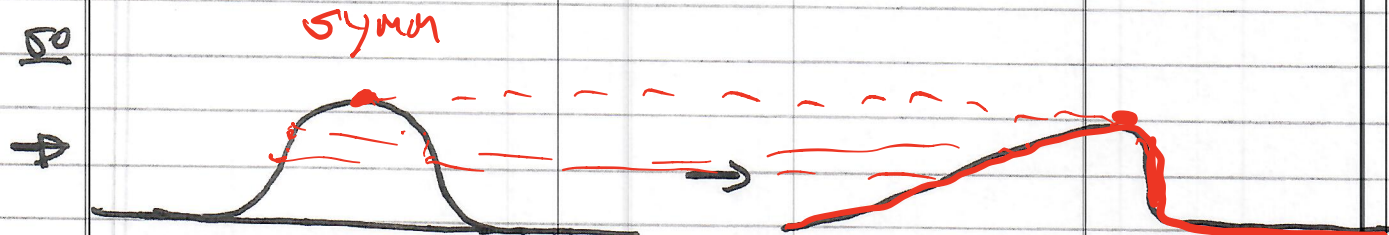
$$u = \frac{\partial x}{\partial t} = v \pm c_s$$

$$dv/d\rho > 0$$

and $dv/d\rho > 0$

→ [speed increases with density!]



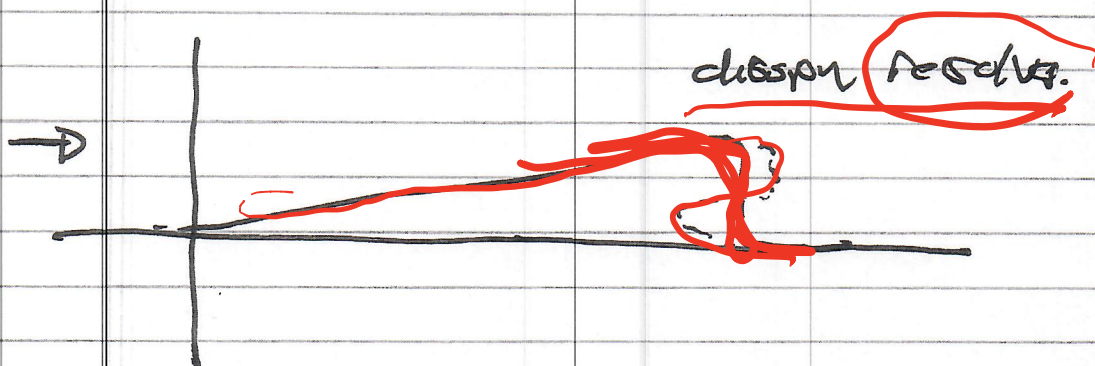
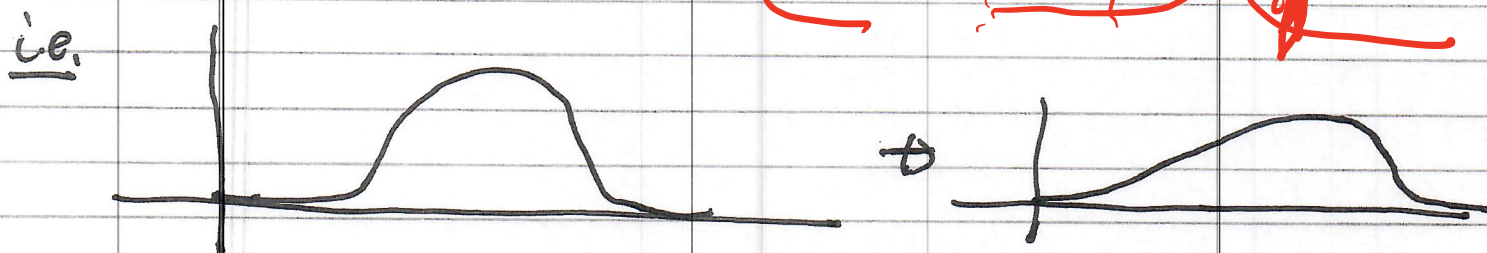


if $\frac{d\phi}{dx} \rightarrow \infty$ anywhere in i.v.d.

$\Rightarrow$ overtaking and shock formation

discontinuity forms

dissipation no longer negligible
(prevents ckt value)



When does breaking / shock formation occur?

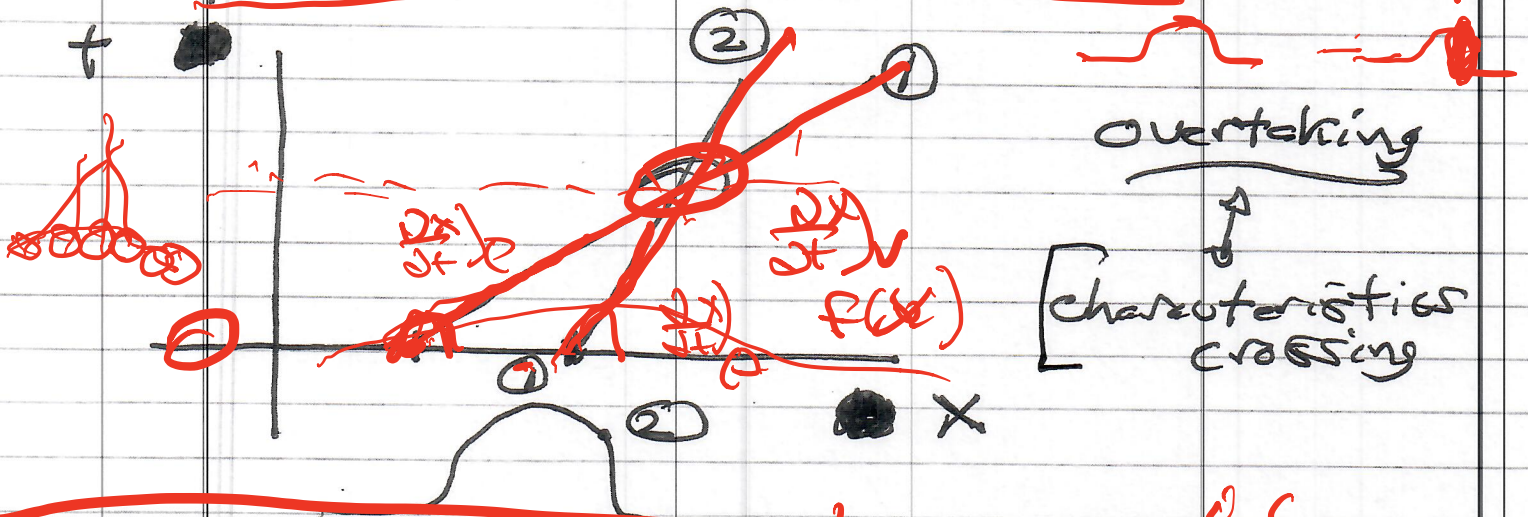
$$V \pm C_s \leftarrow \frac{\partial x}{\partial t} \right)_p$$

$$\rightarrow \text{characteristic eqns} \\ \frac{dx}{dt} = 0 + \frac{1}{\rho} \left(p + \frac{\partial p}{\partial x} \right) \frac{dp}{dx} = 0$$

13

Full nonlinear characteristic eqns.

→ Breaking occurs when 2 critical points arrive at same position at same time.



$$x = t \left[C_0 + \frac{1}{2} (\gamma + 1) v \right] + f(v)$$

$$x = t \left[C_0 + \frac{1}{2} (\gamma + 1) (v + dv) \right] + f(v + dv)$$

intersect:

$$C_0 t + \frac{1}{2} (\gamma + 1) v t + f(v)$$

$$= C_0 t + \frac{1}{2} (\gamma + 1) t v + dv \frac{1}{2} (\gamma + 1) t + f(v) + dv f'(v)$$

$$0 = dv \left[\frac{1}{2} (\gamma + 1) t + f'(v) \right]$$

Difference $\rightarrow 0$ $\equiv$

$$\dot{x} = f(v, I_G) + F(v)$$

$$t_{\text{shock}} = \frac{-2P(v)}{(\gamma+1)}$$

shock formation

time to form shock

need $f'(v) < 0$ for overtaking

all set by i.c.

Generally i need:

$$\left(\frac{\partial x}{\partial v}\right)_t = 0$$

$$\left(\frac{\partial^2 x}{\partial v^2}\right)_t = 0 \quad \text{inflection}$$

HW $\rightarrow$ work all

Geo Dyn. $\rightarrow$ shock

high caught

Xshock

$$\left(\frac{\partial x}{\partial v}\right)_t = \frac{1}{2}(\gamma+1) + \dot{x}P = 0 \quad \checkmark$$

~ formation.

→ Shocks - Flows with Discontinuity

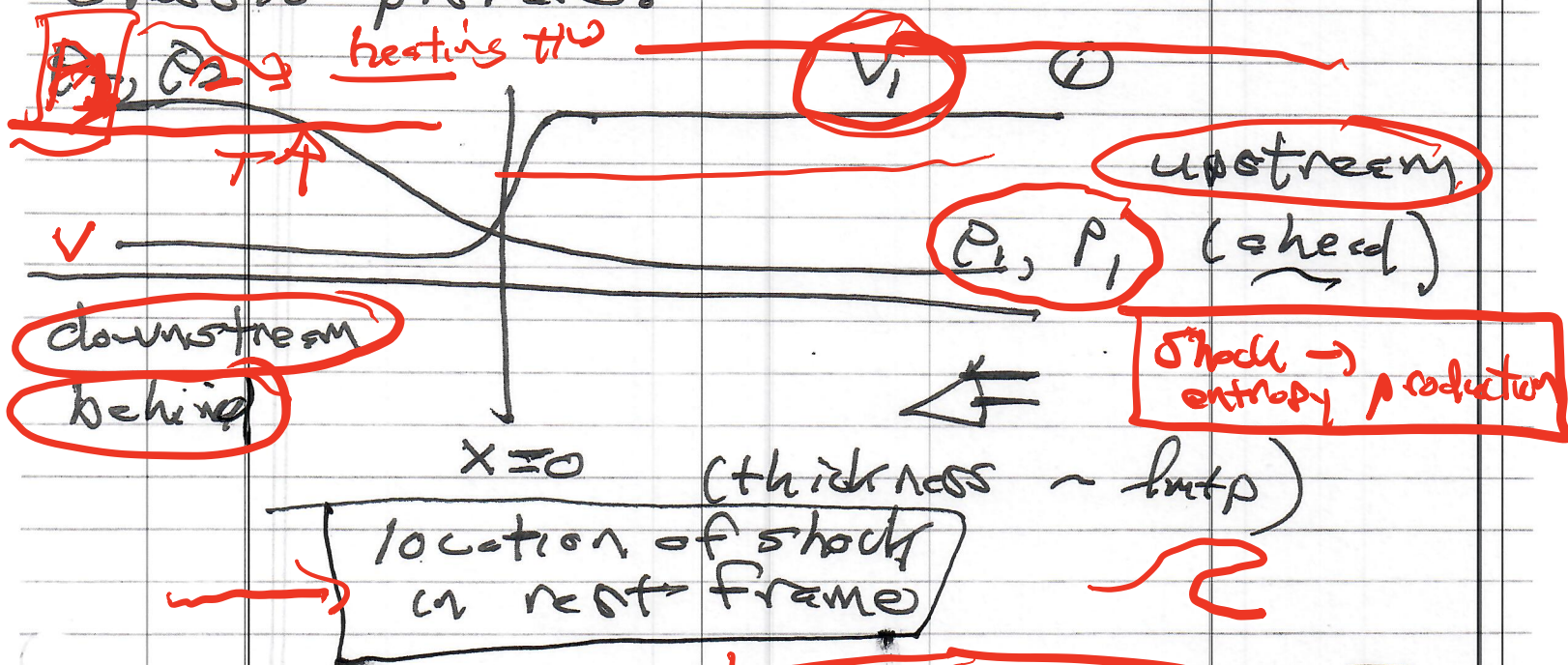
- once wave steepen and break rapid jump

→ discontinuity appears →

- "shock" - localized region of rapid change/discontinuity

- dissipation (thickness essential in shock length) fluid $\lambda \gg \text{length}$

Classic picture:



→ Shock converts kinetic energy into thermal energy, compression (engine)

→ motion
→ per elec -> heat

Entropy Production $\rightarrow$ Insert

1

Supplement: Gas. Dynamic Shocks
Lecture

a.) Scale

$P=0$

HW $\rightarrow$ Solve

(gen.)

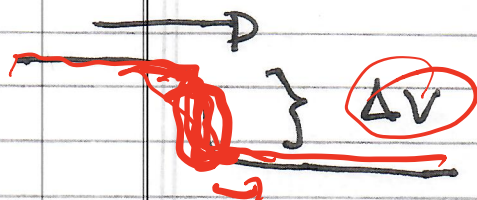
for

Burgers

(with flow)

Consider Burgers Egn.:

$$\partial_t V + v \partial_x V - \nu \partial_x^2 V = 0$$



$$\frac{V}{w} \sim \frac{V}{w^2} V$$

$$w \sim \sqrt{V / \Delta V}$$

for shock width

$$\frac{\Delta V}{w} \sim \nu \frac{\Delta V}{w^2}$$

$$w \sim \sqrt{\nu / \Delta V}$$

discontinuity
sets shock
thickness

for $\nu \sim l_{mfp} c_s$

$$\Delta V \sim c_s$$

$$w \sim l_{mfp}$$

characteristic
thickness of
shock layer.

6.1 Entropy Production

In gas-dynamics

- initially ideal dynamics
- entropy constant

but



pulse steepens and shocks

- sharp gradients produced in shock

⇒ couple to diffusive dissipation

⇒ drive collisional transport ✓

⇒ produce entropy

N.B. : Entropy production required in shock

- sets arrow of time.

- n.b. expansion unstable.

flex force

- to calculate:

$$\frac{dS}{dt} = \left(\frac{1}{k_B} \right) \left(- \Gamma_x \nabla X \right) \rightarrow$$

factor #

$\nabla X \rightarrow$ thermodynamic force

$\Gamma_x \rightarrow$ Flux

Ficks law

i.e.

$$\Gamma_x = - D_x \nabla X$$

$$dS/dt \sim \left(\frac{1}{k_B} \right) D_x (\nabla X)^2$$

entropy production rate density

then, for Burgers shock:

$$dS/dt \sim \nu (\partial_x v)^2 \rightarrow \frac{\partial_x v}{\frac{1}{W} \sim \frac{\Delta v}{\delta x}}$$

$$\sim \nu \frac{(\Delta v)^2}{W^2} \sim \left(\nu (\Delta v)^2 \frac{(\Delta v)^2}{\nu^2} \right)$$

$$\frac{dS}{dt} \sim \frac{(\Delta v)^4}{\nu}$$

local entropy prod

$D_x (\nabla X)^2 d(\text{shock})$

but, total entropy production ~~is~~
is integrated over shock thickness

$$\int dx \frac{dS}{dt} \sim \frac{dS}{dt} \sim \cancel{w} \frac{dS}{dt}$$

$$\frac{1}{w} \sim \left(\frac{v}{\Delta v} \right)^2$$

$$w \sim \left(\frac{\Delta v}{v} \right)^2$$

$$\cancel{w} \sim \frac{v}{\Delta v}$$

$$\sim \frac{v}{\Delta v} \sim (\Delta v)^2 (\Delta v)^2$$

$$\frac{dS_{tot}}{dt} \sim (\Delta v)^3$$

total entropy prod rate indep of v

so total entropy production:

$$dS/dt \sim (\Delta v)^3$$

entropy production independent of v though v required for heating.

- independent v !

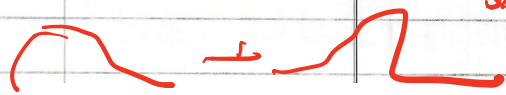
- entropy / heating produced by collisions but total dS/dt independent of v .

shocks $\leftrightarrow$ Turbulence

couple to each other in a similar way

finite time scale

shock



$$f(u) \rightarrow \frac{\partial x}{\partial t} =$$

$$x = t(u \pm f(u)) + f(u)$$

15.

Conservation relations $\Rightarrow$ jump conditions

$$\partial_t \rho + \nabla \cdot (\rho \underline{v}) = 0 \quad \text{continuity, mass}$$

$$\partial_t \left(\frac{1}{2} \rho v^2 + \rho e \right) + \nabla \cdot \left(\rho \underline{v} \left(\frac{v^2}{2} + \frac{\gamma p}{(\gamma-1)\rho} \right) \right)$$

$\Rightarrow$

energy

$$\frac{\gamma p}{\gamma-1} = \frac{p}{\gamma-1} + p$$

↓
energy density

$\hookrightarrow$ PV work on surroundings

$$\frac{\partial \rho v_i}{\partial t} = - \frac{\partial \pi_{ik}}{\partial x_k} \quad \hookrightarrow \text{stress tensor}$$

$$\pi_{ik} = p \delta_{ik} + \rho v_i v_k$$

$\leadsto$ tangential components v continuous

= so integrating,

Work in shock frame - $U=0$

$$\partial_t \rho = -U \partial_x \rho$$

$$-\int U \partial_x \rho = (\rho_2 - \rho_1) U$$

but $U=0$, frame

mass

$$\boxed{\rho v_x \Big|_2 = \rho v_x \Big|_1} \quad (1)$$

continuity
mass flux

energy

$v_n \equiv v_{\perp}$ plane

$$\rho v_n \left(\frac{1}{2} v^2 + \frac{\gamma P/\rho}{\gamma-1} \right) \Big|_2 = \rho v_n \left(\frac{1}{2} v^2 + \frac{\gamma P/\rho}{\gamma-1} \right) \Big|_1$$

but

$$\rho v \Big|_2 = \rho v \Big|_1$$

and continuity parallel $\rightarrow$

$$V_{y,z}^2 \textcircled{2} = V_{y,z}^2 \textcircled{1}$$

02

$$\left(\frac{V_x^2}{2} + \frac{\gamma P / \rho}{\gamma - 1} \right) \textcircled{2} = \left(\frac{V_x^2}{2} + \frac{\gamma P / \rho}{\gamma - 1} \right) \textcircled{1}$$

and momentum conservation:

$$\left(\rho V_x^2 + P \right) \textcircled{2} = \left(\rho V_x^2 + P \right) \textcircled{1}$$

normal component, only, varies.

03, have 3 Rankine-Hugoniot jump/continuity conditions $\rightarrow$

$$[] = () \textcircled{2} - () \textcircled{1}$$

$$\left[\begin{aligned} \rho v_x &= 0 \\ \frac{v_x^2}{2} + \frac{\gamma p}{\gamma - 1} &= 0 \\ \rho v_x^2 + p &= 0 \end{aligned} \right]$$

in shock frame
→ relate
upstream/
downstream
quantities

in Fixed coordinate frame:

$$v_x = v_n - U$$

$\downarrow$ normal v in fixed coords $\rightarrow$ shock velocity

then on to:

- shock adiabatic. $\rightarrow$ entropy production
- $v_2/v_1, \rho_2/\rho_1, p_2/p_1$
 - $\downarrow$ compression
 - $\downarrow$ heating

Shock - illustration - 'sonnet'

x & u.s.c.

Th. is a scalar invariant

Scale - non-dispersive → scale no scale is problem 4

Nonlinear Waves II

gas dyn shock

- critical study: gas-dynamic 'simple waves'
scale free
non-dispersive

- now: dispersive
scale
solitons → solitons
collisionless shock
→ ion acoustic
(simplest, not first)

Point: $\omega^2 = c_s^2 k^2 / (1 + k^2 \lambda_D^2)$

$$\left(\frac{\omega}{k}\right)^2 = c_s^2 / (1 + k^2 \lambda_D^2)$$

dispersion

$\lambda_D \rightarrow 0$: gas dynamic limit
non-dispersive

⇒ all harmonics have same phase velocity

$$\left(\frac{\omega}{k}\right)^2 = c_s^2$$

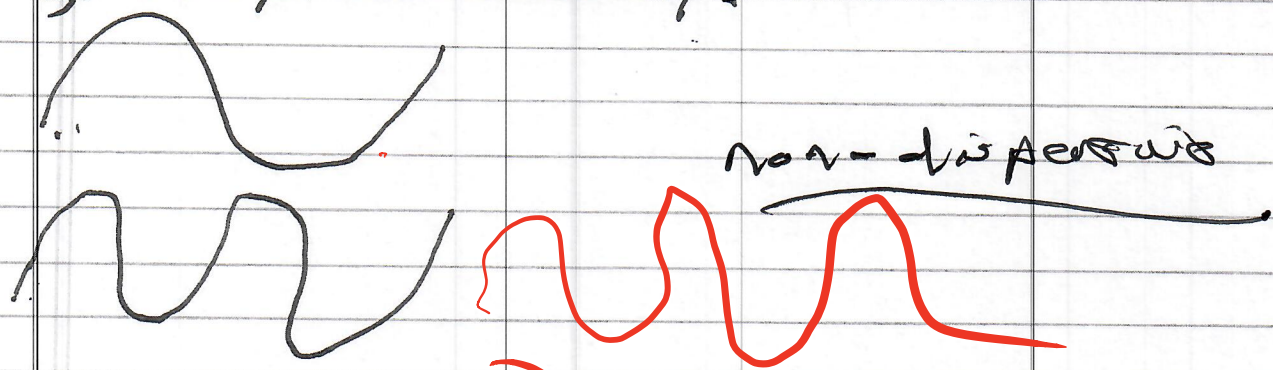
on dispersion

self-best:

$$\left(\frac{\omega}{2k}\right)^2 = c_s^2$$

etc.

As wave steepens by nonlinear interaction, all harmonics move at c_s as phase velocity.



but with dispersion:

$$(\omega/k)^2 = c_s^2 / (1 + k^2 \lambda_D^2)$$

$$\left(\frac{\omega}{k}\right)^2 = c_s^2 / (1 + k^2 \lambda_D^2)$$

$$\omega_{ph}^2 = c_s^2 / (1 + 2n(k) \lambda_D^2)$$

then fundamental and self-beats disperse → spread in speeds.

2 routes to balance of steepened fronts:

n.l. steepening vs. dissipation $\nabla \cdot \mathbf{v}$
 dissipative structure → shock
 (conventional)

n.l. steepening vs. dispersion

→ soliton (1D)

i.e. Schematic: ~~1D~~ 1D called "collisionless shock".

recall 1D compressible hydro-Burgers Eqn.

$$\partial_t V + V \partial_x V = \nu \partial_x^2 V$$

 ΔV

$$\Delta V \frac{\Delta V}{\Delta x} \sim \nu \frac{\Delta V}{(\Delta x)^2}$$

$$\boxed{\Delta x \sim \nu / \Delta V} \rightarrow \text{shock layer thickness}$$

↓
dispn sets → ν vs collisions.

Now, can generalize:

$$\omega^2 \equiv k^2 C_s^2 / (1 + k^2 \lambda_D^2)$$

$$\omega \approx k C_s \left(1 - \frac{k^2 \lambda_D^2}{2} \right)$$

1D

$$\boxed{\frac{\partial \epsilon}{\partial t} + C_s \partial_x \epsilon + \epsilon \partial_x \epsilon = \nu \partial_x^2 \epsilon + \frac{C_s \lambda_D^2}{2} \partial_x^3 \epsilon}$$

~~do not~~

disorder

dispersion

steepness vs. dispersion

4

$C_s \lambda_D \rightarrow \infty \Rightarrow$ Burgers

steepness vs. dispersion

$\gamma \rightarrow 0 \Rightarrow$ KdV Korteweg-de Vries

KdV

$$\partial_t \epsilon + C_s \partial_x \epsilon + \epsilon \partial_x \epsilon = \frac{C_s \lambda_D^2}{2} \partial_x^3 \epsilon$$

dispersion controls steepening

~~do not~~

$\frac{\epsilon}{\Delta} \sim \frac{C_s \lambda_D^2}{2} \frac{\epsilon}{\Delta^3}$

$\Delta^2 \sim \frac{C_s \lambda_D^2}{2 \epsilon}$

$\Delta \sim \left(\frac{C_s}{2 \epsilon} \right)^{1/2} \lambda_D$

Collisionless shock

steepness vs. dispersion

scale

n.b. Where does KdV come from in hydro?

- surface wave: $\omega^2 = gk$

- surface wave with finite depth:

$\omega^2 = gk \tanh(kd)$

then expanding:

$\omega^2 = k^2 g d + g k^3 \left(-\frac{k^2 d^3}{3} \right)$

shallow H_2O

$$\omega^2 = \omega_{SD}^2$$

18

$$\omega^2 = k^2 g d \left(1 - \frac{k^2 d^2}{8} \right)$$

$$\omega^2 = \frac{k^2 c_s^2}{1 + k^2 \lambda_D^2} \rightarrow k^2 c_s^2 \left(1 - \frac{k^2 \lambda_D^2}{2} \right)$$

19

$$\partial_t \Sigma + v_0 \partial_x \Sigma + \Sigma \partial_x \Sigma = - \frac{v_0}{\omega_{pe}^2} \partial_x^2 \Sigma$$

$$v_0 = (gd)^{1/2}$$

$v_0 \partial_x \Sigma$

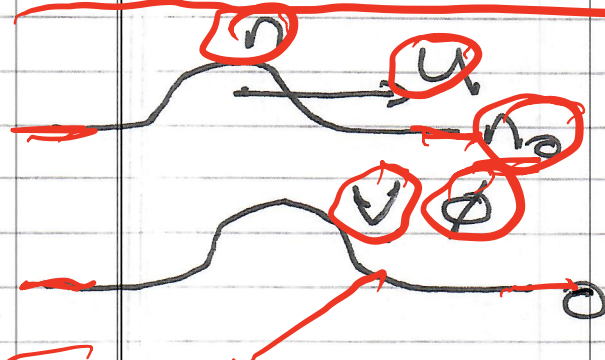
check for solitons

Now, for ion-acoustics:

NL ion-acoustic soliton

$$n_e = n_0 \exp[|e| \phi / T_e]$$

Boltzmann electrons



$$\partial_t n_i + \partial_x (n_i v) = 0$$

$$\partial_t v_i + v_i \partial_x v_i = - \frac{|e|}{m_i} \frac{\partial \phi}{\partial x}$$

Fluid ions

form

$$\left\{ \begin{matrix} n_i \\ v_i \\ \phi \end{matrix} \right\} = F(x - \underset{\substack{\uparrow \\ \text{speed}}}{u} t)$$

standard form of NL pulse.

$$-u n_i' + (n_i v)' = 0$$

$$(v - u) v' = -\frac{q}{m_i} \phi'$$

integrate: $\left. \begin{matrix} \phi \rightarrow 0 \\ v \rightarrow 0 \\ n \rightarrow n_0 = 1 \end{matrix} \right\} |x| \rightarrow \infty$

$$-u n_i + n_0 v = -u \quad (\text{b.c.})$$

$$(u - v) n_i = u$$

$$n_i = u / (u - v)$$

take wire:

$$-\frac{q\phi}{m_i} = \frac{v^2}{2} - \frac{2uv}{2} + \frac{u^2}{2} - \frac{u^2}{2}$$

$$\frac{q\phi}{m_i} = -\frac{1}{2} (u - v)^2 + \frac{u^2}{2}$$

b.c. v.

$$n_i(\phi)$$

$$n_e(\phi)$$

$$\underline{so} \quad \underline{(u-v)} = \left(u^2 - \frac{2q\phi}{m_0} \right)^{1/2}$$

 $\Rightarrow$

$$\partial_x^2 \phi = -4\pi m \Sigma \left(n_i - n_e \right)$$

$$\partial_x^2 \left(\frac{q\phi}{T} \right) = -\frac{1}{\lambda_D^2} \left(\frac{1}{\left(1 - 2q\phi/T \right) \left(c^2/u^2 \right)^{1/2}} \right)$$

$$\exp \left(\frac{q\phi}{T} \right)$$

$$\frac{q\phi}{T} \rightarrow \phi$$

$$\partial_x^2 \phi = -\frac{1}{\lambda_D^2} \left(\frac{1}{\left(1 - \phi \frac{c^2}{u^2} \right)^{1/2}} - e\phi \right)$$

$$M^2 = u^2 / c_s^2$$

NL Wave eqn.

Mach #

Motion in Potential
(Sagdeev)

$$\phi' \partial_x^2 \phi = -\frac{1}{\lambda_D^2} \frac{\phi'}{\left(1 - \frac{c^2}{u^2} \right)^{1/2}} - e\phi \phi'$$

$$= - \left(dV(\phi)/d\phi \right) \phi'$$

and integrate:

Soliton potential

$$V(\phi) = -\frac{1}{\lambda_0^2} \left\{ \frac{m^2}{2} \left(1 - \frac{2\phi}{m^2} \right)^{1/2} - e^\phi \right\} + C$$

~~then~~ $\Rightarrow$

$$\frac{1}{2} \phi'^2 + V(\phi) = 0$$

integration const.

$\rightarrow$ sets possibilities

$$\phi'' = dV/d\phi$$

$\Rightarrow$ conservative problem reduced to particle orbit.

$$\phi'' = dV/d\phi$$

$$\ddot{x} = -dU/dx$$

811

$$\rightarrow m^2 > 2 \text{ vel } \hat{\phi}$$

critical velocity for soliton.

$\rightarrow$ small ϕ

$$V(\phi) \approx -\frac{1}{\lambda_0^2} \left\{ (1 + m^2) + \phi - \phi + \frac{1}{2} \phi^2 \left(\frac{1}{m^2} + 1 \right) \right\}$$

so need $m^2 > 1$

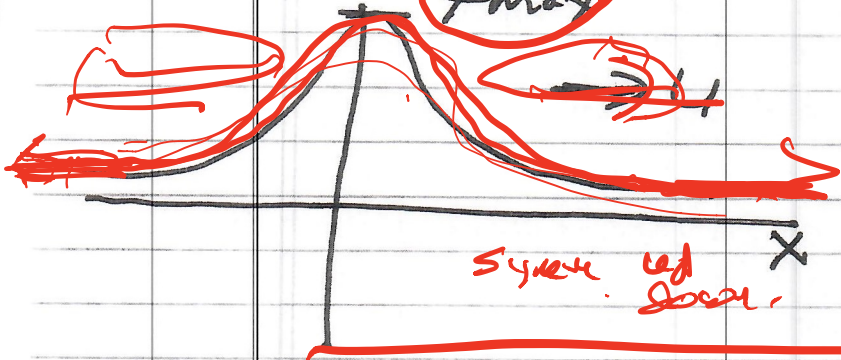
Now, choose:

$\phi' \rightarrow 0$

as

$\phi \rightarrow 0$

ϕ_{max}



$-\frac{1}{2} \frac{d}{dt} \int \phi^2 dx$

$\int \phi^2 dx = \frac{1}{2} \int \phi^2 dx$

Don't

U is

choos

Let's

low stream

UL pulse

→

disorder vs steepening

localized pulse

con-acoustic solution

$$M^2 = \frac{1}{2} \frac{[\exp \phi_{max} - 1]^2}{\exp \phi_{max} - 1 - \phi_{max}}$$

entropy production

Collisionless shock

low amplitude

collisionless shock

2-stream B-O-T decay → entropy

entropy production

$$\phi \approx \frac{3}{2} \left(1 - \frac{1}{M^2} \right) \text{sech}^2 \left\{ \sqrt{\frac{\pi n_0}{2}} \sqrt{1 - \frac{1}{M^2}} x \right\}$$

need $M^2 \geq 1.6$

In Acoustic

collisionless shock

can consider case,

→ Need discuss wave-particle interaction to deal with heating, entropy production etc

→ Sardar → Rev. A. H. S.
Vol. 4 KOL ✓

→ More general discussion (KdV):

Recall, had general: Apply to: Flood waves
(kinematics)
→ here

$$\partial_t \Sigma + (C_0 + \Sigma) \partial_x \Sigma + \frac{C_0}{2} \partial_x^3 \Sigma = 0$$

$$a = \Sigma$$

$$y = x - C_0 t$$

$$\partial_t q + a \partial_y q + \beta \frac{\partial^3 q}{\partial y^3} = 0$$

simple
KdV.

$$L_{\text{soliton}} \sim (\beta/a)^{1/2}$$

└──────────┘
scale amplitude reln.

Solving (reduced) KdV

(integrating)

$$a = a(y - ct)$$

⇒

$$\beta a''' - ca' + a a' = 0$$

invariant
 $a \rightarrow a + V$
 $c \rightarrow c + V$

$$\beta a'' - ca + \frac{a^2}{2} = \frac{\pm 1}{2} C_1$$

4
const

$$h \sim 1/\sqrt{\epsilon}$$

$\epsilon \gg \epsilon_{\text{trunc}}$

$$(2a') * \left(3a'' - c_0 + \frac{a'^2}{2} = \frac{c_1}{2} \right)$$

$$2(3a'a'' - 2c_0a' + \frac{1}{2}a'^2) = c_1a'$$

intr.

$$3a'^2 = -\frac{1}{3}a^3 + C_1a^2 + C_1a + C_2$$

and can now reduce to quadrature.

→ Convenient to factorize:

$$C_0, C_1, C_2 \rightarrow q_1, q_2, q_3$$

$$3a'^2 = -\frac{1}{3}(a-q_1)(a-q_2)(a-q_3)$$

$$C = \frac{1}{3}(q_1 + q_2 + q_3)$$

3 roots
again

For: → bounded $|a(y)|$

need q_1, q_2, q_3 real

$$\Rightarrow \begin{cases} \text{if: } q_1 > q_2 > q_3 \\ q_1 \geq q \geq q_2 \end{cases}$$

$q_3 = 0$ no loss generality

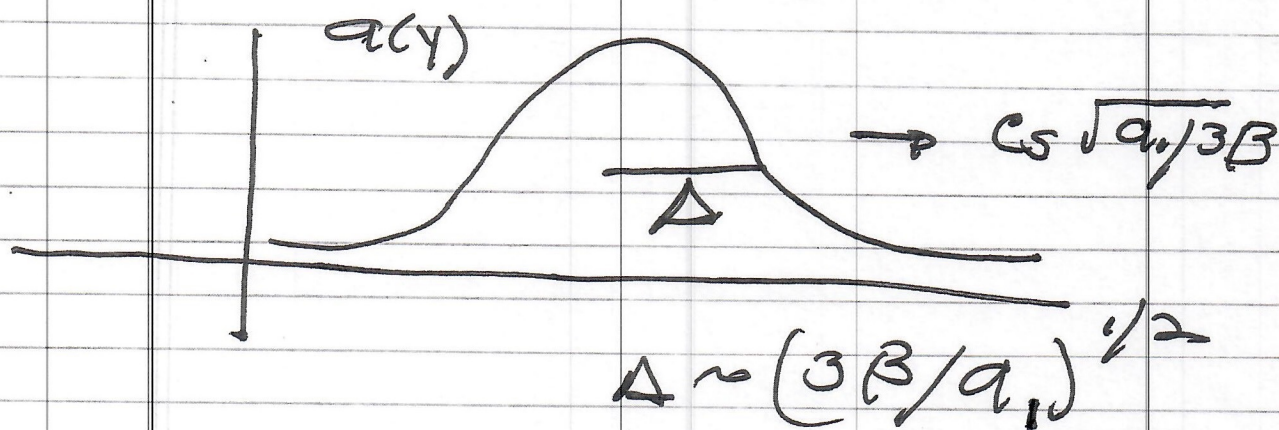
$$\Rightarrow \beta a'^2 = \frac{1}{3} (a_1 - a) (a - a_2) a$$

$$\text{if } a_2 = 0$$

$$a(y) = a_1 \cosh^{-2} \left(\frac{1}{2} y \sqrt{a_1 / 3\beta} \right)$$

$$\rightarrow a_1 \cosh^{-2} \left(\frac{1}{2} (x - ct) \sqrt{a_1 / 3\beta} \right)$$

So, have (as before)



Notes

$$\rightarrow U \sim c_s \sqrt{a_1 / 3\beta}$$

speed - amplitude

$$\rightarrow \Delta \sim (3\beta / a_1)^{1/2}$$

width - amplitude

Note:

→ Soliton has finite width

$$\Delta \sim (3B/a_1)^{1/2} \sim \lambda_{De}$$

contrast shock

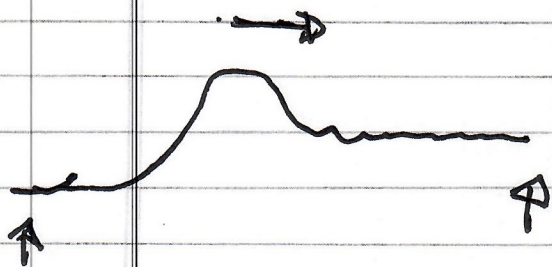
→ bigger solitons go faster

$$V \sim U_0 (a_1/3B)^{1/2}$$

Soliton → symmetric pulse

up/down stream
symmetric.

Collisionless shock → collisionless, asymmetry



entropy production?

TBC