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Periodic Solutions to KdV–Burgers–Kuramoto Equation*

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Abstract *In this paper, a new special ansatz solution, where elliptic equation satisfied by elliptic functions is taken as an intermediate transformation, is applied to solve the KdV-Burgers-Kuramoto equation, and many more new periodic solutions are obtained, including solutions expressed in terms of Jacobi elliptic functions, solution expressed in terms of Weierstrass elliptic function.*

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1 Introduction

Many phenomena are simultaneously involved in nonlinearity, dissipation, dispersion and instability. As Kuramoto[1] suggested, KdV-Burgers-Kuramoto equation[2−4]

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\alpha\frac{\partial^2 u}{\partial x^2}+\beta\frac{\partial^3 u}{\partial x^3}+\gamma\frac{\partial^4 u}{\partial x^4}=0 \tag{1}$$

is an appropriate model to describe these phenomena, especially at the description of turbulence processes,[2,4] where α, β, and γ are constants. Equation (1) is also known as the Kuramoto–Sivashinsky equation[2,3,5] or Benney equation.[4]

In order to well understand various nonlinear phenomena, many methods for obtaining analytical solutions of nonlinear evolution equations have been proposed, among which are the homogeneous balance method,[6] the hyperbolic function expansion method,[7,8] the Jacobi elliptic function expansion method,[9,10] the trial function method,[4,11] the nonlinear transformation method,[5,12−14] the inverse scattering method,[12] truncated expansion method,[15] and so on.

The solutions of KdV-Burgers-Kuramoto equation possess their actual physical application, which is the reason why so many methods, such as trial-function method,[4] Weiss–Tabor–Carnevale transformation method,[5], homogeneous balance method,[6] tanh-function method,[7] and so on, have been applied to obtain the solutions to KdV-Burgers-Kuramoto equation.[15,16] But no method is both convenient and able to be used to get as many solutions as possible. Weiss–Tabor–Carnevale transformation method, tanh-function method, and homogeneous balance method are complex in deducing the solution to KdV-Burgers-Kuramoto equation. No explicit solution and parameter constraint were given in Ref. [7], and only one special case was considered in Ref. [6], so only a special solution was given. Trial-function method is a simple one, some solutions can be easily obtained, too. However, all solutions to KdV-Burgers-Kuramoto equation given in the literature are expressed in terms of hyperbolic functions, exponential function or other elementary functions. To our knowledge, no solutions in terms of special functions have been reported. Generally, there is no solution in terms of special functions to nonlinear equations where odd-order derivative term(s) and even-order derivative term(s) coexist.[10] For example, we cannot obtain solution in terms of Jacobi elliptic function or Weierstrass elliptic function to Burgers equation or KdV-Burgers equation.[17] But, for some of this kind of nonlinear evolution equations, just like KdV-Burgers-Kuramoto equation, under certain conditions, there can exist solutions in terms of special functions, such as Jacobi elliptic function and Weierstrass elliptic function.

Actually, elliptic equation can be taken as an intermediate transformation to solve nonlinear wave equations,[18] and to derive many periodic solutions and solitary wave solutions. In this paper, we will revisit elliptic equation method and apply it to obtain periodic solutions in terms of Jacobi elliptic function and Weierstrass elliptic function to KdV-Burgers-Kuramoto equation and compare them with solutions given in Refs. [4] ∼ [7], [15], and [16].

2 Jacobi Elliptic Function Solutions to KdV-Burgers-Kuramoto Equation

In order to solve Eq. (1), the following transformation

$$u=u(\xi)\,,\quad \xi=k(x-c_0t) \tag{2}$$

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is needed, where k is called wave number and c_0 is wave speed.

Substituting Eq. (2) into Eq. (1) yields

$$-c_0\frac{\mathrm{d}u}{\mathrm{d}\xi}+u\frac{\mathrm{d}u}{\mathrm{d}\xi}+\alpha k\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2}+\beta k^2\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3}+\gamma k^3\frac{\mathrm{d}^4u}{\mathrm{d}\xi^4}=0\,. \quad (3)$$

Integrating Eq. (3) once leads to

$$-c_0u+\frac{1}{2}u^2+\alpha k\frac{\mathrm{d}u}{\mathrm{d}\xi}+\beta k^2\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2}+\gamma k^3\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3}+c_1=0\,, \quad (4)$$

where c_1 is an integration constant.

We suppose that equation (1) takes the following ansatz solution,

$$u(\xi)=A_0+A_1y+A_2y'\,, \quad (5)$$

where y satisfies elliptic equation.[19]

Remark 1 Contrary to the general function basis expansion methods, which only take expansion of different powers of the basis function as ansatz solution, here we take into the derivatives of basis function in the ansatz expansion solution. Doing so is due to the special form of the KdV-Burgers-Kuramoto equation, where odd-order derivative term(s) and even-order derivative term(s) coexist.

First we consider the following case,

$$y'^2=ay+by^2+cy^3\,, \quad (6)$$

then

$$y''=\frac{a}{2}+by+\frac{3}{2}cy^2\,, \quad (7a)$$

$$y'''=(b+3cy)y'\,, \quad (7b)$$

$$y^{(4)}=\frac{1}{2}ab+\Big(b^2+\frac{9}{2}ac\Big)y+\frac{15}{2}bcy^2+\frac{15}{2}c^2y^3\,. \quad (7c)$$

Combining Eqs. (6) and (7) with Eq. (5) leads to

$$u'=\Big(\frac{1}{2}aA_2+A_1y'\Big)+bA_2y+\frac{3}{2}cA_2y^2\,, \quad (8)$$

$$u^2=(A_0^2+2A_0A_2y')+(2A_0A_1+2A_1A_2y'+aA_2^2)y+(A_1^2+bA_2^2)y^2+cA_2^2y^3\,, \quad (9)$$

$$u''=\Big(\frac{1}{2}aA_1+A_2y'\Big)+(bA_1+3cA_2y')y+\frac{3}{2}cA_1y^2\,, \quad (10)$$

$$u'''=\Big(bA_1+\frac{1}{2}abA_2\Big)+\Big[3cA_1y'+\Big(b^2+\frac{9}{2}ac\Big)A_2\Big]y+\frac{15}{2}bcA_2y^2+\frac{15}{2}c^2A_2y^3\,. \quad (11)$$

Substituting Eqs. (5) and (8) $\sim$ (11) into Eq. (4) results in

$$\frac{1}{2}cA_2^2+\frac{15}{2}\gamma k^3c^2A_2=0\,, \quad (12a)$$

$$\frac{1}{2}(A_1^2+bA_2^2)+\frac{3}{2}c\alpha kA_2+\frac{3}{2}c\beta k^2A_1+\frac{15}{2}bc\gamma k^3A_2=0\,, \quad (12b)$$

$$-c_0A_1+\frac{1}{2}(2A_0A_1+2A_1A_2y'+aA_2^2)+b\alpha kA_2+\beta k^2(bA_1+3cA_2y')+\gamma k^3\Big[3cA_1y'+\Big(b^2+\frac{9}{2}ac\Big)A_2\Big]=0\,, \quad (12c)$$

$$c_1-c_0A_0-c_0A_2y'+\frac{1}{2}(A_0^2+2A_0A_2y')+\alpha k\Big(\frac{1}{2}aA_2+A_1y'\Big)+\beta k^2\Big(\frac{1}{2}aA_1+A_2y'\Big)+\gamma k^3\Big(bA_1+\frac{1}{2}abA_2\Big)=0\,, \quad (12d)$$

from which one has

$$A_2=-15c\gamma k^3\,,\quad A_1=-\frac{15}{4}c\beta k^2\,,$$

$$A_0=c_0-\beta k^2-\frac{\alpha\beta}{4\gamma} \quad (13)$$

with k satisfying

$$(b-3ac)k^4+4(5b-4)\frac{\alpha}{\gamma}k^2-4\frac{\alpha^2}{\gamma^2}=0 \quad (14)$$

under the constraint

$$\beta^2=16\alpha\gamma\,. \quad (15)$$

Actually, we know that the solutions to the elliptic equation (6) are[19−21]

$$y=\mathrm{sn}^2(\xi,m) \quad (16)$$

with

$$a=4\,,\quad b=-4(1+m^2)\,,\quad c=4m^2\,, \quad (17)$$

where $0\le m\le 1$ is called modulus of Jacobi elliptic functions and $\mathrm{sn}(\xi,m)$ is Jacobi elliptic sine function;

$$y=\mathrm{cn}^2(\xi,m) \quad (18)$$

with

$$a=4(1-m^2)\,,\quad b=4(2m^2-1)\,,\quad c=-4m^2 \quad (19)$$

where $\mathrm{cn}(\xi,m)$ is Jacobi elliptic cosine function; and

$$y=\mathrm{dn}^2(\xi,m) \quad (20)$$

with

$$a=-4(1-m^2)\,,\quad b=4(2-m^2)\,,\quad c=-4\,, \quad (21)$$

where $\mathrm{dn}(\xi,m)$ is Jacobi elliptic function of the third kind.

For these three basic Jacobi elliptic functions, there are the following relations

$$\mathrm{sn}'(\xi,m)=\mathrm{cn}(\xi,m)\,\mathrm{dn}(\xi,m)\,,$$

$$\mathrm{cn}'(\xi,m)=-\mathrm{sn}(\xi,m)\,\mathrm{dn}(\xi,m)\,,$$

$$\mathrm{dn}'(\xi,m) = -m^2\,\mathrm{sn}(\xi,m)\mathrm{cn}(\xi,m)\,. \tag{22}$$

So when $a = 4$, $b = -4(1+m^2)$, $c = 4m^2$, the solution to Eq. (1) is

$$u_1 = c_0 - \beta k^2 - \frac{\alpha\beta}{4\gamma} - \frac{15}{4}c\beta k^2\,\mathrm{sn}^2(\xi,m) - 30c\gamma k^3\,\mathrm{sn}(\xi,m)\mathrm{cn}(\xi,m)\mathrm{dn}(\xi,m) \tag{23}$$

with k satisfying Eq. (14).

When $a = 4(1-m^2)$, $b = 4(2m^2-1)$, $c = -4m^2$, the solution to Eq. (1) is

$$u_2 = c_0 - \beta k^2 - \frac{\alpha\beta}{4\gamma} - \frac{15}{4}c\beta k^2\,\mathrm{cn}^2(\xi,m) + 30c\gamma k^3\,\mathrm{sn}(\xi,m)\mathrm{cn}(\xi,m)\mathrm{dn}(\xi,m) \tag{24}$$

with k satisfying Eq. (14).

When $a = -4(1-m^2)$, $b = 4(2-m^2)$, $c = -4$, the solution to Eq. (1) is

$$u_3 = c_0 - \beta k^2 - \frac{\alpha\beta}{4\gamma} - \frac{15}{4}c\beta k^2\,\mathrm{dn}^2(\xi,m) + 30c\gamma k^3 m^2\,\mathrm{sn}(\xi,m)\mathrm{cn}(\xi,m)\mathrm{dn}(\xi,m) \tag{25}$$

with k satisfying Eq. (14).

Moreover, it is known that when $m \to 1$, one has $\mathrm{sn}(\xi,m) \to \tanh\xi$, $\mathrm{cn}(\xi,m) \to \mathrm{sech}\,\xi$, $\mathrm{dn}(\xi,m) \to \mathrm{sech}\,\xi$ and when $m \to 0$, one has $\mathrm{sn}(\xi,m) \to \sin\xi$, $\mathrm{cn}(\xi,m) \to \cos\xi$. So we also can derive more kinds of solutions expressed in terms of hyperbolic functions and trigonometric functions.

3 Weierstrass Elliptic Function Solutions to KdV-Burgers-Kuramoto Equation

Secondly, we consider y in ansatz solution (5) satisfying the following case,

$$y'^2 = 4y^3 - g_2 y - g_3\,, \tag{26}$$

then

$$y'' = -\frac{1}{2}g_2 + 6y^2\,, \tag{27a}$$

$$y''' = 12y'y\,, \tag{27b}$$

$$y^{(4)} = -12g_3 - 18g_2 y + 120y^3\,. \tag{27c}$$

Combining Eqs. (26) and (27) with Eq. (5) leads to

$$u' = \Bigl(-\frac{1}{2}g_2A_2 + A_1y'\Bigr) + 6A_2y^2\,, \tag{28}$$

$$u^2 = (A_0^2 + 2A_0A_2y' - g_3A_2^2) + (2A_0A_1 + 2A_1A_2y' - g_2A_2^2)y + A_1^2y^2 + 4A_2^2y^3\,, \tag{29}$$

$$u'' = -\frac{1}{2}g_2A_1 + 12A_2y'y + 6A_1y^2\,, \tag{30}$$

$$u''' = -12g_3A_2 + (12A_1y' - 18g_2A_2)y + 120A_2y^3\,. \tag{31}$$

Substituting Eqs. (5) and (28) $\sim$ (31) into Eq. (4) results in

$$2A_2^2 + 120\gamma k^3A_2 = 0\,, \tag{32a}$$

$$\frac{1}{2}A_1^2 + 6\alpha kA_2 + 6\beta k^2A_1 = 0\,, \tag{32b}$$

$$-c_0A_1 + \frac{1}{2}(2A_0A_1 + 2A_1A_2y' - g_2A_2^2) + 12\beta k^2A_2y' + \gamma k^3(12A_1y' - 18g_2A_2) = 0\,, \tag{32c}$$

$$c_1 - c_0A_0 - c_0A_2y' + \frac{1}{2}(A_0^2 + 2A_0A_2y' - g_3A_2^2) + \alpha k(-\frac{1}{2}g_2A_2 + A_1y') - \frac{1}{2}\beta k^2g_2A_1 - 12\gamma k^3g_3A_2 = 0\,, \tag{32d}$$

from which one has

$$A_2 = -60\gamma k^3\,,\quad A_1 = -15\beta k^2\,,\quad A_0 = c_0 - \frac{48\gamma^2k^4g_2}{\beta}\,,$$
$$k = \pm\sqrt{\frac{\alpha}{2\gamma}\sqrt{\frac{1}{3g_2}}} \tag{33}$$

under the constraint

$$\beta^2 = 16\alpha\gamma\,. \tag{34}$$

So the solution to Eq. (1) is

$$u_4 = A_0 + A_1y + A_2y' = c_0 - \frac{48\gamma^2k^4g_2}{\beta} - 15\beta k^2\wp(\xi,g_2,g_3) - 60\gamma k^3\wp'(\xi,g_2,g_3)\,, \tag{35}$$

where $\wp(\xi,g_2,g_3)$ is Weierstrass elliptic function, which satisfies Eq. (26), see Refs. [19] $\sim$ [21].

Remark 2 To our knowledge, these periodic solutions from u_1 to u_4 have not been reported in the literature.

4 Conclusion and Discussion

If we set

$$\delta \equiv \frac{\beta^2}{\alpha\gamma}\,, \tag{36}$$

then

$$\delta = 16\,. \tag{37}$$

Similar constraint was given in Refs. [4] and [5], where they obtained solutions to Eq. (1) expressed in terms of hyperbolic functions.[4–7,15,16] In this paper, under the constraint (37), we derive more new periodic solutions expressed in terms of Jacobi elliptic functions or Weierstrass elliptic function, whose limiting solutions are those expressed in terms of hyperbolic functions or trigonometric functions. To our knowledge, these solutions have not been reported in the literature. Of course, there are more constraints (see Refs. [4] and [5]), so solutions expressed in terms of Jacobi elliptic functions or Weierstrass elliptic function under these constraints deserve further studies.

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