A Scheme for Atomic Entangled States and Quantum Gate Operations in Cavity QED

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We propose a scheme for controllably entangling the ground states of five-state W-type atoms confined in a cavity and realizing swap gate and phase gate operations. In this scheme the cavity is only virtually excited and the atomic excited states are almost not occupied, so the produced entangled states and quantum logic operations are very robust against the cavity decay and atomic spontaneous emission.

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Entangled states of two or more particles are indeed valuable resources in the field of quantum information.1-3 They have various applications in quantum teleportation, quantum computation, quantum dense coding, and quantum cryptography. Many schemes have been proposed for the generation of entangled states using optical systems,4 cavity QED,5 and ion trap.6 As one of the important candidates for producing entangled states, cavity QED has many advantages.5-7 Cold atoms trapped in resonators are not only the important resource of entanglement, but also well suited for storing the information with the long-lived ground states. Entangled states for two atoms have been realized in cavity QED.7-9 Besides two-particle entanglement, multiparticle entangled states also play an important role. Typical multipartite entangled states are GHZ states10-12 and cluster states.13-15 Experimentally in high-Q resonators the GHZ states of three particles have already been generated.16 Cavity QED is also a good candidate for quantum computation.17 Two qubits can couple to each other via the cavity mediated interactions to realize quantum logic operations.8 However, the main obstacles for implementing the quantum information processing in cavity QED are cavity decay and atomic spontaneous emission.

In this Letter, a scheme is proposed for controllably entangling the ground states of five-level W-type atoms confined in a cavity and realizing swap gate and phase gate operations. The produced entangled states involving the ground states of the atoms, together with the proposed quantum gate mechanism, are very robust and could be implemented with very high fidelity. With the available experimental setups in cavity QED, the implementation of this proposal is feasible.

Consider N identical Λ configuration atoms simultaneously interacting with a single-mode cavity field and two classical fields, assisted by two virtually excited states, which forms a W configuration. As sketched in Fig. 1, each atom has two ground states |g⟩ and |h⟩, which are coupled via Raman transitions involving one classical field of frequency ω1 and the cavity mode of frequency ν. In particular, one classical field drives the transition from level |g⟩ to |e⟩, with Rabi frequency Ω1 and detuning Δ1 = ωeg - ω1. The cavity mode couples level |e⟩ to |h⟩, with coupling constant g1 and detuning Δ2 = ωeh - ν, which is also virtually coupled to the fourth state |s⟩ with coupling constant g2 and detuning Δs = ωsh - ν. The fifth state |r⟩ is virtually excited from state |g⟩ by another largely detuned laser field of frequency ω2. The detuning is Δr = ωrg - ω2. These two virtually excited states are used to add the additional ac-Stark shifts to the states |g⟩ and |h⟩, respectively. In the interaction picture, the associated Hamiltonian under the

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dipole and rotating wave approximations is given by
(let \(h = 1\))
\[
H_I = \sum_{j=1}^{N} \left[ \Omega_1 \hat{\sigma}^j_t \hat{e}^{i\Delta_1 t} + g_1 \hat{a} \hat{\sigma}^j_{eh} \hat{e}^{i\Delta_2 t} + \Omega_2 \hat{\sigma}^j_{ry} \hat{e}^{i\Delta_r t} \\
+ g_2 \hat{a} \hat{\sigma}^j_{sh} \hat{e}^{i\Delta_s t} + H.c. \right],
\]
(1)
where \(\hat{\sigma}^j_{im} = |i\rangle_j \langle m|\) is the atomic transition operator for atom \(j\), and \(a\) is the annihilation operator for the cavity mode. We assume that \(\Omega_1, \Omega_2, g_1, \) and \(g_2\) are real for simplicity. Dispersive detunings of the light fields from the excited states are considered, i.e., \(|\Delta_2|, |\Delta_r|, |\Delta_s| \gg |\Omega_1|, |g_1|, |\Omega_2|, |g_2|\). Therefore, the excited states can be adiabatically eliminated and atomic spontaneous emission can be neglected.[18] Then the effective Hamiltonian describing this Raman system reads
\[
H_{\text{eff}} = \sum_{j=1}^{N} \left[ \left( \frac{\Omega_1^2}{\Delta_1} + \frac{\Omega_2^2}{\Delta_r} \right) \hat{\sigma}^j_{gg} \\
+ \left( \frac{|g_1|^2}{\Delta_2} + \frac{|g_2|^2}{\Delta_s} \right) \hat{a} \hat{\sigma}^j_{hh} \\
+ \frac{\Omega_1 g_1}{\Delta_1} \hat{a} e^{i\delta t} \hat{\sigma}^j_{gh} + \frac{\Omega_1 g_2}{\Delta_1} \hat{e}^{-i\delta t} \hat{\sigma}^j_{gh} \right],
\]
(2)
where \(\delta = \Delta_1 - \Delta_2\). The first two terms correspond to dynamical energy shifts of levels \(|g\rangle\) and \(|h\rangle\), and the last two terms describe the transitions between them, accompanied by creation or annihilation of a photon in the cavity mode.

To appropriately choose the parameters, we assume that \(\frac{|\Omega_1|^2}{\Delta_1} + \frac{|\Omega_2|^2}{\Delta_r} = 0\) and \(\frac{|g_1|^2}{\Delta_2} + \frac{|g_2|^2}{\Delta_s} = 0\), which can be fulfilled as the detunings \(\Delta_1(\Delta_2)\) and \(\Delta_r(\Delta_s)\) have the opposite sign. Then a new effective Hamiltonian can be obtained,
\[
H_{\text{eff}} = \beta \sum_{j=1}^{N} (\hat{a} \hat{\sigma}^j_{+} e^{-i\delta t} + \hat{a}^\dagger \hat{\sigma}^j_{-} e^{i\delta t}),
\]
(3)
where \(\beta = |\Omega_1 g_1/\Delta_1|\) is the Raman coupling strength, \(\hat{\sigma}^j_{+} = \hat{\sigma}^j_{gh}\) and \(\hat{\sigma}^j_{-} = \hat{\sigma}^j_{gh}\) are atomic transition operators. In this Hamiltonian the ac-stark effect terms have been eliminated due to the coupling to the virtually excited states and the cavity mode. Different from the previous work,[8] the effective two-level systems only involve their ground states \(|g\rangle\) and \(|h\rangle\), thus immune to atomic spontaneous emission. In the case of \(\beta \gg \delta\), the Raman transitions between the states \(|g\rangle\) and \(|h\rangle\) are suppressed. Thus there is no energy exchange between the atomic system and the resonator.[8] Then the Hamiltonian (3) can be reduced to
\[
H_{\text{eff}} = \Theta \left[ \sum_{j=1}^{N} (\hat{a} \hat{\sigma}^j_{g} - \hat{a}^\dagger \hat{\sigma}^j_{hh}) + \sum_{j,k=1}^{N} \hat{\sigma}^j_{+} \hat{\sigma}^k_{-} \right],
\]
(4)
where \(\Theta = \beta^2/\delta\). The first and second terms describe the photon-number dependent Stark shifts for each atomic level. The last two terms describe the dipole coupling induced by the exchange of virtual cavity photons between the \(j\)th and \(k\)th atoms. If an atom emits a virtual photon, then this photon is absorbed by another atom immediately, which then also does transition between states \(|g\rangle\) and \(|h\rangle\). Therefore, whenever the atom emits or absorbs a virtual photon into or from the cavity mode, it will transfer from level \(|g\rangle\) to \(|h\rangle\) or vice versa. Now we assume the cavity mode is initially in the vacuum state, so the above Hamiltonian reduces to
\[
H_{\text{eff}} = \Theta \left[ \sum_{j=1}^{N} \hat{\sigma}^j_{g} + \sum_{j,k=1}^{N} \hat{\sigma}^j_{+} \hat{\sigma}^k_{-} \right] \quad j \neq k.
\]
(5)

Equation (5) is the main result of this work, which describes a collection of effective two-level atoms through effective dipole-dipole coupling mediated by the cavity mode. The dynamics of the cavity mode has been eliminated. As a result, the dynamics of the whole system has been confined to only two ground states of the atoms. Compared to the proposals presented in Refs. [11,12], which employ the three-level atoms to generate entanglement and implement quantum logic, this protocol utilizes five-level \(W\)-type atoms, which corresponds to adding two virtually excited states to the \(A\)-type atoms, which can eliminate the dynamical energy shifts of levels \(|g\rangle\) and \(|h\rangle\). Equation (5) is the starting point of the following discussions and we will use this Hamiltonian to produce a variety of entangled states and quantum gates.

**Generation of Entangled States.** We first consider the two atoms case, where the maximally entangled states are the Einstein-Podolsky-Rosen (EPR) state.[19] The evolution operator of the corresponding two-qubit system is given by
\[
U(t) = \exp \left[ -i\Theta t (\hat{\sigma}^1_{gg} + \hat{\sigma}^2_{gg} + \hat{\sigma}^1_{+} \hat{\sigma}^2_{-} + \hat{\sigma}^2_{+} \hat{\sigma}^1_{-}) \right].
\]
(6)
Assume the atoms initially in the state \(|g_1\rangle|h_2\rangle\), then the state evolution of the system is given by
\[
U(t)|g_1\rangle|h_2\rangle = e^{-i\Theta t} \left[ \cos(\Theta t)|g_1\rangle|h_2\rangle \\
- i \sin(\Theta t)|h_1\rangle|g_2\rangle \right].
\]
(7)
If we choose \(\Theta t = \pi/4\), we can obtain the maximally entangled two-atom state
\[
|\psi_{\text{EPR}}\rangle = \frac{1}{\sqrt{2}} (|g_1\rangle|h_2\rangle - i|h_1\rangle|g_2\rangle),
\]
(8)
which is the well-known EPR state.[19]

We next consider the case of three atoms interacting simultaneously with the cavity. It has been
shown that for pure entangled states of tripartite system there are two inequivalent classes, i.e., GHZ state and W state. In what follows, we display these maximally entangled states. In order to generate a three-atom GHZ state we assume that initially each atom stays in the state \( \frac{1}{\sqrt{2}} (|g\rangle_j + |h\rangle_j) \). After an interaction time of \( t = \pi/\Theta \), we have

\[
|\psi(t)\rangle = \frac{1}{\sqrt{2}} (\{+\}_1\{+\}_2\{+\}_3 + \{-\}_1\{-\}_2\{-\}_3),
\]

where \( \{\pm\}_j = \frac{1}{\sqrt{2}} (|g\rangle_j \pm |h\rangle_j) \). If we perform a local unitary transformation, we can obtain the GHZ state for tripartite system in the familiar basis \( \{|g\rangle, |h\rangle\} \).

Next we consider a different class of entangled states for tripartite system. Assume that the three atoms are initially in the state \( |h\rangle_1|h\rangle_2|h\rangle_3 \), then we get to the W state of the system after an interaction time \( t = \pi/\Theta \),

\[
|\psi_W\rangle = \frac{1}{\sqrt{3}} \left( e^{i2\pi/3} |h\rangle_1|h\rangle_2|g\rangle_3 + |g\rangle_1|h\rangle_2|h\rangle_3 + |g\rangle_1|g\rangle_2|h\rangle_3 \right),
\]

where the common phase factor has been discarded.

Now we consider the case of multitatom entanglement. The extension of three-atom entangled GHZ state to the multitatom case is straightforward. Assume that initially each atom stays in the state \( \frac{1}{\sqrt{2}} (|g\rangle_j + |h\rangle_j) \) \( (j = 1, 2, \ldots, N) \). After an interaction time of \( t = \pi/\Theta \), we have

\[
|\psi(t)\rangle = \frac{1}{\sqrt{N}} \left( \{+\}_1 \{+\}_2 \cdots \{+\}_N + \{-\}_1 \{-\}_2 \cdots \{-\}_N \right).
\]

Then if we perform a local unitary transformation, we can obtain the GHZ state for multipartite system in the familiar basis \( \{|g\rangle, |h\rangle\} \).

**Realization of quantum gates.** Next, we focus on the implementation of quantum gates in the context of cavity QED. We now show that the interaction described by Eq. (5) can be used to realize a quantum swap gate in the context of cavity QED. It was first shown by Wang\(^{[20]}\) that the Heisenberg two-qubit XY model can be utilized to realize a swap gate. It is known that a swap gate can be realized by successive three CNOT gates, while in the present work we only need the one-time evolution of the interaction (5) in the two-atom case. Choosing \( \Theta t = \pi/2 \), from Eq. (6) we have

\[
U (\pi/2\Theta) |g\rangle_1 |g\rangle_2 = |g\rangle_1 |g\rangle_2,
\]

\[
U (\pi/4\Theta) |h\rangle_1 |h\rangle_2 = |h\rangle_1 |h\rangle_2,
\]

\[
U (\pi/2\Theta) |g\rangle_1 |h\rangle_2 = -i |h\rangle_1 |g\rangle_2,
\]

\[
U (\pi/4\Theta) |h\rangle_1 |g\rangle_2 = i |g\rangle_1 |h\rangle_2.
\]

These equations show that the operator \( U(\pi/2\Theta) \) acts as a swap gate up to a phase. Another gate \( \sqrt{\text{swap}} \) which is universal can be constructed simply as \( U(\pi/4\Theta) \),

\[
U (\pi/4\Theta) |g\rangle_1 |g\rangle_2 = |g\rangle_1 |g\rangle_2,
\]

\[
U (\pi/4\Theta) |h\rangle_1 |h\rangle_2 = |h\rangle_1 |h\rangle_2,
\]

\[
U (\pi/4\Theta) |g\rangle_1 |h\rangle_2 = \frac{1}{\sqrt{2}} (|g\rangle_1 |h\rangle_2 - |h\rangle_1 |g\rangle_2),
\]

\[
U (\pi/4\Theta) |h\rangle_1 |g\rangle_2 = \frac{1}{\sqrt{2}} (|h\rangle_1 |g\rangle_2 - |g\rangle_1 |h\rangle_2).
\]

In the following we consider the implementation of a quantum phase gate (QPG) in the context of cavity QED. We use the time-evolution operator to realize the two-qubit QPG operation. The evolution operator of the corresponding three-qubit system is given by

\[
U(t) = \exp \left[ -i \Theta t \left( \sum_{j=1}^{3} \hat{\sigma}_j^g + \sum_{j,k} \hat{\sigma}_j^g \hat{\sigma}_k^g \right) \right].
\]

We assume that one of the atoms is in the state \( |h\rangle \) initially, e.g., atom 1. Then after an interaction time \( t = \pi/\Theta \), we get the following relations:

\[
U (\pi/\Theta) |h\rangle_1 |h\rangle_2 |h\rangle_3 = |h\rangle_1 |h\rangle_2 |h\rangle_3,
\]

\[
U (\pi/\Theta) |h\rangle_1 |g\rangle_2 |h\rangle_3 = |h\rangle_1 |g\rangle_2 |h\rangle_3,
\]

\[
U (\pi/\Theta) |h\rangle_1 |h\rangle_2 |g\rangle_3 = |h\rangle_1 |h\rangle_2 |g\rangle_3,
\]

\[
U (\pi/\Theta) |h\rangle_1 |g\rangle_2 |g\rangle_3 = - |h\rangle_1 |g\rangle_2 |g\rangle_3.
\]

Clearly this operation implements a two-qubit QPG between atom 2 and 3 only through one step. The atom 1 is utilized to serve as a data-bus for the other two atoms in the QPG operation. A similar QPG operation has been presented using three superconducting quantum interference device qubits in Ref.\(^{[21]}\). Here we implement the QPG operation in the context of cavity QED, where the atoms coherently interact with the electromagnetic field confined within a resonator. These atomic qubits have the advantages of stability and control only through the time in the experiment.

It is necessary to verify the validity of the above approximations by numerics. We numerically simulate the dynamics generated by the full Hamiltonian \( H_I \) and compare it with the results of the effective model (5). As an example we consider the case of two atoms interacting with the cavity mode simultaneously. Initially the atoms are prepared in the state \( |g\rangle_1 |h\rangle_2 \), and the cavity mode in the vacuum. The parameters are chosen as \( \Omega_1 = \Omega_2 = 2 \text{ GHz}, g_1 = g_2 = 1 \text{ GHz}, \Delta_1 = 20 \text{ GHz}, \Delta_2 = 22 \text{ GHz}, \Delta_s = -20 \text{ GHz}, \) and \( \Delta_s = -22 \text{ GHz} \).\(^{[22]}\) We calculate the occupation probability \( p_{g_1} \) of the state \( |g\rangle_1 \), and the occupation of the excited atomic states \( p_{e_1} = \langle e_1 | e_1 \rangle \) as well as the photon number occupation \( p_N = \langle a^\dagger a \rangle \) both from the
full Hamiltonian and the effective model. The numerical results are shown in Fig. 2. The occupations of the excited states and the photon number are always smaller than 0.01. It can be seen that the effective model can describe the dynamics very well, provided that the parameters are appropriately chosen. Discrepancies between the numerical results for the full Hamiltonian and the effective Hamiltonian are due to the neglect of higher order terms for the detunings and Rabi frequencies. However, these discrepancies are below 5% with respect to the results from the full Hamiltonian.

In summary, we have proposed a scheme for controllably entangling the internal ground states of atoms confined in a cavity and realizations of swap gate and phase gate operations. In this scheme the cavity is only virtually excited and the atomic excited states are almost not occupied. Therefore, it is immune to both cavity dissipation and atomic spontaneous emission, and the produced entangled states and quantum gates are robust. With presently available experimental setups in cavity QED, the implementation of this proposal is feasible.

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References


Fig. 2. The occupation probability $p_{01}$, of state $|g\rangle_1$ (solid line) represents results from the effective model, and dash line represents results from the full Hamiltonian, the occupation of excited atomic states $p_{01} = \langle e | 1 \rangle | e \rangle$, and the photon number occupation $p_N = \langle a | a \rangle$. Parameters are chosen as $\Omega_1 = \Omega_2 = 2$ GHZ, $g_1 = g_2 = 1$ GHZ, $\Delta_1 = 20$ GHZ, $\Delta_2 = 22$ GHZ, $\Delta_t = -20$ GHZ, and $\Delta_s = -22$ GHZ.

We consider some experimental matters. Appropriate candidates for the proposal are microcavities with high cooperativity factor, such as photonic band gap cavities\cite{23} and toroidal or spherical microcavities.\cite{24} The atomic configuration could be realized in, typically, alkali atoms. With the above given parameters we obtain the parameter $\Theta = 0.005$ GHZ. The occupation of the excited state $|e\rangle$ can be estimated to be $\langle e | e \rangle \sim |\Omega_1 / \Delta_1|^2$. Spontaneous emission from the excited state at a rate $\Gamma_e$ thus leads to effective decay rate $\Gamma_{e} = |\Omega_1 / \Delta_1|^2 \gamma_e$. The photon number in the cavity can be estimated to be $N \sim |\Omega_1 g_1 / \Delta_1|^2$. Cavity decay at a rate $\gamma_c$ induces an effective decay rate $\Gamma_{c} = |\Omega_1 g_1 / \Delta_1|^2 \gamma_c$. With the given parameters one can estimate negligible effects of the spontaneous decay of the atoms and the cavity decay on the fidelity of the proposal.