

变系数非线性方程的 Jacobi 椭圆函数展开解*

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把 Jacobi 椭圆函数展开法扩展并应用到求解变系数的非线性演化方程,比较方便地得到新的解析解.

关键词: Jacobi 椭圆函数, 变系数非线性方程, 类椭圆余弦波解, 类孤子解

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1. 引 言

随着科学技术的发展,非线性在自然科学和社会科学领域的作用越来越重要,对于非线性问题的关注也越来越大. 在非线性问题中,对非线性演化方程的求解和定性分析占有很重要的地位. 对于常系数非线性演化方程的求解和分析已经取得了很多突破,发展了很多比较系统的求解方法和分析手段,如齐次平衡法^[1-5],双曲正切函数展开法^[6-8],非线性变换法^[9,10],试探函数法^[11,12],和 sine-cosine 方法^[13]. 这些方法也求得非线性波方程的冲击波解和孤立波解,或仅仅能够得到初等函数构成的周期解^[14-21],Porubov 等^[22-24]应用 Weierstrass 椭圆函数求得了一些非线性波方程的准确周期解. 但是,毕竟常系数非线性演化方程只是现实中的非线性问题的理想化和近似,事实上,这些非线性演化方程的系数是随着时间和空间变化的. 对于这些变系数的非线性演化方程的分析和求解已经引起了越来越多的关注,也取得了一定的进展^[25-28]. 本文把我们^[29-31]最近提出的 Jacobi 椭圆函数展开方法扩展并应用到变系数非线性演化方程的求解,得到了新的解析解.

2. Jacobi 椭圆函数展开法

考虑含变系数的非线性方程

$$N \left[u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 u}{\partial x^2}, \dots \right] = 0, \quad (1)$$

寻求它的行波解为

$$u = u(\xi), \quad \xi = f(t)x + g(t), \quad (2)$$

其中 $f(t)$ 和 $g(t)$ 为待定函数.

将 $u(\xi)$ 展开为下列 Jacobi 椭圆正弦函数 sn 的级数:

$$u(\xi) = \sum_{j=0}^n a_j(t) \text{sn}^j. \quad (3)$$

类似对常系数非线性方程的求解,在(3)式中,选择适当的 n 使得含变系数的非线性波动方程(1)中的非线性项和最高阶导数项平衡,具体的求解步骤可参看文献[29]. 应该指出的是,因为 $m > 1$ 时, $\text{sn} \tanh$, (3)式就退化为

$$u(\xi) = \sum_{j=0}^n a_j(t) \tanh^j, \quad (4)$$

所以,我们的方法包含了求解变系数的非线性方程的双曲正切函数展开法.

下面,把这种方法应用于两类变系数 KdV 方程和柱 KdV 方程进行求解.

3. 第一类变系数 KdV 方程的解

第一类变系数 KdV 方程形式为

$$\frac{\partial u}{\partial t} + (t)u \frac{\partial u}{\partial x} + (t) \frac{\partial^3 u}{\partial x^3} = 0. \quad (5)$$

把(2)式和(3)式代入(5)式使得方程中的非线性项和最高阶导数项平衡,得到

$$n = 2, \quad (6)$$

即方程(5)的形式解为

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$$u = a_0(t) + a_1(t) \operatorname{sn} + a_2(t) \operatorname{sn}^2. \quad (7)$$

由于

$$\frac{\partial u}{\partial t} = a_0 + a_1 \operatorname{sn} + a_2 \operatorname{sn}^2 + (a_1 + 2a_2 \operatorname{sn})(f x + g) \operatorname{cn} \operatorname{dn}, \quad (8)$$

$$\frac{\partial u}{\partial x} = f(a_1 + 2a_2 \operatorname{sn}) \operatorname{cn} \operatorname{dn}, \quad (9)$$

$$u \frac{\partial u}{\partial x} = f[a_0 a_1 + (a_1^2 + 2a_0 a_2) \operatorname{sn} + 3a_1 a_2 \operatorname{sn}^2 + 2a_2^2 \operatorname{sn}^3] \operatorname{cn} \operatorname{dn}, \quad (10)$$

$$\frac{\partial^2 u}{\partial x^2} = f^2[2a_2 - (1 + m^2)a_1 \operatorname{sn} - 4(1 + m^2)a_2 \operatorname{sn}^2 + 2m^2 a_1 \operatorname{sn}^3 + 6m^2 a_2 \operatorname{sn}^4], \quad (11)$$

$$\frac{\partial^3 u}{\partial x^3} = f^3[-(1 + m^2)a_1 - 8(1 + m^2)a_2 \operatorname{sn} + 6m^2 a_1 \operatorname{sn}^2 + 24m^2 a_2 \operatorname{sn}^3] \operatorname{cn} \operatorname{dn}. \quad (12)$$

在(8) —(12)式中 m 为模数 ($0 < m < 1$). 这样把(8), (10)和(12)式代入方程(5)得到

$$\begin{aligned} & a_0 + a_1 \operatorname{sn} + a_2 \operatorname{sn}^2 + a_1[f x + g + f a_0 - (1 + m^2) f^3] \operatorname{cn} \operatorname{dn} \\ & + [2a_2(f x + g) + f(a_1^2 + 2a_0 a_2) - 8(1 + m^2) f^3 a_2] \operatorname{sn} \operatorname{cn} \operatorname{dn} + 3a_1 f[a_2 + 2m^2 f^2] \operatorname{sn}^2 \operatorname{cn} \operatorname{dn} \\ & + 2a_2 f[a_2 + 12m^2 f^2] \operatorname{sn}^3 \operatorname{cn} \operatorname{dn} = 0, \end{aligned} \quad (13)$$

从而有

$$a_0(t) = a_1(t) = a_2(t) = 0, \quad (14)$$

$$a_1[f x + g + f a_0 - (1 + m^2) f^3] = 0, \quad (15)$$

$$2a_2(f x + g) + f(a_1^2 + 2a_0 a_2) - 8(1 + m^2) f^3 a_2 = 0, \quad (16)$$

$$3a_1 f[a_2 + 2m^2 f^2] = 0, \quad (17)$$

$$2a_2 f[a_2 + 12m^2 f^2] = 0. \quad (18)$$

由(14)式求得

$$a_0 = c_0, a_1 = c_1, a_2 = c_2 \quad (c_0, c_1, c_2 \text{ 为常数}). \quad (19)$$

由(15)式和(16)式求得

$$f = 0, \quad (20)$$

即

$$f = k \quad (k \text{ 为常数}). \quad (21)$$

由(18)式得到

$$a_2 = -12m^2 k^2. \quad (22)$$

考虑到(19)式,可知

$$- = (\text{ = 常数}), \quad (23)$$

这样

$$a_2 = -12m^2 k^2. \quad (24)$$

再由(17)式得到

$$a_1 = c_1 = 0. \quad (25)$$

由(16)式、(21)式和(23)式得到

$$a_0 = -\frac{g(t)}{k(t)} + 4(1 + m^2) k^2. \quad (26)$$

考虑到(19)式,则

$$-\frac{g(t)}{k(t)} = c \quad (c \text{ = 常数}), \quad (27)$$

所以

$$g(t) = -kc \int (\) dt, \quad (28)$$

$$a_0 = c + 4(1 + m^2) k^2, \quad (29)$$

从而得到

$$\begin{aligned} u &= c + 4(1 + m^2) k^2 - 12m^2 k^2 \operatorname{sn}^2 \\ &= c + 4(1 - 2m^2) k^2 + 12m^2 k^2 \operatorname{cn}^2, \end{aligned} \quad (30)$$

这就是第一类变系数 KdV 方程(5)的类椭圆余弦波解. 其中

$$\begin{aligned} & = - (\text{ = 常数}), \\ & = k(x - c \int (\) dt) \quad (k, c \text{ 为常数}). \end{aligned} \quad (31)$$

当 $m = 1$ 时, (30)式退化为

$$\begin{aligned} u &= c + 8k^2 - 12k^2 \operatorname{tanh}^2 \\ &= c - 4k^2 + 12k^2 \operatorname{sech}^2, \end{aligned} \quad (32)$$

这就是第一类变系数 KdV 方程(5)的类孤立波或孤子解.

4. 第二类变系数 KdV 方程的解

第二类变系数 KdV 方程形式为

$$\begin{aligned} & \frac{\partial u}{\partial t} + [(t) + \mu(t)x] \frac{\partial u}{\partial x} + (t) u \frac{\partial u}{\partial x} \\ & + (t) \frac{\partial^3 u}{\partial x^3} + (t) u = 0, \end{aligned} \quad (33)$$

把(2)式和(3)式代入(33)式得到方程的形式解仍然为(7)式,把(7)式代入方程(33)得到

$$\begin{aligned} & (a_0 + a_0) + (a_1 + a_1) \operatorname{sn} + (a_2 + a_2) \operatorname{sn}^2 + a_1[f + (f + \mu f)x + g + f a_0 - (1 + m^2) f^3] \operatorname{cn} \operatorname{dn} \\ & + \{2a_2[f + \mu f]x + (g + f)\} \end{aligned}$$

$$\begin{aligned}
 &+ f(a_1^2 + 2a_0 a_2) - 8(1 + m^2) f^3 a_2 \} \\
 &\times \operatorname{sn} \operatorname{cn} \operatorname{dn} + 3 a_1 f[a_2 \\
 &+ 2 m^2 f^2] \operatorname{sn}^2 \operatorname{cn} \operatorname{dn} + 2 a_2 f[a_2 \\
 &+ 12 m^2 f^2] \operatorname{sn}^3 \operatorname{cn} \operatorname{dn} = 0. \tag{34}
 \end{aligned}$$

由此求得

$$\begin{aligned}
 a_0 &= c_0 e^{-\int_0^t \mu(\tau) d\tau}, a_2 = c_2 e^{-\int_0^t \mu(\tau) d\tau} \\
 &(c_0, c_2 \text{ 为常数}), \tag{35}
 \end{aligned}$$

$$a_1(t) = 0, \tag{36}$$

$$f(t) = f_0 e^{-\int_0^t \mu(\tau) d\tau} \quad (f_0 \text{ 为常数}), \tag{37}$$

$$a_2 = -12 m^2 f_0^2 \frac{(t)}{(t)} e^{-2 \int_0^t \mu(\tau) d\tau}, \tag{38}$$

$$\begin{aligned}
 g(t) &= \int_0^t \{ 4(1 + m^2) (t) f^3(t) \\
 &- [(t) - (t) a_0(t)] f(t) \} dt, \tag{39}
 \end{aligned}$$

对比(35)式和(38)式,则有

$$- = (\text{为常数}), (t) = 2\mu(t). \tag{40}$$

所以

$$\begin{aligned}
 u &= [c_0 - 12 m^2 f_0^2 \operatorname{sn}^2] e^{-\int_0^t \mu(\tau) d\tau} \\
 &= [c_0 - 12 m^2 f_0^2 + 12 m^2 f_0^2 \operatorname{cn}^2] e^{-\int_0^t \mu(\tau) d\tau}. \tag{41}
 \end{aligned}$$

这就是第二类变系数 KdV 方程(33)的类椭圆余弦波解. 其中

$$= - = \text{常数}, (t) = 2\mu(t),$$

$$f(t) = f_0 e^{-\int_0^t \mu(\tau) d\tau} \quad (f_0 \text{ 为常数}), \tag{42}$$

$$\begin{aligned}
 g(t) &= \int_0^t \{ 4(1 + m^2) (t) f^3(t) - [(t) \\
 &- (t) a_0(t)] f(t) \} dt, \\
 &= f(t) x + g(t)
 \end{aligned}$$

当 $m = 1$ 时,(41)式退化为

$$u = [c_0 - 12 f_0^2 + 12 f_0^2 \operatorname{sech}^2] e^{-\int_0^t \mu(\tau) d\tau}. \tag{43}$$

这就是第二类变系数 KdV 方程(33)的类孤立波或孤子解.

5. 柱 KdV 方程的解

柱 KdV 方程形式为

$$\frac{\partial}{\partial t} + a \frac{\partial}{\partial y} + b \frac{\partial^3}{\partial y^3} + \frac{1}{2t} = 0, \tag{44}$$

这里 a 和 b 为常数. 很显然,方程(44)不可以直接求解,需要做变换才可以.

首先,令

$$u = t^{1/2}, \tag{45}$$

这样得到

$$\frac{\partial u}{\partial t} + at^{-1/2} u \frac{\partial u}{\partial y} + b \frac{\partial^3 u}{\partial y^3} = 0. \tag{46}$$

再令

$$x = t^{-1/4} y, \tag{47}$$

由方程(47)得到

$$\frac{\partial u}{\partial t} + at^{-3/4} u \frac{\partial u}{\partial x} + bt^{-3/4} \frac{\partial^3 u}{\partial x^3} = 0. \tag{48}$$

若令

$$(t) = at^{-3/4}, (t) = bt^{-3/4}, \tag{49}$$

则

$$\frac{\partial u}{\partial t} + (t) u \frac{\partial u}{\partial x} + (t) \frac{\partial^3 u}{\partial x^3} = 0. \tag{50}$$

这就是第一类的变系数 KdV 方程,很明显 $= (t) / (t) = b/a = \text{常数}$,这样方程(50)就是(30)式. 因此,柱 KdV 方程(44)的类椭圆余弦波解为

$$\begin{aligned}
 &= t^{-1/2} [c + 4(1 + m^2) k^2 - 12 m^2 k^2 \operatorname{sn}^2] \\
 &= t^{-1/2} [c + 4(1 - 2 m^2) k^2 + 12 m^2 k^2 \operatorname{cn}^2], \tag{51}
 \end{aligned}$$

其中

$$\begin{aligned}
 &= \frac{b}{a} \quad (= \text{常数}), \\
 &= k(t^{-1/4} y - 4act^{1/4}) \quad (k, c \text{ 为常数}). \tag{52}
 \end{aligned}$$

当 $m = 1$ 时,(51)式退化为类孤立波或孤子解

$$\begin{aligned}
 &= t^{-1/2} [c + 8 k^2 - 12 k^2 \tanh^2] \\
 &= t^{-1/2} [c - 4 k^2 + 12 k^2 \operatorname{sech}^2]. \tag{53}
 \end{aligned}$$

6. 结 论

本文把 Jacobi 椭圆函数展开法扩展并应用到含变系数的非线性演化方程,得到新的解析解(类椭圆余弦波解). 我们知道当 $m = 1$ 时, $\operatorname{cn} = \operatorname{sech}$, 所以, 这些解析解在一定条件下可以退化为类孤子解.

[1] Wang ML 1995 *Phys. Lett. A* **199** 169

[2] Wang ML, Zhou YB and Li ZB 1996 *Phys. Lett. A* **216** 67



- [3] Yang L, Zhu Z and Wang Y 1999 *Phys. Lett. A* **260** 55
- [4] Fan E G and Zhang H Q 1998 *Acta Phys. Sin.* **47** 353 (in Chinese) [范恩贵、张鸿庆 1998 物理学报 **47** 353]
- [5] Fan E G and Zhang H Q 2000 *Acta Phys. Sin.* **49** 1409 (in Chinese) [范恩贵、张鸿庆 2000 物理学报 **49** 1409]
- [6] Yang L, Liu J and Yang K 2001 *Phys. Lett. A* **278** 267
- [7] Parkes E J and Duffy B R 1997 *Phys. Lett. A* **229** 217
- [8] Fan E G 2000 *Phys. Lett. A* **277** 212
- [9] Hirota R 1973 *J. Math. Phys.* **14** 810
- [10] Kudryashov N A 1990 *Phys. Lett. A* **147** 287
- [11] Otwinowski M, Paul R and Laidlaw W G 1988 *Phys. Lett. A* **128** 483
- [12] Liu S K, Fu Z T, Liu S D and Zhao Q 2001 *Appl. Math. Mech.* **22** 326
- [13] Yan C 1996 *Phys. Lett. A* **224** 77
- [14] Li Z B and Pan S Q 2001 *Acta Phys. Sin.* **50** 402 (in Chinese) [李志斌、潘素起 2001 物理学报 **50** 402]
- [15] Zhang J F 1998 *Acta Phys. Sin.* **47** 1416 (in Chinese) [张解放 1998 物理学报 **47** 1416]
- [16] Yan Z Y and Zhang H Q 1999 *Acta Phys. Sin.* **48** 1962 (in Chinese) [闫振亚、张鸿庆 1999 物理学报 **48** 1962]
- [17] Yan Z Y, Zhang H Q and Fan E G 1999 *Acta Phys. Sin.* **48** 1 (in Chinese) [闫振亚、张鸿庆、范恩贵 1999 物理学报 **48** 1]
- [18] Xia T C, Zhang H Q and Yan Z Y 2001 *Chin. Phys.* **10** 694
- [19] Li Z B and Yao R X 2001 *Acta Phys. Sin.* **50** 2062 (in Chinese) [李志斌、姚若霞 2001 物理学报 **50** 2062]
- [20] Lu K P, Shi T R, Duan W S and Zhao J B 2001 *Acta Phys. Sin.* **50** 2074 (in Chinese) [吕克璞、石太仁、段文山、赵金保 2001 物理学报 **50** 2074]
- [21] Zhang J F 1999 *Chin. Phys.* **8** 326
- [22] Prubov A V 1996 *Phys. Lett. A* **221** 391
- [23] Prubov A V and Velarde M G 1999 *J. Math. Phys.* **40** 884
- [24] Prubov A V and Parker DF 1999 *Wave Motion* **29** 97
- [25] Yan Z Y and Zhang H Q 1999 *Acta Phys. Sin.* **48** 1957 (in Chinese) [闫振亚、张鸿庆 1999 物理学报 **48** 1957]
- [26] Hong W and Jung Y D 1999 *Phys. Lett. A* **257** 149
- [27] Wang M L and Wang Y M 2001 *Phys. Lett. A* **287** 211
- [28] Zhang J F and Chen F Y 2001 *Acta Phys. Sin.* **50** 1648 (in Chinese) [张解放、陈芳跃 1999 物理学报 **50** 1648]
- [29] Liu S K, Fu Z T, Liu S D and Zhao Q 2001 *Acta Phys. Sin.* **50** 2068 (in Chinese) [刘式适、傅遵涛、刘式达、赵强 2001 物理学报 **50** 2068]
- [30] Liu S K, Fu Z T, Liu S D and Zhao Q 2002 *Acta Phys. Sin.* **51** 10 (in Chinese) [刘式适、傅遵涛、刘式达、赵强 2002 物理学报 **51** 10]
- [31] Liu S D, Fu Z T, Liu S K and Zhao Q 2002 *Acta Phys. Sin.* **51** 718 (in Chinese) [刘式达、傅遵涛、刘式适、赵强 2002 物理学报 **51** 718]

Jacobi elliptic function expansion solution to the variable coefficient nonlinear equations^{*}

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Abstract

Jacobi elliptic function expansion method is extended and applied to construct the exact solutions of variable coefficient nonlinear wave equations, new solutions are obtained by this method.

Keywords: Jacobi elliptic function, variable coefficient nonlinear equation, cnoidal wave-like solution, soliton-like solution

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