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Scaling Equation for Invariant Measure*

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Abstract *An iterated function system (IFS) is constructed. It is shown that the invariant measure of IFS satisfies the same equation as scaling equation for wavelet transform (WT). Obviously, IFS and scaling equation of WT both have contraction mapping principle.*

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1 Introduction

It is well known that a contracted iterated function system (IFS) can produce self-similar fractal set.[1] And the scaling equation for wavelet transform (WT)[2] shows that the scaling function $\phi(t)$ for a scale is equal to the sum of a very few scaling function $\phi(2t)$ whose scale is half as large as that of $\phi(t)$. What relation is between IFS and scaling equation of WT? So far, IFS is only considered in chaos and fractal field, and scaling equation only in wavelet field. In this paper, we will show this relation.

2 Iterated Function System

The following map

$$x_{n+1} = \begin{cases} \frac{1}{3}x_n, & x_n \in [0,1]\,, \\ \frac{1}{3}x_n + \frac{2}{3}, & n = 0,1,2\ldots \end{cases} \tag{1}$$

is a contraction mapping in unit interval, i.e. IFS.[3]

Assuming transform in Eq. (1) is

$$w_1(x) = \frac{1}{3}x, \qquad w_2(x) = \frac{1}{3}x + \frac{2}{3}\,, \tag{2}$$

where $w(A)$ is Hutchinson operator[4]

$$w(A) = w_1(A) \cup w_2(A)\,. \tag{3}$$

Then the attractor of IFS

$$A_{n+1} = w(A_n), \quad A_0 \in [0,1], \quad n = 0,1,2\ldots \tag{4}$$

is the well-known Cantor set.

Obviously, $A_1 = w(A_0) = w_1(A_0) \cup w_2(A_0) = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, $A_2 = w(A_1) = w_1(A_1) \cup w_2(A_1) = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{5}{9}] \cup [\frac{8}{9}, \frac{9}{9}]$, ….

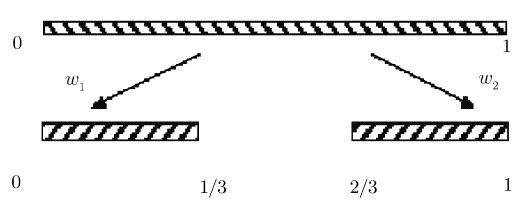

Fig. 1 Similar transform about Cantor set w_1 and w_2.

It is shown in Fig. 1, from which we can see that if we iterate every one step, the scale is one third of the last image, so they form a self-similar structure.

3 Scaling Equation

The scaling equation used to construct wavelet function can be written as[5]

$$\phi(t) = \sum_{k=-\infty}^{+\infty} c_k \phi(2t-k)\,, \tag{5}$$

where $\phi(t)$ is called the scaling function, also the father wavelet.[6]

The advantage of scaling equation (5) lies in that most of the coefficients c_k on the right-hand side are zero, only very few c_k are nonzero.

For example, there is $c_0 = 1$, $c_1 = 1$ for Haar scaling equation. i.e., Haar scaling equation is

$$\phi(t) = \phi(2t) + \phi(2t-1)\,. \tag{6}$$

There is $c_0 = 1/2$, $c_1 = 1$, $c_2 = 1/2$ for hat scaling function, i.e., hat scaling equation is

$$\phi(t) = \frac{1}{2}\phi(2t) + \phi(2t-1) + \frac{1}{2}\phi(2t-2)\,. \tag{7}$$

For quadratic Battle–Lemarie scaling function, there is $c_0 = 1/4$, $c_1 = 3/4$, $c_2 = 3/4$, $c_3 = 1/4$, i.e., the scaling equation is

$$\begin{aligned}\phi(t) =& \frac{1}{4}\phi(2t) + \frac{3}{4}\phi(2t-1) + \frac{3}{4}\phi(2t-2) \\ &+ \frac{1}{4}\phi(2t-3)\,.\end{aligned} \tag{8}$$

The scale of $\phi(2t)$ on the right-hand side of Eqs. (5), (6), (7), and (8) is only half as that of $\phi(t)$ on the left part.

The Haar scaling function is

$$\phi(t) = \begin{cases} 1, & 0 \le t \le 1\,, \\ 0, & \text{otherwise}\,, \end{cases} \tag{9}$$

and $\phi(2t)$ and $\phi(t)$ in Eq. (6) are shown in Fig. 2.

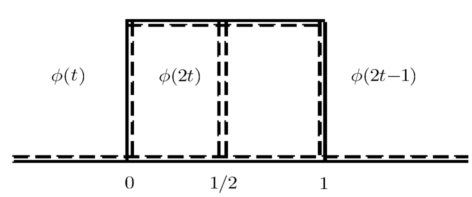

Fig. 2 Haar scaling function.

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From Fig. 2 we can see that the scale of $\phi(2t)$ is half as that of $\phi(t)$, and $\phi(2t-1)$ is the result that $\phi(2t)$ is shifted to the right by half unit. Obviously, the scale of $\phi(4t)$ is half as that of $\phi(2t)$. The process of scale decreasing down forms the self-similar structure.

4 Invariant Measure of Iterated Function System

The research about IFS is considered through orbit of point sets which is divided into fixed point, periodic orbit and chaotic orbit. But the research can also considered through evolution of global density.[7] The equation for density evolution is also an IFS whose fixed point is invariant measure.

The following IFS is constructed,

$$\begin{aligned} w_1(x) &= \tfrac{1}{2}x, & p_1 &= \tfrac{1}{2}\,, \\ w_2(x) &= \tfrac{1}{2}x + \tfrac{1}{2}, & p_2 &= \tfrac{1}{2}\,, \end{aligned} \tag{10}$$

where p_1 and p_2 are probabilities, which control the evolution distribution of $w_1(x)$ and $w_2(x)$.

According to the theory of density evolution,[8] the density $f(x)$ for mapping satisfies the density evolution equation

$$f_{n+1}(x) = Mf_n(x), \qquad n = 0, 1, 2, \ldots \tag{11}$$

with

$$M(f) = p_1 f(w_1^{-1}) + p_2 f(w_2^{-1})\,, \tag{12}$$

which is called Markov operator. Sometimes the operator is also replaced by Frobenius–Perron operator, which is a special case for Markov operators. w_1^{-1} and w_2^{-1} are the inverse mapping of w_1 and w_2.

Now we assume the probability density over the initial interval $[0, 1]$ is

$$f_0(x) = \begin{cases} 1, & x \in [0, 1]\,, \\ 0, & \text{otherwise}\,, \end{cases} \tag{13}$$

then what will happen for $f_0(x)$ under the operator M?

First, we notice that there is difference between mappings (10) and (1), the coefficient before x in $w_1(x)$ and $w_2(x)$ is 1/3 for mapping (1), but 1/2 for mapping (10). According to Eq. (4), the attractor of Eq. (10) is the unit interval, i.e. $A_\infty = [0, 1]$.

Next, for a subset $A \subset [0, 1/2]$, we have $w_1^{-1}(A) \subset [0, 1]$, $w_2^{-1}(A) \subset [-1, 0]$, then

$$f(w_2^{-1}(A)) = 0\,. \tag{14}$$

In the same way, for a subset $A \subset [1/2, 1]$, there is $w_1^{-1}(A) \subset [1, 2]$, $w_2^{-1}(A) \subset [0, 1]$, and

$$f(w_1^{-1}(A)) = 0\,. \tag{15}$$

Thus after the first step, f_0 becomes

$$f_1(x) = \begin{cases} p_1, & x \in [0, \frac{1}{2}]\,, \\ p_2, & x \in [\frac{1}{2}, 1]\,, \\ 0, & \text{otherwise}\,, \end{cases} \tag{16}$$

under the operator M. In other words

$$Mf_0(x) = f_1(2x) + f_1(2x-1)\,, \tag{17}$$

which is shown in Fig. 3.

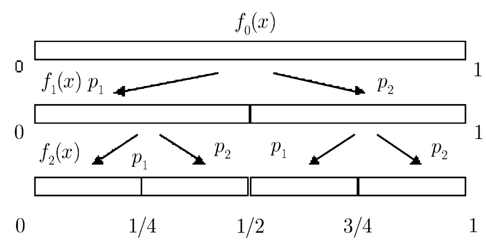

Fig. 3 Transform from p_3 and p_2 over unit interval.

The scale of $f_1(2x)$ on the right-hand side of Eq. (17) is half as that of $f_1(x)$, and $f_1(2x-1)$ is the result of shifting $f_1(2x)$ to right by half interval.

Because the value of the right-hand side of Eq. (17) is 1, so equation (17) can also be written as

$$f_1(x) = f_1(2x) + f_1(2x-1)\,. \tag{18}$$

Thus equation (18) is the same as scaling equation (6).

In the same way, under the operator M acting on $f_1(x)$, there is

$$Mf_1(x) = f_2(2x) + f_2(2x-1) = f_2(x)\,. \tag{19}$$

If mapping (10) is extended to the general mapping

$$w_k(x) = \frac{1}{2}(x+k), \quad p_k = \frac{1}{2}c_k, \quad k = 0, 1, 2, \ldots, K\,, \tag{20}$$

where K is a finite number. Then we can find the evolution equation for density

$$Mf(x) = \sum_{k=-\infty}^{+\infty} c_k f(2x-k)\,, \tag{21}$$

which is the same one as Eq. (5).

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