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Power-law scalings in weakly-interacting Bose gases at quantum criticality

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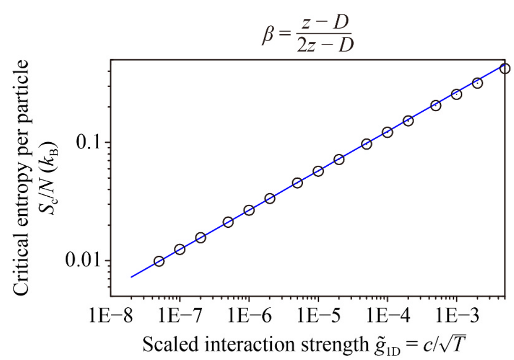
ABSTRACT

Weakly interacting quantum systems in low dimensions have been investigated for a long time, but there still remain a number of open questions and a lack of explicit expressions of physical properties of such systems. In this work, we find power-law scalings of thermodynamic observables in low-dimensional interacting Bose gases at quantum criticality. We present a physical picture for these systems with the repulsive interaction strength approaching zero; namely, the competition between the kinetic and interaction energy scales gives rise to power-law scalings with respect to the interaction strength in characteristic thermodynamic observables. This prediction is supported by exact Bethe ansatz solutions in one dimension, demonstrating a simple 1/3-power-law scaling of the critical entropy per particle. Our method also yields results in agreement with a non-perturbative renormalization-group computation in two dimensions. These results provide a new perspective for understanding many-body phenomena induced by weak interactions in quantum gases.



Keywords power-law scaling, quantum criticality, Bose gases, weak interaction, non-perturbative methods

1 Introduction

The properties of weakly interacting quantum systems are fundamental scientific problems of long interest [1–9]. The interaction effect becomes even more important at lower dimensions, where singularity effects (such as infrared divergence) are present in non-interacting gases [10, 11]. In the context of one-dimensional (1D) interacting gases, the ground-state properties have been extensively discussed based on the integral-equation method [2, 12, 13], but the analyticity of the ground-state energy with respect to a vanishing interacting strength remains an open question [14–21]. Furthermore, explicit expressions of physical properties of weakly interacting quantum gases in low dimensions (especially in 1D) are still very rare [10, 11].

Quantum criticality [22] provides a unique platform for understanding weakly interacting many-body systems. At and near a quantum critical point, quantum fluctuations and correlations play an important role, such that even very weak interaction will have a profound effect, which can be revealed by the dependence of physical quantities on the interaction strength. To reveal such novel interaction-induced many-body effect, however, it is difficult to perform perturbative studies because the fugacity can reach unity at a quantum critical point. Therefore, understanding the physical properties of these weakly interacting systems often requires non-perturbative research approaches such as exact solutions [7, 11], accurate numerical simulations [4, 23] and experiments [24, 25].

The determination of thermodynamic quantities and identification of the corresponding scaling laws have become important tools to access universal properties of interacting quantum systems [2, 26]. Along this route, considerable research progress on interacting atomic gases has been made based on theoretical studies [2, 26–28], numerical simulations [4–6, 29, 30], and ultracold atoms experiments [31–41]. Recently, some of the authors in this paper observed a novel power-law scaling for the critical entropy per particle in 1D and 2D Bose gases with strong and medium interactions, and identified such scaling as a smoking-gun signature of interaction-induced fractional exclusion statistics that emerges at quantum criticality [42]. In the distinct limit of vanishing interaction strength, finding novel quantum scalings can advance our understanding of quantum physics in the many-body regime.

In this work, we show that power-law scalings of characteristic thermodynamic quantities with respect to the interaction strength naturally emerge in low-dimensional Bose gases at quantum criticality under very weak repulsive short-range interactions. We consider the quantum gases that experience a phase transition from a vacuum to a quantum liquid when the chemical potential μ reaches a critical point $\mu_{\rm c}$. Here "quantum liquid" denotes Tomonaga–Luttinger liquid (TLL) [41] in 1D or superfluid in 2D [38, 43]. We explicitly derive the power-law scaling exponents, which simply rely on the dimensionality D and the dynamical critical exponent z. For $D=1$, we find strong evidence of the predicted power-law scalings, including a simple 1/3-power-law scaling of the critical entropy per particle, by exactly solving the Bose gas at the vacuum-to-TLL transition. For $D=2$, we use the same method to obtain scaling results in agreement with an independent non-perturbative renormalization-group study [30].

2 Prediction of a power-law scaling

When a quantum system experiences a quantum phase transition, it exhibits scaling behaviors with respect to temperature [22]. At and near a quantum critical point, the temperature T provides the sole independent energy scale, and a given thermodynamic quantity F can be expressed as a T-independent universal part ($\tilde{F}$) multiplied by a certain power of T, a phenomena known as scale invariance [22, 26, 27, 38]. The function $\tilde{F}$ reveals important information for a system in the quantum critical regime and in general depends on a dimensionless interaction strength $\tilde{g}$ [30, 38, 39]. To further simplify and formulate the physics problem, we denote the value of $\tilde{F}$ as $\tilde{F}_{\rm c}$ at the exact quantum critical point where quantum correlations have maximum influences. To facilitate the subsequent analysis, we will mainly discuss dimensionless physical quantities that are scaled by proper powers of temperature T. For example, the momentum $\boldsymbol{k}$, energy ϵ are scaled into $\tilde{\boldsymbol{k}}\equiv \boldsymbol{k}/T^{1/z}$, $\tilde{\epsilon}\equiv\epsilon/T$, respectively.

Here, we consider a generic D-dimensional quantum system with a dynamical critical exponent z and study the scaling of its thermodynamic quantities $\tilde{F}_{\rm c}$ with respect to a weak repulsive short-range interaction strength $\tilde{g}$. In particular, we focus on the critical entropy per particle, $F_{\rm c}=\tilde{F}_{\rm c}=S_{\rm c}/N$. This quantity $(S_{\rm c}/N)$ measures a critical level below which fluctuations must be suppressed in order for the system to show the onset of a quantum liquid phase, and is thus an important thermodynamic quantity closely related to the critical phase space density $\phi_{\rm c}$ and the critical temperature $T_{\rm c}$ of a Bose–Einstein condensate.

Based on the thermodynamics of a scale-invariant quantum gas, the critical entropy per particle can be obtained from the dimensionless scaled critical pressure $\tilde{p}_{\rm c}\equiv\tilde{p}(\mu=\mu_{\rm c})$ and scaled critical density $\tilde{n}_{\rm c}\equiv\tilde{n}(\mu=\mu_{\rm c})$ [26]:

$$\frac{S_{\rm c}}{N}\equiv\frac{S}{N}(\mu=\mu_{\rm c})=\frac{D+z}{z}\frac{\tilde{p}_{\rm c}}{\tilde{n}_{\rm c}}, \tag{1}$$

where p is the pressure, n the density, $\tilde{p}=p/T^{D/z+1}$ the scaled pressure, $\tilde{n}=n/T^{D/z}$ the scaled density, $\tilde{n}_{\rm c}=n_{\rm c}/T^{D/z}$, $\tilde{p}_{\rm c}=p_{\rm c}/T^{D/z+1}$, and we have set $2m=k_{\rm B}=\hbar=1$, with m being the particle mass, $k_{\rm B}$ the Boltzmann constant, and $\hbar$ the reduced Planck constant. For a gas with very weak interaction, the scaled critical pressure $\tilde{p}_{\rm c}$ can be approximated as that of a D-dimensional ideal gas at $\mu=\mu_{\rm c}=0$ [44], which is independent of the interaction strength $\tilde{g}$:

$$\tilde{p}_{\rm c}\approx\tilde{p}_{\rm ideal}(\mu=0)=A_D\int_0^\infty\frac{\tilde{\epsilon}^{D/z}{\rm d}\tilde{\epsilon}}{{\rm e}^{\tilde{\epsilon}}-1}\equiv\tilde{p}_{0Dz}, \tag{2}$$

where $A_D=\frac{1}{(2\sqrt{\pi})^D\Gamma(\frac{D}{2}+1)}$. For an ideal gas under $D\leq 2$, its scaled critical density $\tilde{n}_{\rm c}$ (defined at $\mu=\mu_{\rm c}=0$) diverges because the low-momentum states can accommodate an infinite number of bosonic particles, and the corresponding critical entropy per particle equals zero. In order to understand how $S_{\rm c}/N$ approaches its ideal-gas limit as $\tilde{g}$ approaches zero, a key task is to compute the way $\tilde{n}_c$ diverges with a vanishing $\tilde{g}$.

Competition between two energy scales and the $\tilde{k}_*$ hypothesis. We solve the dependence of $\tilde{n}_c$ with respect to $\tilde{g}$ by properly taking into account the competition between kinetic energy and interaction energy of a quantum system. We identify two different regimes in the momentum space and treat them separately: in one regime, kinetic energy dominates over interaction energy; in the other regime, interaction energy prevails over kinetic energy. While an ideal gas stays in the former regime over the entire momentum space, an interacting gas will enter the latter regime when the momentum is low enough. To further quantify this physical picture, we introduce a characteristic parameter, $\tilde{k}_*$, for the scaled momentum and make the following key hypothesis based on $\tilde{k}_*$: the kinetic energy dominates over interaction when $\tilde{k} \equiv |\tilde{\boldsymbol{k}}| > \tilde{k}_*$, and the interaction energy dominates when $\tilde{k} < \tilde{k}_*$. This parameter $\tilde{k}_*$ depends on $\tilde{g}$ and must satisfy the following necessary condition in the non-interacting limit:

$$\lim_{\tilde{g}\to 0} \tilde{k}_* = 0. \tag{3}$$

In the limit of $\tilde{g} \to 0$, the interaction-dominating regime shrinks towards a single point, and we expect the above $\tilde{k}_*$ hypothesis to become accurate in reproducing the scaling of physical quantities.

The $\tilde{k}_*$ parameter can be determined from the condition when the kinetic energy per particle equals the interaction energy per particle:

$$\tilde{E}_{\mathrm{K}}\Big|_{\tilde{k}=\tilde{k}_*} = \tilde{E}_{\mathrm{int}}\Big|_{\tilde{k}=\tilde{k}_*}. \tag{4}$$

Here, the scaled kinetic energy per particle at $\tilde{k}_*$ is

$$\tilde{E}_{\mathrm{K}} = \tilde{k}_*^z. \tag{5}$$

In the limit of $\tilde{g} = 0$ (the non-interacting case), Eqs. (4) and (5) give rise to $\tilde{k}_*^z = 0$ and thus $\tilde{k}_* = 0$, which is consistent with Eq. (3).

Under weak repulsive interactions, the quantum states of the system can still be depicted by the momentum $\tilde{\boldsymbol{k}}$, but the total energy per particle of a given state will increase because of interaction, which leads to a reduction of the occupation for this state as compared to the non-interacting case. At quantum criticality ($\mu = \mu_c = 0$), the occupation number of a given state with momentum $\tilde{\boldsymbol{k}}$ can be approximately given by the Bose distribution $1/[\mathrm{e}^{\epsilon(k)/T} - 1] = 1/[\mathrm{e}^{\tilde{\epsilon}(\tilde{k})} - 1]$, where $\tilde{\epsilon}(\tilde{k}) = \tilde{E}_{\mathrm{K}} + \tilde{E}_{\mathrm{int}}$ is the total energy per particle.

Furthermore, under weak interaction strength, the interaction energy per particle is approximately proportional to the total particle density, namely $\tilde{E}_{\mathrm{int}} \propto \tilde{g}\tilde{n}_c$. Since the total density $\tilde{n}_c$ can be separated into two parts ($\tilde{n}_{c,1} + \tilde{n}_{c,2}$), one part $\tilde{n}_{c,1}$ contributed by states with $\tilde{k} > \tilde{k}_*$ and the other part $\tilde{n}_{c,2}$ contributed by states with $\tilde{k} < \tilde{k}_*$, the interaction energy per particle will accordingly have two contributions as determined below.

The first contribution ($\tilde{E}_{\mathrm{int},1}$) to the interaction energy per particle is proportional to $\tilde{g}\tilde{n}_{c,1}$ and comes from all the particles occupying the momentum states in the kinetic-energy-dominating regime ($\tilde{k} > \tilde{k}_*$):

$$\tilde{E}_{\mathrm{int},1} = A\tilde{g}\int_{\tilde{k}_*}^{\infty} \mathrm{d}\tilde{k}\, \frac{\tilde{k}^{D-1}}{\mathrm{e}^{\tilde{k}^z} - 1}, \tag{6}$$

where we apply the Bose–Einstein distribution of an ideal gas at $\mu = \mu_c = 0$ because kinetic energy dominates in this regime, and A is a proportionality constant. For $D \le z$, this term has a major part that diverges with a vanishing $\tilde{k}_*$:

$$\tilde{E}_{\mathrm{int},1} \approx A\tilde{g}\int_{\tilde{k}_*}^{1} \mathrm{d}\tilde{k}\, \frac{\tilde{k}^{D-1}}{\tilde{k}^z} \tag{7}$$

$$\approx \begin{cases} \dfrac{A}{z-D}\tilde{g}\tilde{k}_*^{-(z-D)}, & D < z. \\ A\tilde{g}\ln\dfrac{1}{\tilde{k}_*}, & D = z. \end{cases} \tag{8}$$

The second contribution ($\tilde{E}_{\mathrm{int},2}$) to the interaction energy per particle is proportional to $\tilde{g}\tilde{n}_{c,2}$ and comes from particles occupying low momentum states in the interaction-dominating regime. In this regime, the kinetic energy is neglected, and the occupation at a given state with momentum $\tilde{\boldsymbol{k}}$ is determined by the Bose distribution $1/(\mathrm{e}^{\tilde{E}_{\mathrm{int}}} - 1)$. Here, $\tilde{E}_{\mathrm{int}}$ is insensitive to $\tilde{\boldsymbol{k}}$ and one can use Eq. (4) to evaluate $\tilde{E}_{\mathrm{int}}$ at $\tilde{k} = \tilde{k}_*$: $\tilde{E}_{\mathrm{int}} = \tilde{E}_{\mathrm{K}} = \tilde{k}_*^z$. Therefore, this second contribution is given by

$$\tilde{E}_{\mathrm{int},2} \approx A\tilde{g}\int_{0}^{\tilde{k}_*} \mathrm{d}\tilde{k}\, \frac{\tilde{k}^{D-1}}{\mathrm{e}^{\tilde{k}_*^z} - 1} \approx \frac{A\tilde{g}}{D}\tilde{k}_*^{-(z-D)}. \tag{9}$$

For $D < z$, the total interaction energy per particle can thus be computed:

$$\tilde{E}_{\mathrm{int}} = \tilde{E}_{\mathrm{int},1} + \tilde{E}_{\mathrm{int},2} \approx B\tilde{g}\tilde{k}_*^{-(z-D)}, \tag{10}$$

where $B = A\left(\frac{1}{z-D} + \frac{1}{D}\right)$. Based on Eq. (4), the characteristic parameter $\tilde{k}_*$ can be determined:

$$\tilde{k}_* \approx B'\tilde{g}^{\frac{1}{2z-D}}, \tag{11}$$

where $B' = B^{\frac{1}{2z-D}}$. Accordingly, the critical scaled density $\tilde{n}_c$ can be obtained:

$$\tilde{n}_{\rm c} \approx A_2\left(\int_{\tilde{k}_*}^{\infty} {\rm d}\tilde{k}\frac{\tilde{k}^{D-1}}{{\rm e}^{\tilde{k}^z}-1} + \int_0^{\tilde{k}_*} {\rm d}\tilde{k}\frac{\tilde{k}^{D-1}}{{\rm e}^{\tilde{k}_*^z}-1}\right)$$
$$\approx \frac{B_2}{(B')^{z-D}}\tilde{g}^{-\frac{z-D}{2z-D}}, \qquad (12)$$

where A_2 is a constant and $B_2 = A_2\left(\frac{1}{z-D}+\frac{1}{D}\right)$. We further apply Eq. (1) to derive the critical entropy per particle as follows:

$$\frac{S_{\rm c}}{N} = \frac{D+z}{z}\frac{\tilde{p}_{0Dz}}{B_2}(B')^{z-D}\times\tilde{g}^{\beta} \propto \tilde{g}^{\beta}, \qquad (13)$$

where the exponent β is given by

$$\beta = \frac{z-D}{2z-D}, \quad D<z. \qquad (14)$$

In a generic weakly interacting Bose gas at its vacuum-to-quantum liquid quantum critical point under $D<z$, Eq. (14) remarkably depicts the power-law scaling form in which, as $\tilde{g}\to 0$, $\tilde{n}_{\rm c}$ diverges, and $S_{\rm c}/N$ monotonically approaches its ideal-gas limit of zero. The scaling of $S_{\rm c}/N$ with $\tilde{g}$ quantitatively depicts a physical scenario that in low dimensions, interactions are indispensable for a thermal gas to reduce its entropy per particle below a realistic finite threshold to become a quantum liquid (a more coherent phase). This scenario of "interaction dictates coherence" is in stark contrast to the case in three dimensions, where an ideal gas can condense into a Bose-Einstein condensate and interactions merely cause shifts of the critical temperature $T_{\rm c}$ [45].

Similarly, one can derive that for $D\le z$, the critical scaled compressibility $\tilde{\kappa}_{\rm c}$ scales as

$$\tilde{\kappa}_{\rm c} \equiv \left.\frac{{\rm d}\tilde{n}}{{\rm d}\tilde{\mu}}\right|_{\tilde{\mu}=\tilde{\mu}_{\rm c}}$$
$$= A_2\left(\frac{1}{2z-D}+\frac{1}{D}\right)\tilde{k}_*^{-(2z-D)}, \quad D\le z. \qquad (15)$$

Combining Eq. (15) with Eq. (11) for $D<z$, one obtain

$$\tilde{\kappa}_{\rm c} \propto \frac{1}{\tilde{g}}, \quad D<z. \qquad (16)$$

For a non-relativistic interacting Bose gas under one dimension $(D=1)$, the dynamical critical exponent is given by $z=2$, and Eq. (14) predicts an exponent of $\beta=1/3$. In the following section, we verify this one-third power-law scaling of $S_{\rm c}/N$ with respect to $\tilde{g}$ based on exact Bethe ansatz solutions of 1D gases at the vacuum-to-TLL transition.

3 Scaling in 1D

The 1D Bose gases with repulsive delta-function interaction are described by [2, 46]

$$\mathcal{H} = \sum_{i=1}^{N}\left(-\nabla_i^2-\mu\right) + c\sum_{i\neq j}\delta(\boldsymbol{r}_i-\boldsymbol{r}_j), \qquad (17)$$

where c is the repulsive elastic interaction strength and N the particle number. The quantum criticality of such interacting 1D gases for the vacuum-to-TLL phase transition $(\mu_{\rm c}=0)$ was studied in Refs. [41, 47] based on the thermodynamic Bethe ansatz (TBA) equation [2]:

$$\varepsilon(q) = q^2-\mu-\frac{T}{2\pi}\int b(q-l)\ln\left(1+{\rm e}^{-\frac{\varepsilon(l)}{T}}\right){\rm d}l, \qquad (18)$$

where q and l are quasi-momenta, $b(q)=2c/(c^2+q^2)$, and the pressure is given by $p(\mu,T)=\frac{T}{2\pi}\int\ln\left(1+{\rm e}^{-\varepsilon(k)/T}\right){\rm d}k$. We compute the critical entropy per particle $S_{\rm c}/N$ as a function of the dimensionless interaction strength $\tilde{g}_{\rm 1D}\equiv\tilde{c}=c/\sqrt{T}$ by numerically solving Eq. (18).

As shown in Fig. 1(a), the $S_{\rm c}/N$ increases with $\tilde{g}_{\rm 1D}$ and reaches $A_{\infty,{\rm 1D}}\approx 1.89738$ in the strong interaction limit $(\tilde{g}_{\rm 1D}\to\infty)$. This limit $A_{\infty,{\rm 1D}}$ exactly matches the $S_{\rm c}/N$ for a non-interacting Fermi gas, as predicted and observed for Tonks–Girardeau gases [46, 48–51].

On the other hand, in the weakly interacting regime as shown in Fig. 1(b), exact solutions of $S_{\rm c}/N$ conform

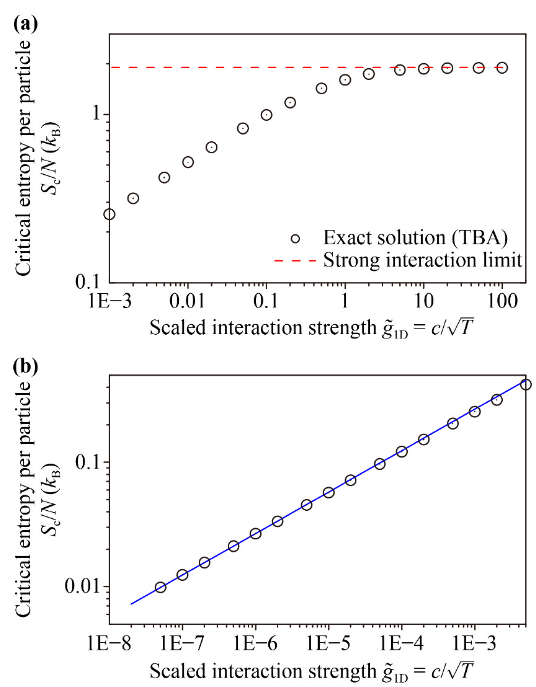


Fig. 1 Critical entropy per particle as a function of scaled interaction strength $\tilde{g}_{\rm 1D}$. **(a)** Critical entropy per particle $S_{\rm c}/N$ as a function of scaled interaction strength $\tilde{g}_{\rm 1D}=c/\sqrt{T}$. Exact solutions (circles) agree well with the strong interaction limit $A_{\infty,{\rm 1D}}$ (dashed line). Here the error bars are much smaller than the symbols. **(b)** A power-law scaling of $S_{\rm c}/N$ with respect to $\tilde{g}_{\rm 1D}$. A blue line shows a power-law scaling with an exponent of $1/3$, which agrees excellently with the exact solutions, as shown over a large range of $\tilde{g}_{\rm 1D}$.

excellently to a $\beta = 1/3$ power-law scaling with $\tilde{g}_{1D}$, where the fractional accuracy of the power-law scaling with respect to the exact solutions are better than 0.8% for the smallest $\tilde{g}_{1D}$ values in $[5\times10^{-8}, 1\times10^{-5})$. Furthermore, we compute a "running" exponent β_R by fitting the slope in the log-log plot (similar to Fig. 1(b)) for each group of three adjacent data points and show β_R as a function of the geometric mean of the three interaction strengths. As shown in Fig. 2, the running exponent β_R approaches 1/3 as the scaled interaction strength $\tilde{g}_{1D}$ decreases towards zero. At the smallest several mean values of $\tilde{g}_{1D}$, β_R has a larger uncertainty due to the higher cost of computation time; the relative deviations of β_R from 1/3 (normalized by the corresponding numerical uncertainties) are fairly small and similar. In addition, we also compute the critical scaled compressibility $\tilde{\kappa}_c$ as a function of $\tilde{g}_{1D} \in [5\times10^{-8}, 1\times10^{-5})$, and observe a power-law scaling $\tilde{\kappa}_c \propto \tilde{g}_{1D}^{\beta_{\tilde{\kappa}}}$ with $\beta_{\tilde{\kappa}} = -0.993 \pm 0.005$ [52]. These observations agree well with the generic predictions by Eqs. (13), (14) and (16) in the previous section. The generic predictions are further supported by our analytical computation that approximately solves the TBA equation under weak interactions [52], where the critical entropy per particle is given explicitly by

$$\frac{S_c}{N} \approx \frac{3I_0}{2}\left(\frac{\pi^2}{2}\right)^{1/3}\tilde{g}_{1D}^{1/3}, \tag{19}$$

with I_0 being a constant given in Ref. [52]. We observe that the critical exponent of 1/3 in Eq. (19) precisely agrees with the generic result of Eqs. (13) and (14). These agreements thus provide strong evidence for the underlying physical picture of a characteristic $\tilde{k}_*$ parameter separating a kinetic-energy-dominating (namely, effectively non-interacting) regime and an interaction-dominating regime.

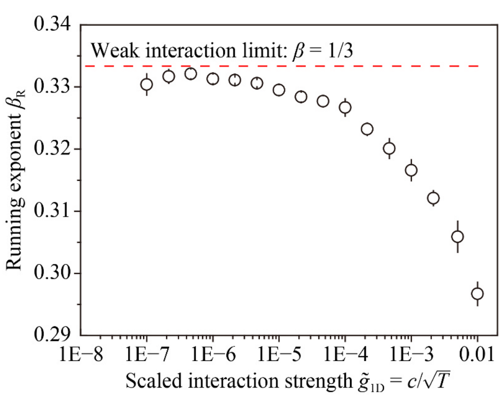


Fig. 2 Running power-law scaling exponent as a function of scaled interaction strength. As $\tilde{g}_{1D}$ decreases towards zero, the running exponent β_R approaches 1/3 from below. Uncertainties are quoted as 1σ standard errors.

4 Scaling in 2D

We further verify our generic predictions in two dimensions (2D). For a two-dimensional quantum system within the dilute Bose gas universality class ($D = z = 2$), Eqs. (8) and (9) mean that the major part of interaction energy per particle comes from $\tilde{E}_{\text{int},1}$. Based on Eq. (4), one derives

$$\tilde{k}_* \approx \sqrt{\frac{A}{2}}\tilde{g}_{2D}^{1/2}\sqrt{\ln\frac{1}{\tilde{g}_{2D}}}, \tag{20}$$

where we have neglected higher-order terms. Therefore, the critical scaled density is given by

$$\tilde{n}_c \approx A_2\int_{\tilde{k}_*}^{\infty} d\tilde{k}\frac{\tilde{k}^{D-1}}{e^{\tilde{k}^z}-1} \approx \frac{A_2}{2}\ln\frac{1}{\tilde{g}_{2D}}, \tag{21}$$

and the critical entropy per particle is given by

$$\frac{S_c}{N} = \frac{2\tilde{p}_{0Dz}}{\tilde{n}_c} \propto \frac{1}{\ln\frac{1}{\tilde{g}_{2D}}}. \tag{22}$$

Besides, using Eq. (15), one obtain

$$\tilde{\kappa}_{c,2D} \propto \frac{1}{\tilde{k}_*^2} \propto \frac{1}{\tilde{g}_{2D}\ln\frac{1}{\tilde{g}_{2D}}}. \tag{23}$$

Eq. (22) shows a logarithmic scaling of S_c/N with respect to $\tilde{g}_{2D}$, which is slower than any power-law scaling with a positive exponent. This result can be formally denoted as the residual logarithmic dependence when a power-law scaling with "$\beta = 0$" (as dictated by Eq. (14) at $D = z = 2$) stops being the dominant contribution. The predictions in Eqs. (21), (22) and (23) are well supported by an independent non-perturbative renormalization-group (NPRG) computation for 2D interacting Bose gases at a vacuum-to-superfluid quantum critical point [30].

5 Conclusion and discussion

In summary, under generic physical conditions ($D \le z = 2$), we found a simple physical picture for predicting power-law scalings of characteristic thermodynamic quantities with respect to the weak repulsive interaction strength in low-dimensional Bose gases at quantum criticality. This prediction has been supported by the Bethe ansatz exact solutions in 1D and a NPRG computation in 2D. Our non-perturbative approach goes well beyond simple dimensional analysis by properly introducing an infrared momentum cut-off [10]. The explicit relation between this cut-off and the interaction strength shows an analogue of the Wilsonian renormalization approach in quantum field theory [30, 53]. The success of our physical picture may shed light on the renormalizability [54, 55] of theories for quantum systems with vanishing interaction. Our result also allows for precisely calibrating and experimentally controlling the interaction strength based on thermodynamic measurements, which can promote the construction of interaction-driven many-particle quantum heat engine and refrigeration [56, 57].

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