Exact Jacobian Elliptic Function Solutions to sinh-Gordon Equation*

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Abstract In this paper, two transformations are introduced to solve sinh-Gordon equation by using the knowledge of elliptic equation and Jacobian elliptic functions. It is shown that different transformations are required in order to obtain more kinds of solutions to the sinh-Gordon equation.

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1 Introduction

The sinh-Gordon (ShG for short) equation\[1\] of noncommutative field theories,[2] \[3\] fluid dynamics,[3] and so on. Due to the wide applications of sinh-Gordon equation, many achievements have been obtained in different aspects.[3–9] For instance, ShG equation is known to be completely integrable[4] because it possesses similarity reductions to the third Painlevé equation.[5]

Due to the special form of the sinh-Gordon equation, it is very difficult to solve them directly, so one needs some transformations. In this paper, based on the introduced transformations, we will show systematical results about solutions to ShG equation (1) by using the knowledge of elliptic equation and Jacobian elliptic functions.[10–12]

2 The First Kind of Transformation and Solutions to ShG Equation

The first transformation is introduced in the form

\[ u_{xt} = \alpha \sinh u \]  

(1)

is widely applied in physics and engineering, for example, integrable quantum field theories,[1] noncommutative field theories,[2] fluid dynamics,[3] and so on. Due to the wide applications of sinh-Gordon equation, many achievements have been obtained in different aspects.[3–9] For instance, ShG equation is known to be completely integrable[4] because it possesses similarity reductions to the third Painlevé equation.[5]

Equation (5) can be solved in the frame

\[ v = v(\xi), \quad \alpha = k(x - ct), \]  

(6)

where \( k \) and \( c \) are wave number and wave speed, respectively.

Substituting Eqs. (6) into Eqs. (5), we have

\[ (1 + v^2) \frac{d^2v}{d\xi^2} - v \left( \frac{dv}{d\xi} \right)^2 + \alpha_1 v(1 + v^2) = 0, \]

(7)

and then we suppose equation (7) has the following solution:

\[ v = v(y) = \sum_{j=0}^{n} b_j y^j, \quad b_n \neq 0, \quad y = y(\xi), \]

(8)

where \( y \) satisfies elliptic equation[10,13]

\[ y'^2 = a_0 + a_2 y^2 + a_4 y^4, \quad a_4 \neq 0, \quad y' = \frac{dy}{d\xi}. \]

(9)

There \( n \) in Eq. (8) can be determined by the partial balance between the highest order derivative terms and the highest degree nonlinear term in Eq. (7). Here we know that the degree of \( v \) is

\[ O(v) = O(y^n) = n, \]

(10)

and from Eq. (9), one has

\[ O(y^2) = O(y^4) = 4, \]

\[ O(y'^4) = O(y'^3) = 3, \]

\[ O(y'^6) = l + 1. \]  

(11)

So one has

\[ O(v) = n, \quad O(v') = n + 1, \]

(12)


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\( O(v''') = n + 2, \quad O(v^{(l)}) = n + l. \) (12)

For ShG equation (1), we have \( n = 1 \), so the ansatz solution of Eq. (7) can be rewritten as
\[
v = b_0 + b_1 y, \quad b_1 \neq 0.\quad (13)
\]

Substituting Eq. (13) into Eq. (7) results in an algebraic equation for \( y \), which can be used to determine expansion coefficients in Eq. (13) and some constraints can also be obtained. Here we have
\[
b_0 = 0, \quad b_1 = \pm \sqrt{\frac{a_4}{\alpha}}\quad (14)
\]
\[
a_1 = \frac{\alpha}{k^2c} = \frac{-a_2 \pm \sqrt{a_2^2 - 4a_0a_4}}{2}.\quad (15)
\]

In order to derive real solutions, there are two constraints,
\[
-\frac{a_4}{a_1} > 0, \quad a_2^2 - 4a_0a_4 \geq 0.\quad (16)
\]

There are some cases to be considered. For example,

**Case 1** If \( a_0 = 1 - m^2, a_2 = 2m^2 - 1, a_4 = -m^2 \), where \( 0 \leq m \leq 1 \) is called modulus of Jacobian elliptic functions,\(^{[13-15]}\) then

\[
y = \frac{a_4}{a_1} \xi, \quad b_0 = 0, \quad b_1 = \pm \frac{m}{\sqrt{1 - m^2}}, \quad c = \frac{\alpha}{k^2(1 - m^2)}, \quad 0 < m < 1,\quad (17)
\]

where \( k \) is an arbitrary constant, and \( c \xi \) is Jacobian elliptic cosine function.\(^{[13-15]}\) So the solution to ShG equation (1) is

\[
u_1 = 2 \sinh^{-1}\left(\pm \frac{m}{\sqrt{1 - m^2}} c \xi \right).\quad (18)
\]

**Case 2** If \( a_0 = -m^2, a_2 = 2m^2 - 1, a_4 = 1 - m^2 \), then

\[
y = \frac{a_4}{a_1} \xi, \quad b_0 = 0, \quad b_1 = \pm \frac{m}{\sqrt{1 - m^2}}, \quad c = -\frac{\alpha}{m^2 k^2}, \quad 0 < m < 1,\quad (19)
\]

where \( k \) is an arbitrary constant, and then the solution to ShG equation (1) is

\[
u_2 = 2 \sinh^{-1}\left(\pm \frac{m}{\sqrt{1 - m^2}} c \xi \right).\quad (20)
\]

**Case 3** If \( a_0 = 1, a_2 = 2 - m^2, a_4 = 1 - m^2 \), then

\[
y = \frac{a_4}{a_1} \xi, \quad b_0 = 0, \quad b_1 = \pm \frac{1}{\sqrt{1 - m^2}}, \quad c = -\frac{\alpha}{(1 - m^2) k^2}, \quad 0 < m < 1,\quad (21)
\]

where \( k \) is an arbitrary constant, and \( \frac{a_4}{a_1} \xi \) is Jacobian elliptic sine function.\(^{[13-15]}\) So the solution to ShG equation (1) is

\[
u_3 = 2 \sinh^{-1}\left(\pm \frac{1}{\sqrt{1 - m^2}} \xi \right).\quad (22)
\]

**Case 4** If \( a_0 = 1, a_2 = 2 - m^2, a_4 = 1 - m^2 \), then

\[
y = \frac{a_4}{a_1} \xi, \quad b_0 = 0, \quad b_1 = \pm \frac{1}{\sqrt{1 - m^2}}, \quad c = -\frac{\alpha}{k^2}, \quad 0 < m < 1,\quad (23)
\]

where \( k \) is an arbitrary constant, and the solution to ShG equation (1) is

\[
u_4 = 2 \sinh^{-1}\left(\pm \frac{1}{\sqrt{1 - m^2} m \xi} \right).\quad (24)
\]

**Case 5** If \( a_0 = 1, a_2 = 2m^2 - 1, a_4 = (m^2 - 1) m^2 \), then

\[
y = \frac{a_4}{a_1} \xi, \quad b_0 = 0, \quad b_1 = \pm m, \quad c = -\frac{\alpha}{(1 - m^2) k^2}, \quad 0 < m < 1,\quad (25)
\]

where \( k \) is an arbitrary constant, and \( \frac{a_4}{a_1} \xi \) is Jacobian elliptic cosine function.\(^{[13-15]}\) So the solution to ShG equation (1) is

\[
u_5 = 2 \sinh^{-1}\left(\pm m \xi \right).\quad (26)
\]

When \( m \rightarrow 1 \), \( u_6 \) recovers

\[
u_6 = 2 \sinh^{-1}\left(\pm \sinh \xi \right).\quad (27)
\]

**Case 6** If \( a_0 = 1 - m^2, a_2 = 2 - m^2, a_4 = 1 \), then

\[
y = \frac{a_4}{a_1} \xi, \quad b_0 = 0, \quad b_1 = \pm 1, \quad c = -\frac{\alpha}{k^2}, \quad 0 < m \leq 1,\quad (28)
\]

where \( k \) is an arbitrary constant, and then the solution to ShG equation (1) is

\[
u_7 = 2 \sinh^{-1}\left(\pm \sinh \xi \right).\quad (29)
\]

**Case 7** If \( a_0 = 1 - m^2, a_2 = 2 - m^2, a_4 = 1 \), then

\[
y = \frac{a_4}{a_1} \xi, \quad b_0 = 0, \quad b_1 = \pm 1, \quad c = -\frac{\alpha}{(1 - m^2) k^2}, \quad 0 < m \leq 1,\quad (30)
\]

where \( k \) is an arbitrary constant, and the solution to ShG equation (1) is

\[
u_8 = 2 \sinh^{-1}\left(\pm \frac{1}{\sqrt{1 - m^2}} \xi \right).\quad (31)
\]

**Case 8** If \( a_0 = m^2 (m^2 - 1), a_2 = 2m^2 - 1, a_4 = 1 \), then

\[
y = \frac{a_4}{a_1} \xi, \quad b_0 = 0, \quad b_1 = \pm \frac{1}{m}, \quad c = -\frac{\alpha}{(m^2) k^2}, \quad 0 < m \leq 1,\quad (32)
\]

where \( k \) is an arbitrary constant, and then the solution to ShG equation (1) is

\[
u_9 = 2 \sinh^{-1}\left(\pm \frac{1}{m} \xi \right).\quad (33)
\]
When $m \to 1$, $u_9$ recovers $u_7$.

Apart from these Jacobian elliptic solutions, $y$ also has some rational solutions in terms of Jacobian elliptic functions, such as the following.

**Case 9** If $a_0 = (1 - m^2)/4$, $a_2 = (1 + m^2)/2$, $a_4 = (1 - m^2)/4$, then

$$
\begin{align*}
    y &= \frac{\text{cn} \xi}{1 \pm \text{sn} \xi}, \\
    b_0 &= 0, \\
    b_1 &= \pm \frac{\sqrt{1 - m^2}}{1 + m}, \\
    c &= -\frac{4\alpha}{(1 + m^2)k^2}, \\
    0 &< m < 1,
\end{align*}
$$
(33)

where $k$ is an arbitrary constant. So the solution to ShG equation (1) is

$$
u_{10} = 2 \sinh^{-1} \left( \pm \frac{\sqrt{1 - m^2}}{1 + m} \frac{\text{cn} \xi}{1 \pm \text{sn} \xi} \right),
$$
(34)

**Case 10** If $a_0 = (1 - m^2)/4$, $a_2 = (1 + m^2)/2$, $a_4 = (1 - m^2)/4$, then

$$
\begin{align*}
    y &= \frac{\text{cn} \xi}{1 \pm \text{sn} \xi}, \\
    b_0 &= 0, \\
    b_1 &= \pm \frac{\sqrt{1 - m^2}}{1 - m}, \\
    c &= -\frac{4\alpha}{(1 - m^2)k^2}, \\
    0 &< m < 1,
\end{align*}
$$
(35)

where $k$ is an arbitrary constant, and the solution to ShG equation (1) is

$$
u_{11} = 2 \sinh^{-1} \left( \pm \frac{\sqrt{1 - m^2}}{1 - m} \frac{\text{cn} \xi}{1 \pm \text{sn} \xi} \right),
$$
(36)

**Remark**: Transformation (2) and the solutions from $u_1$ to $u_{11}$ in terms of Jacobian elliptic functions have not been reported in the literature.

### 3 The Second Kind of Transformation and Solutions to ShG Equation

The second transformation is introduced in the form

$$
u = 2 \cosh^{-1} v \quad \text{or} \quad v = \cosh \frac{u}{2},
$$
(37)

and then

$$
\sinh u = 2 \sinh \frac{u}{2} \cosh \frac{u}{2} = 2v \sqrt{v^2 - 1},
$$
(38)

and

$$
u_{tx} = \sqrt{v^2 - 1} v_x - \frac{2v}{(v^2 - 1)^{1/2}} v v_x.
$$
(39)

Combining Eqs. (38) and (39) with Eq. (1), the ShG equation can be rewritten as

$$
(v^2 - 1)v_x - v v_x - \alpha v(v^2 - 1)^2 = 0.
$$
(40)

We can see that equation (40) is similar to Eq. (5), so it can be easily solved just like what we have done to Eq. (5). Here we have

$$
b_0 = 0, \quad b_1 = \pm \sqrt{\frac{\alpha k^2 c}{\alpha}},
$$

$$
\frac{\alpha}{k^2 c} = \frac{a_2 \pm \sqrt{a_2^2 - 4a_0 a_4}}{2},
$$
(41)

with constraints

$$
\frac{a_1 k^2 c}{\alpha} > 0, \quad a_2^2 - 4a_0 a_4 \geq 0.
$$
(42)

Similarly, there are some cases to be considered. For example,

**Case 1** If $a_0 = 1$, $a_2 = -(1 + m^2)$, $a_4 = m^2$, then

$$
\begin{align*}
    y &= \text{sn} \xi, \\
    b_0 &= 0, \\
    b_1 &= \pm 1,
\end{align*}
$$
(43)

where $k$ is an arbitrary constant, and the solution to ShG equation (1) is

$$
u_{12} = 2 \cosh^{-1}(\pm \text{sn} \xi),
$$
(44)

Actually, when $m \to 1$, $u_{12}$ and $u_{13}$ both recover to

$$
u_{14} = 2 \cosh^{-1}(\pm \tanh \xi), \quad c = -\frac{\alpha}{k^2},
$$
(47)

**Case 3** If $a_0 = 1 - m^2$, $a_2 = 2m^2 - 1$, $a_4 = -m^2$, then

$$
\begin{align*}
    y &= \text{cn} \xi, \\
    b_0 &= 0, \\
    b_1 &= \pm 1,
\end{align*}
$$
(48)

where $k$ is an arbitrary constant, and the solution to ShG equation (1) is

$$
u_{15} = 2 \cosh^{-1}(\pm \text{cn} \xi),
$$
(49)

Actually, when $m \to 1$, $u_{15}$ recovers to

$$
u_{16} = 2 \cosh^{-1}(\pm \text{sech} \xi), \quad c = \frac{\alpha}{k^2},
$$
(50)

**Case 4** If $a_0 = m^2 - 1$, $a_2 = 2 - m^2$, $a_4 = -1$, then

$$
\begin{align*}
    y &= \text{dn} \xi, \\
    b_0 &= 0, \\
    b_1 &= \pm 1,
\end{align*}
$$
(51)

where $k$ is an arbitrary constant, and the solution to ShG equation (1) is

$$
u_{17} = 2 \cosh^{-1}(\pm \text{dn} \xi).
$$
(52)

Actually, when $m \to 1$, $u_{17}$ recovers $u_{16}$.

**Case 5** If $a_0 = m^2 - 1$, $a_2 = 2 - m^2$, $a_4 = -1$, then

$$
\begin{align*}
    y &= \text{dn} \xi, \\
    b_0 &= 0, \\
    b_1 &= \pm \frac{1}{\sqrt{1 - m^2}},
\end{align*}
$$
\[ c = \frac{\alpha}{(1-m^2)k^2}, \quad 0 < m < 1, \quad (53) \]

where \( k \) is an arbitrary constant, and the solution to ShG equation (1) is

\[ u_{18} = 2 \cosh^{-1} \left( \pm \frac{1}{\sqrt{1-m^2}} \right). \quad (54) \]

**Case 6** If \( a_0 = m^2, a_2 = -(1+m^2), a_4 = 1 \), then

\[ y = \text{ns} \xi, \quad b_0 = 0, \quad b_1 = \pm 1, \quad c = -\frac{\alpha}{k^2}, \quad 0 < m \leq 1, \quad (55) \]

where \( k \) is an arbitrary constant, and the solution to ShG equation (1) is

\[ u_{19} = 2 \cosh^{-1} \left( \pm \frac{1}{m} \text{ns} \xi \right). \quad (56) \]

**Case 7** If \( a_0 = m^2, a_2 = -(1+m^2), a_4 = 1 \), then

\[ y = \text{ns} \xi, \quad b_0 = 0, \quad b_1 = \pm 1, \quad c = -\frac{\alpha}{k^2}, \quad 0 < m \leq 1, \quad (57) \]

where \( k \) is an arbitrary constant, and the solution to ShG equation (1) is

\[ u_{20} = 2 \cosh^{-1} (\pm \text{ns} \xi). \quad (58) \]

Actually, when \( m \to 1 \), \( u_{19} \) and \( u_{20} \) both recover to

\[ u_{21} = 2 \cosh^{-1} (\pm \text{coth} \xi), \quad c = -\frac{\alpha}{k^2}. \quad (59) \]

**Case 8** If \( a_0 = -m^2, a_2 = 2m^2 - 1, a_4 = 1 - m^2, \) then

\[ y = \text{nc} \xi, \quad b_0 = 0, \quad b_1 = \pm 1, \quad c = -\frac{\alpha}{(1-m^2)k^2}, \quad 0 < m < 1, \quad (60) \]

where \( k \) is an arbitrary constant, and the solution to ShG equation (1) is

\[ u_{22} = 2 \cosh^{-1} (\pm \text{nc} \xi). \quad (61) \]

**Case 9** If \( a_0 = -1, a_2 = 2 - m^2, a_4 = m^2 - 1, \) then

\[ y = \text{nd} \xi, \quad b_0 = 0, \quad b_1 = \pm 1, \quad c = -\frac{\alpha}{(m^2 - 1)k^2}, \quad 0 < m < 1, \quad (62) \]

where \( k \) is an arbitrary constant, and the solution to ShG equation (1) is

\[ u_{23} = 2 \cosh^{-1} (\pm \text{nd} \xi). \quad (63) \]

**Case 10** If \( a_0 = -1, a_2 = 2 - m^2, a_4 = m^2 - 1, \) then

\[ y = \text{nd} \xi, \quad b_0 = 0, \quad b_1 = \pm \sqrt{1-m^2}, \quad c = \frac{\alpha}{k^2}, \quad 0 < m < 1, \quad (64) \]

where \( k \) is an arbitrary constant, and the solution to ShG equation (1) is

\[ u_{24} = 2 \cosh^{-1} (\pm \sqrt{1-m^2} \text{nd} \xi). \quad (65) \]

**Case 11** If \( a_0 = 1, a_2 = 2m^2 - 1, a_4 = (m^2 - 1)m^2, \) then

\[ y = \text{sd} \xi, \quad b_0 = 0, \quad b_1 = \pm \sqrt{1-m^2}, \quad c = -\frac{\alpha}{m^2k^2}, \quad 0 < m < 1, \quad (66) \]

where \( k \) is an arbitrary constant, and the solution to ShG equation (1) is

\[ u_{25} = 2 \cosh^{-1} (\pm \sqrt{1-m^2} \text{sd} \xi). \quad (67) \]

**Case 12** If \( a_0 = 1, a_2 = -(1+m^2), a_4 = m^2, \) then

\[ y = \text{cd} \xi, \quad b_0 = 0, \quad b_1 = \pm m, \quad c = -\frac{\alpha}{k^2}, \quad 0 < m \leq 1, \quad (68) \]

where \( k \) is an arbitrary constant, and the solution to ShG equation (1) is

\[ u_{26} = 2 \cosh^{-1} (\pm m \text{cd} \xi). \quad (69) \]

**Case 13** If \( a_0 = 1, a_2 = -(1+m^2), a_4 = m^2, \) then

\[ y = \text{cd} \xi, \quad b_0 = 0, \quad b_1 = \pm 1, \quad c = -\frac{\alpha}{m^2k^2}, \quad 0 < m \leq 1, \quad (70) \]

where \( k \) is an arbitrary constant, and the solution to ShG equation (1) is

\[ u_{27} = 2 \cosh^{-1} (\pm \text{cd} \xi). \quad (71) \]

**Case 14** If \( a_0 = m^2(m^2 - 1), a_2 = 2m^2 - 1, a_4 = 1, \) then

\[ y = \text{sd} \xi, \quad b_0 = 0, \quad b_1 = \pm \frac{1}{\sqrt{1-m^2}}, \quad c = -\frac{\alpha}{(1-m^2)k^2}, \quad 0 < m < 1, \quad (72) \]

where \( k \) is an arbitrary constant, and the solution to ShG equation (1) is

\[ u_{28} = 2 \cosh^{-1} (\pm \frac{1}{m} \text{sd} \xi). \quad (73) \]

**Case 15** If \( a_0 = m^2, a_2 = -(1+m^2), a_4 = 1, \) then

\[ y = \text{dc} \xi, \quad b_0 = 0, \quad b_1 = \pm \frac{1}{m}, \quad c = -\frac{\alpha}{m^2k^2}, \quad 0 < m \leq 1, \quad (74) \]

where \( k \) is an arbitrary constant, and the solution to ShG equation (1) is

\[ u_{29} = 2 \cosh^{-1} (\pm \frac{1}{m} \text{dc} \xi). \quad (75) \]

**Case 16** If \( a_0 = m^2, a_2 = -(1+m^2), a_4 = 1, \) then

\[ y = \text{dc} \xi, \quad b_0 = 0, \quad b_1 = \pm 1, \quad c = -\frac{\alpha}{k^2}, \quad 0 < m \leq 1, \quad (76) \]
where \( k \) is an arbitrary constant, and then the solution to ShG equation (1) is
\[
u_{30} = 2 \cosh^{-1}(\pm dc \xi). \tag{77}
\]

**Case 17** If \( a_0 = -(1 - m^2)/4, a_2 = (1 + m^2)/2, a_4 = -(1 - m^2)/4, \) then
\[
y = \frac{\text{dn} \xi}{1 \pm m \text{sn} \xi}, \quad b_0 = 0, \quad b_1 = \pm \frac{\sqrt{1 - m^2}}{1 + m},
\]
\[
c = \frac{4a}{(1 + m)^2 k^2}, \quad 0 < m < 1, \tag{78}
\]
where \( k \) is an arbitrary constant, and then the solution to ShG equation (1) is
\[
u_{31} = 2 \cosh^{-1}\left(\pm \frac{\sqrt{1 - m^2}}{1 + m} \frac{\text{dn} \xi}{1 \pm m \text{sn} \xi}\right). \tag{79}
\]

**Case 18** If \( a_0 = -(1 - m^2)/4, a_2 = (1 + m^2)/2, a_4 = -(1 - m^2)/4, \) then
\[
y = \frac{\text{dn} \xi}{1 \pm m \text{sn} \xi}, \quad b_0 = 0, \quad b_1 = \pm \frac{\sqrt{1 - m^2}}{1 + m},
\]
\[
c = \frac{4a}{(1 + m)^2 k^2}, \quad 0 < m < 1, \tag{80}
\]
where \( k \) is an arbitrary constant, and then the solution to ShG equation (1) is
\[
u_{32} = 2 \cosh^{-1}\left(\pm \frac{\sqrt{1 - m^2}}{1 + m} \frac{\text{dn} \xi}{1 \pm m \text{sn} \xi}\right). \tag{81}
\]

**Case 19** If \( a_0 = m^2/4, a_2 = -(2 - m^2)/2, a_4 = m^2/4, \) then
\[
y = \frac{m \text{sn} \xi}{1 \pm \text{dn} \xi}, \quad b_0 = 0,
\]
\[
b_1 = \pm \frac{m}{\sqrt{2 - m^2 + 2\sqrt{1 - m^2}}},
\]
\[
c = \frac{4a}{(\sqrt{2 - m^2 + 2\sqrt{1 - m^2})^2}}, \quad 0 < m \leq 1, \tag{82}
\]
where \( k \) is an arbitrary constant, and then the solution to ShG equation (1) is
\[
u_{33} = 2 \cosh^{-1}\left(\pm \frac{m}{\sqrt{2 - m^2 + 2\sqrt{1 - m^2}}} \frac{m \text{sn} \xi}{1 \pm \text{dn} \xi}\right). \tag{83}
\]

**Case 20** If \( a_0 = m^2/4, a_2 = -(2 - m^2)/2, a_4 = m^2/4, \) then
\[
y = \frac{m \text{sn} \xi}{1 \pm \text{dn} \xi}, \quad b_0 = 0,
\]
\[
b_1 = \pm \frac{m}{\sqrt{2 - m^2 - 2\sqrt{1 - m^2}}},
\]
\[
c = \frac{4a}{(\sqrt{2 - m^2 - 2\sqrt{1 - m^2})^2}}, \quad 0 < m \leq 1, \tag{84}
\]
where \( k \) is an arbitrary constant, and then the solution to ShG equation (1) is
\[
u_{34} = 2 \cosh^{-1}\left(\pm \frac{m}{\sqrt{2 - m^2 - 2\sqrt{1 - m^2}}} \frac{m \text{sn} \xi}{1 \pm \text{dn} \xi}\right). \tag{85}
\]

When \( m \to 1, \) \( u_{33} \) and \( u_{34} \) both recover to
\[
u_{35} = 2 \cosh^{-1}\left(\pm \frac{\tanh \xi}{1 \pm \text{sech} \xi}\right), \quad c = -\frac{4a}{k^2}. \tag{86}
\]

**Case 21** If \( a_0 = 1/4, a_2 = -(2 - m^2)/2, a_4 = m^4/4, \) then
\[
y = \frac{\text{sn} \xi}{1 \pm \text{dn} \xi}, \quad b_0 = 0, \quad
\]
\[
b_1 = \pm \frac{m^2}{\sqrt{2 - m^2 + 2\sqrt{1 - m^2}}},
\]
\[
c = -\frac{4a}{(\sqrt{2 - m^2 + 2\sqrt{1 - m^2})^2}}, \quad 0 < m \leq 1, \tag{87}
\]
where \( k \) is an arbitrary constant.

So the solution to ShG equation (1) is
\[
u_{36} = 2 \cosh^{-1}\left(\frac{m^2}{\sqrt{2 - m^2 + 2\sqrt{1 - m^2}}} \frac{\text{sn} \xi}{1 \pm \text{dn} \xi}\right). \tag{88}
\]

**Case 22** If \( a_0 = 1/4, a_2 = -(2 - m^2)/2, a_4 = m^4/4, \) then
\[
y = \frac{\text{sn} \xi}{1 \pm \text{dn} \xi}, \quad b_0 = 0, \quad
\]
\[
b_1 = \pm \frac{m^2}{\sqrt{2 - m^2 - 2\sqrt{1 - m^2}}},
\]
\[
c = -\frac{4a}{(\sqrt{2 - m^2 - 2\sqrt{1 - m^2})^2}}, \quad 0 < m \leq 1, \tag{89}
\]
where \( k \) is an arbitrary constant, and then the solution to ShG equation (1) is
\[
u_{37} = 2 \cosh^{-1}\left(\frac{m^2}{\sqrt{2 - m^2 - 2\sqrt{1 - m^2}}} \frac{\text{sn} \xi}{1 \pm \text{dn} \xi}\right). \tag{90}
\]

When \( m \to 1, \) \( u_{36} \) and \( u_{37} \) both recover to \( u_{35}. \)

**Remark** Transformation (37) and the solutions from \( u_{12} \) to \( u_{37} \) in terms of Jacobian elliptic functions have not been given in the literature.

**4 Conclusion**

In this paper, two transformations are introduced to solve sinh-Gordon equation by using the knowledge of elliptic equation and Jacobian elliptic functions. It is shown that different transformations are required in order to obtain more kinds of solutions to the sinh-Gordon equation. Here some new solutions have not been reported in the literature. It is shown that different transformations play different roles in obtaining exact solutions, some transformations may not work for a specific parameter of ShG equation. Of course, still more efforts are needed to explore what kinds of transformations are more suitable
to solving sinh-Gordon equation, because different transformations result in different partial balances for sinh-Gordon equation, which will lead to different expansion truncations in the elliptic equation expansion method. Finally, these will result in different solutions of the sine-Gordon equation.

References