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Periodic Solutions for a Class of Nonlinear Differential-Difference Equations*

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Abstract *In this paper, by applying the Jacobi elliptic function expansion method, the periodic solutions for three nonlinear differential-difference equations are obtained.*

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1 Introduction

Finding the exact solutions of nonlinear differential-difference equations plays an important role in modern physics. However, there only some soliton-like solutions were derived.[1−8] Recently, Zhu and Ma obtained the periodic solutions for the discrete mKdV equation by use of the extended Jacobi elliptic function method.[9] In this paper, by using the elliptic function expansion method,[10−12] we obtain the periodic solutions for a class of nonlinear differential-difference equations.

2 Periodic Solutions to the Hybrid Lattice Equation

The hybrid lattice equation[8,13,14] reads

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = (1+\alpha u_n+\beta u_n^2)(u_{n-1}-u_{n+1})\,, \tag{1}$$

where α and β are constants.

We seek the travelling wave solution of Eq. (1) in the form

$$u_n = u_n(\xi), \qquad \xi = an - bt - \xi_0\,, \tag{2}$$

where a is a non-dimensional wave number, b is the angular frequency and ξ_0 is a constant.

Substituting Eq. (2) into Eq. (1) yields

$$b\frac{\mathrm{d}u_n}{\mathrm{d}\xi} = (1+\alpha u_n+\beta u_n^2)(u_{n-1}-u_{n+1})\,. \tag{3}$$

By using the Jacobi elliptic function expansion method,[10−12] u_n can be expressed as

$$u_n = a_0 + a_1\,\mathrm{sn}\,\xi\,, \tag{4}$$

with the modulus m ($0\le m\le 1$, and $\mathrm{sn}\,\xi$ is Jacobi elliptic sine function.[15,16]

Notice that

$$u_{n+1} = a_0 + a_1\,\mathrm{sn}(\xi+a)\,,$$
$$u_{n-1} = a_0 + a_1\,\mathrm{sn}(\xi-a)\,,$$
$$u_{n+1}-u_{n-1} = \frac{2a_1\,\mathrm{sn}\,a\,\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi}{1-m^2\,\mathrm{sn}^2 a\,\mathrm{sn}^2\xi}\,, \tag{5}$$

where $\mathrm{cn}\,\xi$ and $\mathrm{dn}\,\xi$ are Jacobi elliptic cosine function and Jacobi elliptic function of the third kind.[15,16]

Substituting Eq. (4) into Eq. (3) yields

$$b(1-m^2\,\mathrm{sn}^2 a\,\mathrm{sn}^2\xi) = 2\,\mathrm{sn}\,a[(1+\alpha a_0+\beta a_0^2)+(\alpha+2\beta a_0)a_1\,\mathrm{sn}\,\xi+\beta a_1^2\,\mathrm{sn}^2\xi]\,. \tag{6}$$

From Eq. (6), we have

$$a_0 = -\frac{\alpha}{2\beta}, \quad b = \frac{4\beta-\alpha^2}{2\beta}\,\mathrm{sn}\,a, \quad a_1^2 = \frac{\alpha^2-4\beta}{4\beta^2}m^2\,\mathrm{sn}^2 a, \quad (\alpha^2-4\beta>0)\,, \tag{7}$$

where the second formula is the dispersion relation.

So, the periodic solution to the hybrid lattice equation (1) is given by

$$u_n = -\frac{\alpha}{2\beta} \pm \frac{\sqrt{\alpha^2-4\beta}m\,\mathrm{sn}\,a}{2\beta}\,\mathrm{sn}(an-bt-\xi_0)\,, \quad (\alpha^2-4\beta>0)\,. \tag{8}$$

When $m\to 1$, equation (8) reduces to the following generalized solitary wave solution,

$$u_n = -\frac{\alpha}{2\beta} \pm \frac{\sqrt{\alpha^2-4\beta}\;\tanh a}{2\beta}\tanh(an-b\,t-\xi_0) \qquad \Big(\alpha^2-4\beta>0, \quad b=\frac{4\beta-\alpha^2}{2\beta}\tanh a\Big)\,. \tag{9}$$

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Similar to Eq. (4), we have

$$u_n = b_0 + b_1 \operatorname{cn} \xi \,, \tag{10}$$

with

$$b_0 = -\frac{\alpha}{2\beta}, \quad b = \frac{4\beta - \alpha^2}{2\beta} \frac{\operatorname{sn} a}{\operatorname{dn} a}, \quad b_1^2 = \frac{(4\beta - \alpha^2)m^2}{4\beta^2} \frac{\operatorname{sn}^2 a}{\operatorname{dn}^2 a}, \quad (\alpha^2 - 4\beta < 0)\,, \tag{11}$$

and

$$u_n = c_0 + c_1 \operatorname{dn} \xi \,, \tag{12}$$

with

$$c_0 = -\frac{\alpha}{2\beta}, \quad b = \frac{4\beta - \alpha^2}{2\beta m^2} \frac{\operatorname{sn} a}{\operatorname{cn} a}, \quad c_1^2 = \frac{4\beta - \alpha^2}{4\beta^2} \frac{\operatorname{sn}^2 a}{\operatorname{cn}^2 a}, \quad (\alpha^2 - 4\beta < 0)\,. \tag{13}$$

When $m \to 1$, equations (10) and (12) reduce to the following generalized solitary wave solution

$$u_n = -\frac{\alpha}{2\beta} \pm \frac{\sqrt{4\beta - \alpha^2} \sinh a}{2\beta} \operatorname{sech}(an - b\,t - \xi_0) \quad \Big(\alpha^2 - 4\beta < 0, \quad b = \frac{4\beta - \alpha^2}{2\beta} \sinh a\Big)\,. \tag{14}$$

For the discrete mKdV equation,[9,17]

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = (\alpha - u_n^2)(u_{n-1} - u_{n+1})\,, \tag{15}$$

where α is a constant.

We introduce the transformation

$$\tau = -\alpha\, t\,, \tag{16}$$

then equation (15) can be rewritten as

$$\frac{\mathrm{d}u_n}{\mathrm{d}\tau} = \Big(1 - \frac{1}{\alpha} u_n^2)(u_{n-1} - u_{n+1}\Big)\,. \tag{17}$$

Obviously, equation (17) is a special case of the hybrid lattice equation (1). Thus, we can obtain the periodic solutions to the discrete mKdV equation directly from above results for the hybrid lattice equation (1), they are

$$u_n = \pm(m\sqrt{\alpha} \operatorname{sn} a) \operatorname{sn}(an - b\,t - \xi_0), \quad (b = -2\alpha \operatorname{sn} a, \quad \alpha > 0)\,, \tag{18}$$

$$u_n = \pm m\sqrt{-\alpha} \frac{\operatorname{sn} a}{\operatorname{dn} a} \operatorname{cn}(an - b\,t - \xi_0), \quad \Big(b = -2\alpha \frac{\operatorname{sn} a}{\operatorname{dn} a}, \quad \alpha < 0\Big)\,, \tag{19}$$

and

$$u_n = \pm\sqrt{-\alpha} \frac{\operatorname{sn} a}{\operatorname{cn} a} \operatorname{dn}(an - b\,t - \xi_0), \quad \Big(b = -\frac{2\alpha}{m^2} \frac{\operatorname{sn} a}{\operatorname{cn} a}, \quad \alpha < 0\Big)\,. \tag{20}$$

When $m \to 1$, equations (18), (19), and (20) reduce to

$$u_n = \pm(\sqrt{\alpha} \tanh a) \tanh(an - b\,t - \xi_0), \quad (b = -2\alpha \tanh a, \quad \alpha > 0)\,, \tag{21}$$

and

$$u_n = \pm(\sqrt{-\alpha} \sinh a) \operatorname{sech}(an - b\,t - \xi_0), \quad (b = -2\alpha \sinh a, \quad \alpha < 0)\,. \tag{22}$$

respectively.

3 Periodic Solutions to the Toda Lattice Equation

The Toda lattice equation[1,2,18] reads

$$\frac{\mathrm{d}^2 r_n}{\mathrm{d}t^2} = \alpha(2\,\mathrm{e}^{-\beta r_n} - \mathrm{e}^{-\beta r_{n+1}} - \mathrm{e}^{-\beta r_{n-1}})\,. \tag{23}$$

Set

$$\mathrm{e}^{-\beta r_n} = 1 + \frac{s_n}{\alpha}\,, \tag{24}$$

then the Toda lattice equation (23) can be rewritten as

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \ln\Big(1 + \frac{s_n}{\alpha}\Big) = \beta(s_{n+1} - 2s_n + s_{n-1})\,. \tag{25}$$

Assuming that

$$s_n = s_n(\xi), \qquad \xi = an - b\,t + \xi_0\,, \tag{26}$$

then equation (25) becomes

$$b^2\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\ln\Bigl(1+\frac{s_n}{\alpha}\Bigr)=\beta(s_{n+1}-2s_n+s_{n-1})\,. \tag{27}$$

By using the Jacobi elliptic function expansion method,[10−12] s_n can be written as

$$s_n(\xi)=a_0+a_1\,\mathrm{sn}\,\xi+a_2\,\mathrm{sn}^2\,\xi\,, \tag{28}$$

with the modulus m $(0\le m\le 1)$.

Notice that

$$s_{n+1}=a_0+a_1\,\mathrm{sn}(\xi+a)+\mathrm{sn}^2(\xi+a),\quad s_{n-1}=a_0+a_1\,\mathrm{sn}(\xi-a)+\mathrm{sn}^2(\xi-a)\,,$$

$$\begin{aligned}s_{n+1}-2s_n+s_{n-1}&=2a_1\Bigl(\frac{\mathrm{cn}\,a\,\mathrm{dn}\,a}{1-m^2\,\mathrm{sn}^2\,a\,\mathrm{sn}^2\,\xi}-1\Bigr)\mathrm{sn}\,\xi\\&\quad+2a_2\frac{\mathrm{sn}^2\,a+[-2(1+m^2)\,\mathrm{sn}^2\,a+m^2\,\mathrm{sn}^4\,a]\,\mathrm{sn}^2\,\xi+(3m^2\,\mathrm{sn}^2\,a)\,\mathrm{sn}^4\,\xi-(m^4\,\mathrm{sn}^4\,a)\,\mathrm{sn}^6\,\xi}{1-m^2\,\mathrm{sn}^2\,a\,\mathrm{sn}^2\,\xi}\,,\end{aligned} \tag{29}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\ln\Bigl(1+\frac{s_n}{\alpha}\Bigr)=\frac{a_1\,\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi+2a_2\,\mathrm{sn}\,\xi\,\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi}{(\alpha+a_0)+a_1\,\mathrm{sn}\,\xi+a_2\,\mathrm{sn}^2\,\xi}\,,$$

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\ln\Bigl(1+\frac{s_n}{\alpha}\Bigr)&=\frac{a_1[-(\alpha+a_0)(1+m^2)\,\mathrm{sn}\,\xi+2m^2(\alpha+a_0)\,\mathrm{sn}^3\,\xi]+a_1^2(-1+m^2\,\mathrm{sn}^4\,\xi)}{[(\alpha+a_0)+a_1\,\mathrm{sn}\,\xi+a_2\,\mathrm{sn}^2\,\xi]^2}\\&\quad+\frac{a_1a_2[-(1+m^2)\,\mathrm{sn}^3\,\xi+4m^2\,\mathrm{sn}^5\,\xi]-2a_2[a_2-2(\alpha+a_0)(1+m^2)]\,\mathrm{sn}^2\,\xi}{[(\alpha+a_0)+a_1\,\mathrm{sn}\,\xi+a_2\,\mathrm{sn}^2\,\xi]^2}\\&\quad+\frac{2a_2[(\alpha+a_0)+3(\alpha+a_0)m^2\,\mathrm{sn}^4\,\xi+m^2a_2\,\mathrm{sn}^6\,\xi]}{[(\alpha+a_0)+a_1\,\mathrm{sn}\,\xi+a_2\,\mathrm{sn}^2\,\xi]^2}\,.\end{aligned} \tag{30}$$

Substituting Eq. (28) into Eq. (27) yields

$$a_1=0\,, \tag{31}$$

and

$$\begin{aligned}&b^2\frac{(\alpha+a_0)-[a_2+2(1+m^2)(\alpha+a_0)]\,\mathrm{sn}^2\,\xi+3m^2(\alpha+a_0)\,\mathrm{sn}^4\,\xi+m^2a_2\,\mathrm{sn}^6\,\xi}{[(\alpha+a_0)+a_2\,\mathrm{sn}^2\,\xi]^2}\\&\quad=\beta\frac{\mathrm{sn}^2\,a+[-2(1+m^2)\,\mathrm{sn}^2\,a+m^2\,\mathrm{sn}^4\,a]\,\mathrm{sn}^2\,\xi+(3m^2\,\mathrm{sn}^2\,a)\,\mathrm{sn}^4\,\xi-(m^4\,\mathrm{sn}^4\,a)\,\mathrm{sn}^6\,\xi}{1-m^2\,\mathrm{sn}^2\,a\,\mathrm{sn}^2\,\xi}\,.\end{aligned} \tag{32}$$

From Eq. (32), we have

$$a_2=-\frac{m^2b^2}{\beta},\qquad a_0=-\alpha+\frac{b^2}{\beta\,\mathrm{sn}^2\,a}\,. \tag{33}$$

Thus we ob tain the periodic solution of Eq. (25), i.e.

$$s_n=-\alpha+\frac{b^2}{\beta\,\mathrm{sn}^2\,a}-\frac{m^2b^2}{\beta}\mathrm{sn}^2(an-b\,t+\xi_0)=-\alpha-\frac{b^2}{\beta}\Bigl(m^2-\frac{1}{\mathrm{sn}^2a}\Bigr)+\frac{m^2b^2}{\beta}\,\mathrm{cn}^2(an-b\,t+\xi_0)\,. \tag{34}$$

Taking

$$\alpha=-\frac{b^2}{\beta}\Bigl(m^2-\frac{1}{\mathrm{sn}^2\,a}\Bigr)\,, \tag{35}$$

then we ob tain the dispersion relation,

$$b^2=\alpha\beta\frac{\mathrm{sn}^2\,a}{\mathrm{dn}^2\,a}\,, \tag{36}$$

and

$$s_n=\frac{m^2b^2}{\beta}\,\mathrm{cn}^2(an-b\,t+\xi_0)\,. \tag{37}$$

Substituting Eq. (37) into Eq. (24) leads to

$$r_n=-\frac{1}{\beta}\ln\Bigl[1+\frac{m^2b^2}{\alpha\beta}\mathrm{cn}^2(an-b\,t+\xi_0)\Bigr]\,. \tag{38}$$

When $m \to 1$, equations (37) and (38) reduce to

$$s_n = \frac{b^2}{\beta} \sinh^2(an - b\,t + \xi_0), \quad (b^2 = \alpha\beta \sinh^2 a)\,, \tag{39}$$

and

$$r_n = -\frac{1}{\beta} \ln\Big[1 + \frac{b^2}{\alpha\beta} \mathrm{sech}^2(an - b\,t + \xi_0)\Big], \quad (b^2 = \alpha\beta \sinh^2 a)\,, \tag{40}$$

which is a solitary wave solution to the Toda lattice equation.

4 Conclusion

In this paper, we apply the Jacobi elliptic function expansion to solve three nonlinear differential-difference equations, and many periodic wave solutions and solitary wave solutions are derived. These solutions are helpful in understanding the problems in modern physics.

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