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Fractional transformation and new solutions to mKdV equation

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Abstract

A fractional transformation is introduced to solve modified KdV (mKdV for short) equation, where this fractional transformation is used to map the solutions of an elliptic equation to another elliptic equation. Thus, more new kinds of solutions are obtained, such as rational periodic wave solutions, rational solitary wave solutions and so on. It is shown that this method is more powerful to give more kinds of solutions.

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1. Introduction

Since much attention has been paid to the study how to solve nonlinear models, many methods have been proposed to construct exact solutions to nonlinear equations. Among them are the sine–cosine method [1], the homogeneous balance method [2,3], the hyperbolic tangent expansion method [4,5], the Jacobi elliptic function expansion method [6,7], the nonlinear transformation method [8,9], the trial function method [10,11] and others [12–14].

Apart from methods mentioned above, direct algebra method [15,16] has its own advantages: it is simple and has a strong operability, where the solutions of nonlinear wave equations are mapped to those of simple equations. In Refs. [17–19], elliptic equations have been applied as a mapping to obtain many kinds of periodic solutions.

In this Letter, we will reconsider elliptic equation [20]

$$y'^2 = a_0 + a_1 y^2 + a_2 y^4, \tag{1}$$

i.e.,

$$y'' = a_1 y + 2a_2 y^3, \tag{2}$$

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where the prime denotes the derivatives in terms of its argument and apply it to solve mKdV equation for more kinds of solutions.

2. mKdV equation

mKdV equation reads [20]:

$$u_t + \alpha u^2 u_x + \beta u_{xxx} = 0 \tag{3}$$

and it is met in many fields, such as shallow water model, plasma science, biophysics and etc.

We seek its traveling wave solutions in the following frame

$$u = u(\xi), \quad \xi = k(x - ct), \tag{4}$$

here c is wave velocity, k is wave number.

Substituting Eq. (4) into Eq. (3) and integrating once yield

$$-cu + \frac{\alpha}{3}u^3 + \beta k^2 u'' = D, \tag{5}$$

where D is an integration constant.

For Eq. (5), there are two cases needed to consider, the first one is $D = 0$, Eq. (5) can be rewritten as

$$u'' = \frac{c}{\beta k^2}u - \frac{\alpha}{3\beta k^2}u^3 \tag{6}$$

obviously, this is just the elliptic equation (2) with

$$a_1 = \frac{c}{\beta k^2}, \qquad a_2 = -\frac{\alpha}{6\beta k^2} \tag{7}$$

it has many more kinds of solutions, we will show some next expressed in terms of different Jacobi elliptic functions [20].

(1) If $a_0 = 1$, $a_1 = c/(\beta k^2) = -(1 + m^2)$ and $a_2 = -\alpha/(6\beta k^2) = m^2$, then the solution is

$$u_1 = \mathrm{sn}(\xi, m), \tag{8}$$

where $0 \leqslant m \leqslant 1$, is called modulus of Jacobi elliptic functions, see [20–24], and $\mathrm{sn}(\xi, m)$ is Jacobi elliptic sine function, see [20–24].

(2) If $a_0 = 1 - m^2$, $a_1 = c/(\beta k^2) = 2m^2 - 1$ and $a_2 = -\alpha/(6\beta k^2) = -m^2$, then the solution is

$$u_2 = \mathrm{cn}(\xi, m), \tag{9}$$

where $\mathrm{cn}(\xi, m)$ is Jacobi elliptic cosine function, see [20–24].

(3) If $a_0 = 1 - m^2$, $a_1 = c/(\beta k^2) = 2 - m^2$ and $a_2 = -\alpha/(6\beta k^2) = -1$, then the solution is

$$u_3 = \mathrm{dn}(\xi, m), \tag{10}$$

where $\mathrm{dn}(\xi, m)$ is Jacobi elliptic function of the third kind, see [20–24].

(4) If $a_0 = m^2$, $a_1 = c/(\beta k^2) = -(1 + m^2)$ and $a_2 = -\alpha/(6\beta k^2) = 1$, then the solution is

$$u_4 = \mathrm{ns}(\xi, m) \equiv \frac{1}{\mathrm{sn}(\xi, m)}. \tag{11}$$

(5) If $a_0 = -m^2$, $a_1 = c/(\beta k^2) = 2m^2 - 1$ and $a_2 = -\alpha/(6\beta k^2) = 1 - m^2$, then the solution is

$$u_5 = \mathrm{nc}(\xi, m) \equiv \frac{1}{\mathrm{cn}(\xi, m)}. \tag{12}$$

(6) If $a_0 = -1$, $a_1 = c/(\beta k^2) = 2 - m^2$ and $a_2 = -\alpha/(6\beta k^2) = m^2 - 1$, then the solution is

$$u_6 = \mathrm{nd}(\xi, m) \equiv \frac{1}{\mathrm{dn}(\xi, m)}. \tag{13}$$

(7) If $a_0 = 1$, $a_1 = c/(\beta k^2) = 2 - m^2$ and $a_2 = -\alpha/(6\beta k^2) = 1 - m^2$, then the solution is

$$u_7 = \mathrm{sc}(\xi, m) \equiv \frac{\mathrm{sn}(\xi, m)}{\mathrm{cn}(\xi, m)}. \tag{14}$$

(8) If $a_0 = 1$, $a_1 = c/(\beta k^2) = 2m^2 - 1$ and $a_2 = -\alpha/(6\beta k^2) = (m^2 - 1)m^2$, then the solution is

$$u_8 = \mathrm{sd}(\xi, m) \equiv \frac{\mathrm{sn}(\xi, m)}{\mathrm{dn}(\xi, m)}. \tag{15}$$

(9) If $a_0 = 1 - m^2$, $a_1 = c/(\beta k^2) = 2 - m^2$ and $a_2 = -\alpha/(6\beta k^2) = 1$, then the solution is

$$u_9 = \mathrm{cs}(\xi, m) \equiv \frac{\mathrm{cn}(\xi, m)}{\mathrm{sn}(\xi, m)}. \tag{16}$$

(10) If $a_0 = 1$, $a_1 = c/(\beta k^2) = -(1 + m^2)$ and $a_2 = -\alpha/(6\beta k^2) = m^2$, then the solution is

$$u_{10} = \mathrm{cd}(\xi, m) \equiv \frac{\mathrm{cn}(\xi, m)}{\mathrm{dn}(\xi, m)}. \tag{17}$$

(11) If $a_0 = m^2(m^2 - 1)$, $a_1 = c/(\beta k^2) = 2m^2 - 1$ and $a_2 = -\alpha/(6\beta k^2) = 1$, then the solution is

$$u_{11} = \mathrm{ds}(\xi, m) \equiv \frac{\mathrm{dn}(\xi, m)}{\mathrm{sn}(\xi, m)}. \tag{18}$$

(12) If $a_0 = m^2$, $a_1 = c/(\beta k^2) = -(1 + m^2)$ and $a_2 = -\alpha/(6\beta k^2) = 1$, then the solution is

$$u_{12} = \mathrm{dc}(\xi, m) \equiv \frac{\mathrm{dn}(\xi, m)}{\mathrm{cn}(\xi, m)}. \tag{19}$$

There still exist many other kinds of Jacobi elliptic functions, we do not show here. It is known that when $m \to 1$, $\mathrm{sn}(\xi, m) \to \tanh\xi$, $\mathrm{cn}(\xi, m) \to \mathrm{sech}\,\xi$, $\mathrm{dn}(\xi, m) \to \mathrm{sech}\,\xi$ and when $m \to 0$, $\mathrm{sn}(\xi, m) \to \sin\xi$, $\mathrm{cn}(\xi, m) \to \cos\xi$, so we also can derive solutions expressed in terms of hyperbolic functions and trigonometric functions.

The second case for Eq. (5) is $D \neq 0$, there will exist different kinds of solutions. In order to solve Eq. (5), we introduce a fractional transformation, i.e.,

$$u(\xi) = \frac{b_0 + b_1 y^2(\xi)}{1 + b_2 y^2(\xi)}, \tag{20}$$

where $y(\xi)$ is given by Eqs. (1) and (2).

In order to obtain nontrivial solutions, there is a constraint

$$b_0 b_2 - b_1 \neq 0 \tag{21}$$

for the fractional transformation. Through the fractional transformation (20), the solutions of Eq. (5) with $D \neq 0$ are mapped to those of the elliptic equation (1) or (2).

Considering the fractional transformation (20) and the elliptic equation (1) or (2), we have

$$u'' = 2(b_1 - b_0b_2)\frac{a_0 + (2a_1 - 3a_0b_2)y^2 + (3a_2 - 2a_1b_2)y^4 - a_2b_2y^6}{(1+b_2y^2)^3}, \tag{22}$$

$$u^3 = \frac{b_0^3 + 3b_0^2b_1y^2 + 3b_0b_1^2y^4 + b_1^3y^6}{1 + 3b_2y^2 + 3b_2^2y^4 + b_2^3y^6}. \tag{23}$$

Substituting Eqs. (20), (22) and (23) into Eq. (5) results in

$$\begin{aligned}
&\left[2a_0\beta k^2(b_1 - b_0b_2) + \frac{\alpha}{3}b_0^3 - cb_0 - D\right] \\
&\quad + \left[2\beta k^2(b_1 - b_0b_2)(2a_1 - 3a_0b_2) + \alpha b_0^2b_1 - c(2b_0b_2 + b_1) - 3Db_2\right]y^2 \\
&\quad + \left[2\beta k^2(b_1 - b_0b_2)(3a_2 - 2a_1b_2) + \alpha b_0b_1^2 - c\left(b_0b_2^2 + 2b_1b_2\right) - 3Db_2^2\right]y^4 \\
&\quad + \left[-2\beta k^2(b_1 - b_0b_2)a_2b_2 + \frac{\alpha}{3}b_1^3 - cb_1b_2^2 - Db_2^3\right]y^6 = 0.
\end{aligned} \tag{24}$$

The arbitrariness of argument ξ for function $y(\xi)$ leads to the consistency conditions

$$2a_0\beta k^2(b_1 - b_0b_2) + \frac{\alpha}{3}b_0^3 - cb_0 - D = 0, \tag{25}$$

$$2\beta k^2(b_1 - b_0b_2)(2a_1 - 3a_0b_2) + \alpha b_0^2b_1 - c(2b_0b_2 + b_1) - 3Db_2 = 0, \tag{26}$$

$$2\beta k^2(b_1 - b_0b_2)(3a_2 - 2a_1b_2) + \alpha b_0b_1^2 - c\left(b_0b_2^2 + 2b_1b_2\right) - 3Db_2^2 = 0, \tag{27}$$

$$-2\beta k^2(b_1 - b_0b_2)a_2b_2 + \frac{\alpha}{3}b_1^3 - cb_1b_2^2 - Db_2^3 = 0. \tag{28}$$

We can see that there are rich structures resulted from Eqs. (25), (26), (27) and (28) in the range of parameter values of (5). Here we show two special cases.

Case 1. $b_0 = 0$, $b_1 \neq 0$ *and* $b_2 \neq 0$

In this case, Eqs. (25), (26), (27) and (28) are rewritten as

$$2a_0\beta k^2b_1 - D = 0, \tag{29}$$

$$2\beta k^2b_1(2a_1 - 3a_0b_2) - cb_1 - 3Db_2 = 0, \tag{30}$$

$$2\beta k^2b_1(3a_2 - 2a_1b_2) - 2cb_1b_2 - 3Db_2^2 = 0, \tag{31}$$

$$-2\beta k^2a_2b_1b_2 + \frac{\alpha}{3}b_1^3 - cb_1b_2^2 - Db_2^3 = 0, \tag{32}$$

from which we can obtain

$$b_1 = \frac{D}{2a_0\beta k^2}, \qquad b_2 = \frac{4a_1\beta k^2 - c}{12a_0\beta k^2} \tag{33}$$

with constraints

$$c^2 = 16\beta^2k^4\left(a_1^2 - 3a_0a_2\right) \tag{34}$$

and

$$\left(4a_1\beta k^2 - c\right)^3 + 6c\left(4a_1\beta k^2 - c\right)^2 + 144a_0a_2\beta^2k^4\left(4a_1\beta k^2 - c\right) - 72\alpha D^2 = 0. \tag{35}$$

From constraint (34), we know that

$$a_1^2 - 3a_0a_2 \geqslant 0. \tag{36}$$

Recalling the solutions to Eqs. (1) or (2), i.e., solutions from u_1 to u_{12}, we can obtain another new rational periodic solutions.

(1) If $a_0 = 1$, $a_1 = -(1+m^2)$ and $a_2 = m^2$, then $a_1^2 - 3a_0a_2 = 1 - m^2 + m^4 > 0$ and the solution is

$$y_1 = \mathrm{sn}(\xi, m), \tag{37}$$

$$u_{1a} = \frac{6Dy^2(\xi)}{12a_0\beta k^2 + (4a_1\beta k^2 - c)y^2(\xi)} = \frac{6D\,\mathrm{sn}^2(\xi, m)}{12a_0\beta k^2 + (4a_1\beta k^2 - c)\,\mathrm{sn}^2(\xi, m)}. \tag{38}$$

(2) If $a_0 = 1 - m^2$, $a_1 = 2m^2 - 1$ and $a_2 = -m^2$, then $a_1^2 - 3a_0a_2 = 1 - m^2 + m^4 > 0$ and the solution is

$$y_2 = \mathrm{cn}(\xi, m), \tag{39}$$

$$u_{2a} = \frac{6Dy^2(\xi)}{12a_0\beta k^2 + (4a_1\beta k^2 - c)y^2(\xi)} = \frac{6D\,\mathrm{cn}^2(\xi, m)}{12a_0\beta k^2 + (4a_1\beta k^2 - c)\,\mathrm{cn}^2(\xi, m)}. \tag{40}$$

(3) If $a_0 = 1 - m^2$, $a_1 = 2 - m^2$ and $a_2 = -1$, then $a_1^2 - 3a_0a_2 = 7 - 7m^2 + m^4 > 0$ and the solution is

$$y_3 = \mathrm{dn}(\xi, m), \tag{41}$$

$$u_{3a} = \frac{6Dy^2(\xi)}{12a_0\beta k^2 + (4a_1\beta k^2 - c)y^2(\xi)} = \frac{6D\,\mathrm{dn}^2(\xi, m)}{12a_0\beta k^2 + (4a_1\beta k^2 - c)\,\mathrm{dn}^2(\xi, m)}. \tag{42}$$

(4) If $a_0 = m^2$, $a_1 = -(1+m^2)$ and $a_2 = 1$, then $a_1^2 - 3a_0a_2 = 1 - m^2 + m^4 > 0$ and the solution is

$$y_4 = \mathrm{ns}(\xi, m) \equiv \frac{1}{\mathrm{sn}(\xi, m)}, \tag{43}$$

$$u_{4a} = \frac{6Dy^2(\xi)}{12a_0\beta k^2 + (4a_1\beta k^2 - c)y^2(\xi)} = \frac{6D\,\mathrm{ns}^2(\xi, m)}{12a_0\beta k^2 + (4a_1\beta k^2 - c)\,\mathrm{ns}^2(\xi, m)}. \tag{44}$$

(5) If $a_0 = -m^2$, $a_1 = 2m^2 - 1$ and $a_2 = 1 - m^2$, then $a_1^2 - 3a_0a_2 = 1 - m^2 + m^4 > 0$ and the solution is

$$y_5 = \mathrm{nc}(\xi, m) \equiv \frac{1}{\mathrm{cn}(\xi, m)}, \tag{45}$$

$$u_{5a} = \frac{6Dy^2(\xi)}{12a_0\beta k^2 + (4a_1\beta k^2 - c)y^2(\xi)} = \frac{6D\,\mathrm{nc}^2(\xi, m)}{12a_0\beta k^2 + (4a_1\beta k^2 - c)\,\mathrm{nc}^2(\xi, m)}. \tag{46}$$

(6) If $a_0 = -1$, $a_1 = 2 - m^2$ and $a_2 = m^2 - 1$, then $a_1^2 - 3a_0a_2 = 1 - m^2 + m^4 > 0$ and the solution is

$$y_6 = \mathrm{nd}(\xi, m) \equiv \frac{1}{\mathrm{dn}(\xi, m)}, \tag{47}$$

$$u_{6a} = \frac{6Dy^2(\xi)}{12a_0\beta k^2 + (4a_1\beta k^2 - c)y^2(\xi)} = \frac{6D\,\mathrm{nd}^2(\xi, m)}{12a_0\beta k^2 + (4a_1\beta k^2 - c)\,\mathrm{nd}^2(\xi, m)}. \tag{48}$$

(7) If $a_0 = 1$, $a_1 = 2 - m^2$ and $a_2 = 1 - m^2$, then the $a_1^2 - 3a_0a_2 = 1 - m^2 + m^4 > 0$ and solution is

$$y_7 = \mathrm{sc}(\xi, m) \equiv \frac{\mathrm{sn}(\xi, m)}{\mathrm{cn}(\xi, m)}, \tag{49}$$

$$u_{7a} = \frac{6Dy^2(\xi)}{12a_0\beta k^2 + (4a_1\beta k^2 - c)y^2(\xi)} = \frac{6D\,\mathrm{sc}^2(\xi, m)}{12a_0\beta k^2 + (4a_1\beta k^2 - c)\,\mathrm{sc}^2(\xi, m)}. \tag{50}$$

(8) If $a_0 = 1$, $a_1 = 2m^2 - 1$ and $a_2 = (m^2 - 1)m^2$, then $a_1^2 - 3a_0a_2 = 1 - m^2 + m^4 > 0$ and the solution is

$$y_8 = \mathrm{sd}(\xi, m) \equiv \frac{\mathrm{sn}(\xi, m)}{\mathrm{dn}(\xi, m)}, \tag{51}$$

$$u_{8a} = \frac{6Dy^2(\xi)}{12a_0\beta k^2 + (4a_1\beta k^2 - c)y^2(\xi)} = \frac{6D\,\mathrm{sd}^2(\xi, m)}{12a_0\beta k^2 + (4a_1\beta k^2 - c)\,\mathrm{sd}^2(\xi, m)}. \tag{52}$$

(9) If $a_0 = 1 - m^2$, $a_1 = 2 - m^2$ and $a_2 = 1$, then $a_1^2 - 3a_0a_2 = 1 - m^2 + m^4 > 0$ and the solution is

$$y_9 = \mathrm{cs}(\xi, m) \equiv \frac{\mathrm{cn}(\xi, m)}{\mathrm{sn}(\xi, m)}, \tag{53}$$

$$u_{9a} = \frac{6Dy^2(\xi)}{12a_0\beta k^2 + (4a_1\beta k^2 - c)y^2(\xi)} = \frac{6D\,\mathrm{cs}^2(\xi, m)}{12a_0\beta k^2 + (4a_1\beta k^2 - c)\,\mathrm{cs}^2(\xi, m)}. \tag{54}$$

(10) If $a_0 = 1$, $a_1 = -(1 + m^2)$ and $a_2 = m^2$, then $a_1^2 - 3a_0a_2 = 1 - m^2 + m^4 > 0$ and the solution is

$$y_{10} = \mathrm{cd}(\xi, m) \equiv \frac{\mathrm{cn}(\xi, m)}{\mathrm{dn}(\xi, m)}, \tag{55}$$

$$u_{10a} = \frac{6Dy^2(\xi)}{12a_0\beta k^2 + (4a_1\beta k^2 - c)y^2(\xi)} = \frac{6D\,\mathrm{cd}^2(\xi, m)}{12a_0\beta k^2 + (4a_1\beta k^2 - c)\,\mathrm{cd}^2(\xi, m)}. \tag{56}$$

(11) If $a_0 = m^2(m^2 - 1)$, $a_1 = 2m^2 - 1$ and $a_2 = 1$, then $a_1^2 - 3a_0a_2 = 1 - m^2 + m^4 > 0$ and the solution is

$$y_{11} = \mathrm{ds}(\xi, m) \equiv \frac{\mathrm{dn}(\xi, m)}{\mathrm{sn}(\xi, m)}, \tag{57}$$

$$u_{11a} = \frac{6Dy^2(\xi)}{12a_0\beta k^2 + (4a_1\beta k^2 - c)y^2(\xi)} = \frac{6D\,\mathrm{ds}^2(\xi, m)}{12a_0\beta k^2 + (4a_1\beta k^2 - c)\,\mathrm{ds}^2(\xi, m)}. \tag{58}$$

(12) If $a_0 = m^2$, $a_1 = -(1 + m^2)$ and $a_2 = 1$, then $a_1^2 - 3a_0a_2 = 1 - m^2 + m^4 > 0$ and the solution is

$$y_{12} = \mathrm{dc}(\xi, m) \equiv \frac{\mathrm{dn}(\xi, m)}{\mathrm{cn}(\xi, m)}, \tag{59}$$

$$u_{12a} = \frac{6Dy^2(\xi)}{12a_0\beta k^2 + (4a_1\beta k^2 - c)y^2(\xi)} = \frac{6D\,\mathrm{dc}^2(\xi, m)}{12a_0\beta k^2 + (4a_1\beta k^2 - c)\,\mathrm{dc}^2(\xi, m)}. \tag{60}$$

Case 2. $b_0 \neq 0$, $b_1 = 0$ *and* $b_2 \neq 0$

In this case, from Eqs. (25), (26), (27) and (28), we can derive

$$b_0 = \frac{D}{2a_2\beta k^2} b_2, \quad b_2 = \frac{12a_2\beta k^2}{4a_1\beta k^2 - c} \tag{61}$$

with constraints (34) and (35). Similarly, we can obtain solutions just similar to solutions from u_{1a} to u_{12a}, here we omit the details.

3. Conclusion

In this Letter, we reconsider elliptic equation in applying to solve nonlinear wave equations, taking mKdV equation as an example, more kinds of solutions are derived from there, including rational periodic solutions, rational solitary wave solutions constructed in terms of hyperbolic functions, periodic solutions expressed by trigonometric functions and periodic solutions dealing with elliptic functions. In order to derive rational type solutions, a fractional transformation must be introduced. Of course, whether there is application of similar fractional transformation to other nonlinear wave equations is still an open question.

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