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Periodic Structure of Equatorial Envelope Rossby Wave Under Influence of Diabatic Heating*

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Abstract *A simple shallow-water model with influence of diabatic heating on a β-plane is applied to investigate the nonlinear equatorial Rossby waves in a shear flow. By the asymptotic method of multiple scales, the cubic nonlinear Schrödinger (NLS for short) equation with an external heating source is derived for large amplitude equatorial envelope Rossby wave in a shear flow. And then various periodic structures for these equatorial envelope Rossby waves are obtained with the help of Jacobi elliptic functions and elliptic equation. It is shown that phase-locked diabatic heating plays an important role in periodic structures of rational form.*

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Key words: nonlinear Schrödinger equation, periodic structure, diabatic heating, Jacobi elliptic function

1 Introduction

In the last decades, the theory of equatorial waves has attracted much more attention on equatorial atmospheric dynamics and nonlinear dynamics. It provides a dynamical frame to analyze the slowly evolving large-scale phenomena in low latitudes and underlining dynamics. These theories of equatorial waves have been used for various purposes, especially in explaining some fundamental features of tropical climate and global changes, such as Walker circulation,[1] the low-frequency Madden–Julian oscillation,[2] and ENSO.[3]

Among the nonlinear theories for equatorial waves, many are related to nonlinear Rossby wave activity, for it can manifest some of the prime events of geophysical fluid flows, and this activity often leads to large-scale localized coherent structures that have remarkable permanence and stability. When the zonal flow shear is taken to be nonuniform, one can derive Rossby solitary waves and envelope Rossby solitary waves. Benney,[4] Yamagata,[5] and Zhao[6] investigated envelope Rossby solitary waves in barotropic shear and uniform or nonuniform flows, independently. However, none of them considered the effect of external sources, especially the influence of diabatic heating from oceans. In this paper, we will address this issue by the method of multi-scale to derive the nonlinear Schrödinger (NLS) equation with an external heating source satisfied by the large-amplitude equatorial Rossby waves. And then the basic structures of this NLS equation without and with phase-locked source are obtained by using knowledge of Jacobi elliptic functions and elliptic equation.

2 Derivation of NLS Equation with an External Heating Source

The governing equation is quasi-geostrophic potential vorticity equation of shallow-water model on an equatorial β-plane with an external heating source, i.e.

$$\Big(\frac{\partial}{\partial t_*}-\frac{\partial\psi_*}{\partial y_*}\frac{\partial}{\partial x_*}+\frac{\partial\psi_*}{\partial x_*}\frac{\partial}{\partial y_*}\Big)\Big(\beta y_*+\nabla_*^2\psi_*-\frac{\beta^2y_*^2}{c_0^2}\psi_*\Big)=Q_*(x_*,y_*,t_*)\,, \tag{1}$$

where ψ_* is the stream function, $\beta>0$ is the planetary-vorticity gradient, c_0 is velocity of pure gravity waves, and $Q_*(x_*,y_*,t_*)$ is the diabatic heating due to the tropical ocean, and ∇_*^2 is the horizontal Laplacian operator, which is defined as

$$\nabla_*^2=\frac{\partial^2}{\partial x_*^2}+\frac{\partial^2}{\partial y_*^2}\,. \tag{2}$$

Since the equatorial waves are trapped near the equator, the appropriate boundary condition can be given as

$$\frac{\partial\psi_*}{\partial x_*}\to 0\,,\quad \text{as } y_*\to\pm\infty\,. \tag{3}$$

Equation (3) can be nondimensionalized as

$$\frac{\partial}{\partial t}\bar{\nabla}^2\psi+\varepsilon J(\psi,\bar{\nabla}^2\psi)+\frac{\partial\psi}{\partial x}=\mu Q(x,y,t) \tag{4}$$

with

$$J(a,b)=\frac{\partial a}{\partial x}\frac{\partial b}{\partial y}-\frac{\partial a}{\partial y}\frac{\partial b}{\partial x}\,,$$

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$$\bar{\nabla}^2 = \nabla^2 - y^2\,, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\,, \tag{5}$$

where ε is the equatorial Rossby number, μ is an amplitude parameter, and they are all small numbers, near the equator $\varepsilon \sim O(10^{-2})$. Here it is obvious that ε denotes the magnitude of nonlinearity and μ represents the strength of external forcing.

Due to the existence of these small parameters, the multi-scale expansion method can be applied to solve the problem Eq. (4), where the stretching coordinates of the following form

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial T_1} + \varepsilon^2\frac{\partial}{\partial T_2}\,, \qquad \frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial X_1} + \varepsilon^2\frac{\partial}{\partial X_2} \tag{6}$$

are introduced with

$$T_1 = \varepsilon t\,, \quad T_2 = \varepsilon^2 t\,, \quad X_1 = \varepsilon x\,, \quad X_2 = \varepsilon^2 x\,. \tag{7}$$

Substituting Eq. (6) into Eq. (4) results in

$$\begin{aligned}
&\Big\{\Big(\frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial T_1} + \varepsilon^2\frac{\partial}{\partial T_2}\Big) + \varepsilon\Big[\Big(\frac{\partial\psi}{\partial x} + \varepsilon\frac{\partial\psi}{\partial X_1} + \varepsilon^2\frac{\partial\psi}{\partial X_2}\Big)\frac{\partial}{\partial y} - \frac{\partial\psi}{\partial y}\Big(\frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial X_1} + \varepsilon^2\frac{\partial}{\partial X_2}\Big)\Big]\Big\}\\
&\quad\times\Big[\bar{\nabla}^2\psi + 2\varepsilon\frac{\partial^2\psi}{\partial x\partial X_1} + \varepsilon^2\Big(\frac{\partial^2\psi}{\partial X_1^2} + 2\frac{\partial^2\psi}{\partial x\partial X_2}\Big) + 2\varepsilon^3\frac{\partial^2\psi}{\partial X_1\partial X_2} + \varepsilon^4\frac{\partial^2\psi}{\partial X_2^2}\Big]\\
&\quad+\Big(\frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial X_1} + \varepsilon^2\frac{\partial}{\partial X_2}\Big)\psi = \mu Q(x,y,t;X_1,T_1;X_2,T_2)\,.
\end{aligned} \tag{8}$$

In the presence of small parameters, the total stream function can be written as

$$\psi = -\int^y \bar{u}(s)\,\mathrm{d}s + \sum_{n=1}^{\infty}\varepsilon^n\psi_n(x,y,t;X_1,T_1;X_2,T_2)\,. \tag{9}$$

If the external forcing is weak, i.e. $O(\mu) \sim O(\varepsilon^3)$, then combining Eq. (8) with Eq. (9) leads to

$$O(\varepsilon)\ :\ \wp(\psi_1) = 0\,, \tag{10}$$

$$O(\varepsilon^2)\ :\ \wp(\psi_2) = -\Big(\frac{\partial}{\partial T_1} + \bar{u}\frac{\partial}{\partial X_1}\Big)\bar{\nabla}^2\psi_1 - (1-\bar{u}'')\frac{\partial\psi_1}{\partial X_1} - 2\Big(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\Big)\frac{\partial^2\psi_1}{\partial x\partial X_1} - J(\psi_1,\bar{\nabla}^2\psi_1)\,, \tag{11}$$

$$\begin{aligned}
O(\varepsilon^3)\ :\ \wp(\psi_3) = &-\Big(\frac{\partial}{\partial T_2} + \bar{u}\frac{\partial}{\partial X_2}\Big)\bar{\nabla}^2\psi_1 - (1-\bar{u}'')\frac{\partial\psi_1}{\partial X_2} - 2\Big(\frac{\partial}{\partial T_1} + \bar{u}\frac{\partial}{\partial X_1}\Big)\frac{\partial^2\psi_1}{\partial x\partial X_1}\\
&- J\Big(\psi_1, 2\frac{\partial^2\psi_1}{\partial x\partial X_1}\Big) - \Big(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\Big)\Big(\frac{\partial^2\psi_1}{\partial X_1^2} + 2\frac{\partial^2\psi_1}{\partial x\partial X_2} + 2\frac{\partial^2\psi_2}{\partial x\partial X_1}\Big)\\
&- \frac{\partial\psi_1}{\partial X_1}\frac{\partial}{\partial y}\bar{\nabla}^2\psi_1 + \frac{\partial\psi_1}{\partial y}\frac{\partial}{\partial X_1}\bar{\nabla}^2\psi_1 - \Big(\frac{\partial}{\partial T_1} + \bar{u}\frac{\partial}{\partial X_1}\Big)\bar{\nabla}^2\psi_2\\
&- (1-\bar{u}'')\frac{\partial\psi_2}{\partial X_1} - J(\psi_1,\bar{\nabla}^2\psi_2) - J(\psi_2,\bar{\nabla}^2\psi_1) + Q
\end{aligned} \tag{12}$$

with the operator $\wp$ is defined as

$$\wp(\quad) = \Big(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\Big)\bar{\nabla}^2 + (1-\bar{u}'')\frac{\partial}{\partial x}\,. \tag{13}$$

Obviously, ψ_1 satisfies the linear equation (10), whose solution can be taken as

$$\psi_1 = A(X_1,T_1,X_2,T_2)\Phi_1(y)\exp[\mathrm{i}(\lambda x - \bar{\omega}t)] + \text{c.c.}\,, \tag{14}$$

where λ is the zonal wave number, $\bar{\omega}$ is the angular frequency, c.c. is an abbreviation for "complex conjugate" of its preceding term. The latitudinal structure $\Phi_1(y)$ can be determined by substituting Eq. (14) into Eq. (10), but the wave amplitude $A(X_1,T_1,X_2,T_2)$ can only be obtained from higher order equations. Substituting the first-order solution (14) into the second-order expansion equation (11), we have

$$\wp(\psi_2) = \frac{1-\bar{u}''}{\bar{u}-c}\Phi_1\Big(\frac{\partial A}{\partial T_1} + c_1\frac{\partial A}{\partial X_1}\Big)\exp[\mathrm{i}(\lambda x - \bar{\omega}t)] + \mathrm{i}\lambda P(y)A^2\exp[2\mathrm{i}(\lambda x - \bar{\omega}t)] + \text{c.c.} \tag{15}$$

with

$$c = \frac{\bar{\omega}}{\lambda}\,, \qquad c_1 = c + \frac{2\lambda^2(\bar{u}-c)^2}{1-\bar{u}''}\,, \qquad P(y) = \Phi_1^2\frac{\mathrm{d}}{\mathrm{d}y}\Big(\frac{1-\bar{u}''}{\bar{u}-c}\Big)\,. \tag{16}$$

It is obvious that the homogeneous part of Eq. (15) is identical to Eq. (10), and the solution to this portion is similar to Eq. (14). The first part of inhomogeneous terms on the right hand is resonant with the homogeneous solution, thus in order to avoid secular growth, orthogonality must exist for the inhomogeneous terms, i.e.

$$\lim_{\tau\to\infty}\lim_{x\to\infty}\int_{-x}^{x}\int_{-\infty}^{\infty}\int_{0}^{\tau}\psi_1[\text{the inhomogeneous terms of Eq. (15)}]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,. \tag{17}$$

And this orthogonality condition yields the following equation

$$\frac{\partial A}{\partial T_1}+c_g\frac{\partial A}{\partial X_1}=0 \tag{18}$$

with

$$c_g=c+\frac{I_1}{I}\,,\qquad I=\int_{-\infty}^{\infty}\frac{1-\bar{u}''}{(\bar{u}-c)^2}\Phi_1^2\mathrm{d}y\,,\qquad I_1=2\lambda^2\int_{-\infty}^{\infty}\Phi_1^2\mathrm{d}y\,, \tag{19}$$

and special solution to Eq. (11) is

$$\psi_2=B(X_1,T_1,X_2,T_2)\Phi_2(y)\exp[2\mathrm{i}(\lambda x-\bar{\omega}t)]+\text{c.c.} \tag{20}$$

with the relation

$$B=A^2\,. \tag{21}$$

If the external forcing takes the following form,

$$Q=\sum_{n=0}^{\infty}Q_n\exp[n\mathrm{i}(\lambda x-\bar{\omega}t)]+\text{c.c.}\,, \tag{22}$$

then combining the results for ψ_1 and ψ_2 with Eq. (12) reaches

$$\begin{aligned}\wp(\psi_3)=\Big\{&\frac{1-\bar{u}''}{\bar{u}-c}\Big[\frac{\partial A}{\partial T_2}+c_1\frac{\partial A}{\partial X_2}+\mathrm{i}\lambda\frac{\bar{u}-c}{1-\bar{u}''}(c+2c_g-3\bar{u})\frac{\partial^2 A}{\partial X_1^2}\Big]\Phi_1\\&+\mathrm{i}\lambda\Big[\frac{\Phi_1}{2}\frac{\mathrm{d}}{\mathrm{d}y}\Big(\frac{P}{\bar{u}-c}\Big)+\frac{P}{\bar{u}-c}\frac{\mathrm{d}\Phi_1}{\mathrm{d}y}\Big]|A|^2A+Q_1\Big\}\exp[\mathrm{i}(\lambda x-\bar{\omega}t)]+\text{c.c.}+\Diamond\,,\end{aligned} \tag{23}$$

where $\Diamond$ are terms associated with $\exp[\pm2\mathrm{i}(\lambda x-\bar{\omega}t)]$, $\exp[\pm3\mathrm{i}(\lambda x-\bar{\omega}t)]$, and so on.

Similarly, the solvability of Eq. (23) results in the following NLS equation with an external heating source

$$\mathrm{i}\Big(\frac{\partial A}{\partial T_2}+c_g\frac{\partial A}{\partial X_2}\Big)+\alpha\frac{\partial^2 A}{\partial X_1^2}+\delta|A|^2A=\eta Q_{11}(X_1,T_1,X_2,T_2) \tag{24}$$

with

$$\begin{aligned}&\alpha=\frac{I_2}{I}\,,\qquad \delta=\frac{I_3}{I}\,,\qquad \eta=\frac{I_4}{I}\,,\qquad I_2=-\lambda\int_{-\infty}^{\infty}\frac{(c+2c_g-3\bar{u})}{\bar{u}-c}\Phi_1^2\mathrm{d}y\,,\\&I_3=-\lambda\int_{-\infty}^{\infty}\frac{1}{\bar{u}-c}\Big[\frac{\Phi_1^2}{2}\frac{\mathrm{d}}{\mathrm{d}y}\Big(\frac{P}{\bar{u}-c}\Big)+\frac{P\Phi_1}{\bar{u}-c}\frac{\mathrm{d}\Phi_1}{\mathrm{d}y}\Big]\mathrm{d}y\,,\qquad I_4=-\mathrm{i}\int_{-\infty}^{\infty}\frac{1}{\bar{u}-c}Q_1\Phi_1\mathrm{d}y\,.\end{aligned} \tag{25}$$

If we introduce the following coordinates transformation defined by Jeffrey,[7]

$$T=T_2\,,\qquad X=\frac{1}{\varepsilon}(X_2-c_gT_2)=X_1-c_gT_1\,, \tag{26}$$

then the NLS equation with an external heating source (24) can be rewritten as the canonical form,

$$\mathrm{i}\frac{\partial A}{\partial T}+\alpha\frac{\partial^2 A}{\partial X^2}+\delta|A|^2A=\eta Q_{11}(X,T)\,, \tag{27}$$

where $Q_{11}(X,T)$ is the slowly varying external heating source, and η denotes its strength.

Actually, there are multiple structures controlled by Eq. (27), and in the next sections we will show some special cases.

3 Solutions to NLS Equation Without a Source

First of all, if there is no external heating source, i.e. $\eta=0$, then equation (27) reduces to

$$\mathrm{i}\frac{\partial A}{\partial T}+\alpha\frac{\partial^2 A}{\partial X^2}+\delta|A|^2A=0\,. \tag{28}$$

Equation (28) can be solved in the frame of travelling wave,

$$\begin{aligned}&A(X,T)=\phi(\xi)\exp[\mathrm{i}(kX-\omega T)]\,,\\&\xi=s(X-C_gT)\,.\end{aligned} \tag{29}$$

Then equation (28) is rewritten as

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2}=\frac{\gamma}{\alpha s^2}\phi-\frac{\delta}{\alpha s^2}\phi^3 \tag{30}$$

with

$$C_g=2\alpha k\,,\qquad -\gamma=\omega-\alpha k^2\,. \tag{31}$$

Obviously, equation (30) is an elliptic equation,[8]

$$z'^2=a_0+a_1z^2+a_2z^4\,,\quad\text{or}\quad z''=a_1z+2a_2z^3 \tag{32}$$

with

$$a_1 = \frac{\gamma}{\alpha s^2}\,, \qquad a_2 = -\frac{\delta}{2\alpha s^2}\,, \tag{33}$$

where the prime denotes the derivatives with respect to its argument.

Equation (30) or (32) has many more kinds of solutions, we will show some next, expressed in terms of different Jacobi elliptic functions.[8]

(i) If $a_0 = 1$, $a_1 = \gamma/\alpha s^2 = -(1+m^2)$, and $a_2 = -\delta/2\alpha s^2 = m^2$, then the solution is

$$\phi_1 = \mathrm{sn}(\xi, m)\,, \tag{34}$$

where $0 \le m \le 1$, is called modulus of Jacobi elliptic functions, and $\mathrm{sn}(\xi, m)$ is Jacobi elliptic sine function (see Refs. [8] $\sim$ [12]).

(ii) If $a_0 = 1 - m^2$, $a_1 = \gamma/\alpha s^2 = 2m^2 - 1$, and $a_2 = -\delta/2\alpha s^2 = -m^2$, then the solution is

$$\phi_2 = \mathrm{cn}(\xi, m)\,, \tag{35}$$

where $\mathrm{cn}(\xi, m)$ is Jacobi elliptic cosine function (see Refs. [8] $\sim$ [12]).

(iii) If $a_0 = 1 - m^2$, $a_1 = \gamma/\alpha s^2 = 2 - m^2$, and $a_2 = -\delta/2\alpha s^2 = -1$, then the solution is

$$\phi_3 = \mathrm{dn}(\xi, m)\,, \tag{36}$$

where $\mathrm{dn}(\xi, m)$ is Jacobi elliptic function of the third kind, (see Refs. [8] $\sim$ [12].)

(iv) If $a_0 = m^2$, $a_1 = \gamma/\alpha s^2 = -(1+m^2)$, and $a_2 = -\delta/2\alpha s^2 = 1$, then the solution is

$$\phi_4 = \mathrm{ns}(\xi, m) \equiv \frac{1}{\mathrm{sn}(\xi, m)}\,. \tag{37}$$

(v) If $a_0 = -m^2$, $a_1 = \gamma/\alpha s^2 = 2m^2 - 1$, and $a_2 = -\delta/2\alpha s^2 = 1 - m^2$, then the solution is

$$\phi_5 = \mathrm{nc}(\xi, m) \equiv \frac{1}{\mathrm{cn}(\xi, m)}\,. \tag{38}$$

(vi) If $a_0 = -1$, $a_1 = \gamma/\alpha s^2 = 2 - m^2$, and $a_2 = -\delta/2\alpha s^2 = m^2 - 1$, then the solution is

$$\phi_6 = \mathrm{nd}(\xi, m) \equiv \frac{1}{\mathrm{dn}(\xi, m)}\,. \tag{39}$$

(vii) If $a_0 = 1$, $a_1 = \gamma/\alpha s^2 = 2 - m^2$, and $a_2 = -\delta/2\alpha s^2 = 1 - m^2$, then the solution is

$$\phi_7 = \mathrm{sc}(\xi, m) \equiv \frac{\mathrm{sn}(\xi, m)}{\mathrm{cn}(\xi, m)}\,. \tag{40}$$

(viii) If $a_0 = 1$, $a_1 = \gamma/\alpha s^2 = 2m^2 - 1$, and $a_2 = -\delta/2\alpha s^2 = (m^2 - 1)m^2$, then the solution is

$$\phi_8 = \mathrm{sd}(\xi, m) \equiv \frac{\mathrm{sn}(\xi, m)}{\mathrm{dn}(\xi, m)}\,. \tag{41}$$

(ix) If $a_0 = 1 - m^2$, $a_1 = \gamma/\alpha s^2 = 2 - m^2$, and $a_2 = -\delta/2\alpha s^2 = 1$, then the solution is

$$\phi_9 = \mathrm{cs}(\xi, m) \equiv \frac{\mathrm{cn}(\xi, m)}{\mathrm{sn}(\xi, m)}\,. \tag{42}$$

(x) If $a_0 = 1$, $a_1 = \gamma/\alpha s^2 = -(1+m^2)$, and $a_2 = -\delta/2\alpha s^2 = m^2$, then the solution is

$$\phi_{10} = \mathrm{cd}(\xi, m) \equiv \frac{\mathrm{cn}(\xi, m)}{\mathrm{dn}(\xi, m)}\,. \tag{43}$$

(xi) If $a_0 = m^2(m^2 - 1)$, $a_1 = \gamma/\alpha s^2 = 2m^2 - 1$, and $a_2 = -\delta/2\alpha s^2 = 1$, then the solution is

$$\phi_{11} = \mathrm{ds}(\xi, m) \equiv \frac{\mathrm{dn}(\xi, m)}{\mathrm{sn}(\xi, m)}\,. \tag{44}$$

(xii) If $a_0 = m^2$, $a_1 = \gamma/\alpha s^2 = -(1+m^2)$, and $a_2 = -\delta/2\alpha s^2 = 1$, then the solution is

$$\phi_{12} = \mathrm{dc}(\xi, m) \equiv \frac{\mathrm{dn}(\xi, m)}{\mathrm{cn}(\xi, m)}\,. \tag{45}$$

There still exist many other kinds of solutions in terms of Jacobi elliptic functions,[13−15] but we do not show here. It is known that when $m \to 1$, $\mathrm{sn}(\xi, m) \to \tanh\xi$, $\mathrm{cn}(\xi, m) \to \mathrm{sech}\,\xi$, $\mathrm{dn}(\xi, m) \to \mathrm{sech}\,\xi$, and when $m \to 0$, $\mathrm{sn}(\xi, m) \to \sin\xi$, $\mathrm{cn}(\xi, m) \to \cos\xi$, so we also can derive solutions expressed in terms of hyperbolic functions and trigonometric functions.

4 Periodic Structures to NLS Equation with a Phase-Locked Source

The second case for the external heating source is an external travelling wave source, i.e.

$$Q_{11}(X, T) = \mathrm{e}^{\mathrm{i}(kX - \omega T)}\,, \tag{46}$$

then equation (27) reduces to

$$\mathrm{i}\frac{\partial A}{\partial T} + \alpha\frac{\partial^2 A}{\partial X^2} + \delta|A|^2 A = \eta\,\mathrm{e}^{\mathrm{i}(kX - \omega T)}\,, \tag{47}$$

whose travelling wave solution, phase-locked with the external source, is taken as

$$A(X, T) = \phi(\xi)\,\mathrm{e}^{\mathrm{i}(kX - \omega T]}\,, \quad \xi = s(X - C_g T)\,, \tag{48}$$

where k is the wave number, and ω is the angular frequency in the space of (X, T).

Separating the real and imaginary parts of Eq. (47), one has

$$\alpha s^2\phi'' + \delta\phi^3 - \gamma\phi - \eta = 0 \tag{49}$$

with

$$C_g = 2\alpha k, \quad \gamma = -(\omega - \alpha k^2)\,. \tag{50}$$

Based on the knowledge of Jacobi elliptic functions and elliptic equations, we cannot directly find the solution to Eq. (49), here some special transformations must be introduced. In fact, $\eta \neq 0$ will result in different structures. In order to solve Eq. (49), we introduce a fractional transformation, i.e.

$$\phi(\xi) = \frac{b_0 + b_1 z^2(\xi)}{1 + b_2 z^2(\xi)}\,, \tag{51}$$

where $z(\xi)$ is given by Eq. (32).

In order to obtain nontrivial solutions, there is a constraint

$$b_0 b_2 - b_1 \neq 0 \tag{52}$$

for the fractional transformation. Through the fractional transformation (51), the solutions of Eq. (49) with $\eta \neq 0$ are mapped to those of the elliptic equation (32).

We can see that there are rich structures resulted from Eq. (49) in the range of parameter values. Here we show two special cases.

Case 1 $b_0 = 0$, $b_1 \neq 0$, and $b_2 \neq 0$

In this case, we can obtain

$$b_1 = \frac{\eta}{2a_0\alpha s^2}\,, \qquad b_2 = \frac{4a_1\alpha s^2 - \gamma}{12a_0\alpha s^2} \tag{53}$$

with constraints

$$\gamma^2 = 16\alpha^2 s^4 (a_1^2 - 3a_0a_2)\,, \tag{54}$$

and

$$(4a_1\alpha s^2 - \gamma)^3 + 6\gamma(4a_1\alpha s^2 - \gamma)^2 + 144a_0a_2\alpha^2 s^4(4a_1\alpha s^2 - \gamma) - 216\delta\eta^2 = 0. \tag{55}$$

From constraint (54), we know that

$$a_1^2 - 3a_0a_2 \geq 0\,. \tag{56}$$

Recalling the solutions from ϕ_1 to ϕ_{12}, we can obtain another new rational periodic solutions.

(i) If $a_0 = 1$, $a_1 = -(1+m^2)$, and $a_2 = m^2$, then $a_1^2 - 3a_0a_2 = 1 - m^2 + m^4 > 0$, and the solution is

$$z_1 = \mathrm{sn}(\xi, m)\,, \tag{57}$$

$$\phi_{1a} = \frac{6\eta z^2(\xi)}{12a_0\alpha s^2 + (4a_1\alpha s^2 - \gamma)z^2(\xi)} = \frac{6\eta \mathrm{sn}^2(\xi, m)}{12a_0\alpha s^2 + (4a_1\alpha s^2 - \gamma)\mathrm{sn}^2(\xi, m)}\,. \tag{58}$$

(ii) If $a_0 = 1 - m^2$, $a_1 = 2m^2 - 1$, and $a_2 = -m^2$, then $a_1^2 - 3a_0a_2 = 1 - m^2 + m^4 > 0$, and the solution is

$$z_2 = \mathrm{cn}(\xi, m)\,, \tag{59}$$

$$\phi_{2a} = \frac{6\eta z^2(\xi)}{12a_0\alpha s^2 + (4a_1\alpha s^2 - \gamma)z^2(\xi)} = \frac{6\eta \mathrm{cn}^2(\xi, m)}{12a_0\alpha s^2 + (4a_1\alpha s^2 - \gamma)\mathrm{cn}^2(\xi, m)}\,. \tag{60}$$

(iii) If $a_0 = 1 - m^2$, $a_1 = 2 - m^2$, and $a_2 = -1$, then $a_1^2 - 3a_0a_2 = 7 - 7m^2 + m^4 > 0$, and the solution is

$$z_3 = \mathrm{dn}(\xi, m)\,, \tag{61}$$

$$\phi_{3a} = \frac{6\eta z^2(\xi)}{12a_0\alpha s^2 + (4a_1\alpha s^2 - \gamma)z^2(\xi)} = \frac{6\eta\, \mathrm{dn}^2(\xi, m)}{12a_0\alpha s^2 + (4a_1\alpha s^2 - \gamma)\mathrm{dn}^2(\xi, m)}\,. \tag{62}$$

(iv) If $a_0 = m^2$, $a_1 = -(1+m^2)$, and $a_2 = 1$, then $a_1^2 - 3a_0a_2 = 1 - m^2 + m^4 > 0$, and the solution is

$$z_4 = \mathrm{ns}(\xi, m) \equiv \frac{1}{\mathrm{sn}(\xi, m)}\,, \tag{63}$$

$$\phi_{4a} = \frac{6\eta z^2(\xi)}{12a_0\alpha s^2 + (4a_1\alpha s^2 - \gamma)z^2(\xi)} = \frac{6\eta\, \mathrm{ns}^2(\xi, m)}{12a_0\alpha s^2 + (4a_1\alpha s^2 - \gamma)\mathrm{ns}^2(\xi, m)}\,. \tag{64}$$

(v) If $a_0 = -m^2$, $a_1 = 2m^2 - 1$, and $a_2 = 1 - m^2$, then $a_1^2 - 3a_0a_2 = 1 - m^2 + m^4 > 0$, and the solution is

$$z_5 = \mathrm{nc}(\xi, m) \equiv \frac{1}{\mathrm{cn}(\xi, m)}\,, \tag{65}$$

$$\phi_{5a} = \frac{6\eta z^2(\xi)}{12a_0\alpha s^2 + (4a_1\alpha s^2 - \gamma)z^2(\xi)} = \frac{6\eta\, \mathrm{nc}^2(\xi, m)}{12a_0\alpha s^2 + (4a_1\alpha s^2 - \gamma)\mathrm{nc}^2(\xi, m)}\,. \tag{66}$$

(vi) If $a_0 = -1$, $a_1 = 2 - m^2$, and $a_2 = m^2 - 1$, then $a_1^2 - 3a_0a_2 = 1 - m^2 + m^4 > 0$, and the solution is

$$z_6 = \mathrm{nd}(\xi, m) \equiv \frac{1}{\mathrm{dn}(\xi, m)}\,, \tag{67}$$

$$\phi_{6a} = \frac{6\eta z^2(\xi)}{12a_0\alpha s^2 + (4a_1\alpha s^2 - \gamma)z^2(\xi)} = \frac{6\eta\, \mathrm{nd}^2(\xi, m)}{12a_0\alpha s^2 + (4a_1\alpha s^2 - \gamma)\mathrm{nd}^2(\xi, m)}\,. \tag{68}$$

(vii) If $a_0 = 1$, $a_1 = 2 - m^2$, and $a_2 = 1 - m^2$, then the $a_1^2 - 3a_0a_2 = 1 - m^2 + m^4 > 0$, and solution is

$$z_7 = \mathrm{sc}(\xi, m) \equiv \frac{\mathrm{sn}(\xi, m)}{\mathrm{cn}(\xi, m)}\,, \tag{69}$$

$$\phi_{7a} = \frac{6\eta z^2(\xi)}{12a_0\alpha s^2 + (4a_1\alpha s^2 - \gamma)z^2(\xi)} = \frac{6\eta\, \mathrm{sc}^2(\xi, m)}{12a_0\alpha s^2 + (4a_1\alpha s^2 - \gamma)\mathrm{sc}^2(\xi, m)}\,. \tag{70}$$

(viii) If $a_0 = 1$, $a_1 = 2m^2 - 1$, and $a_2 = (m^2 - 1)m^2$, then $a_1^2 - 3a_0a_2 = 1 - m^2 + m^4 > 0$, and the solution is

$$z_8 = \mathrm{sd}(\xi, m) \equiv \frac{\mathrm{sn}(\xi, m)}{\mathrm{dn}(\xi, m)}\,, \tag{71}$$

$$\phi_{8a} = \frac{6\eta z^2(\xi)}{12a_0\alpha s^2 + (4a_1\alpha s^2 - \gamma)z^2(\xi)} = \frac{6\eta\, \mathrm{sd}^2(\xi, m)}{12a_0\alpha s^2 + (4a_1\alpha s^2 - \gamma)\mathrm{sd}^2(\xi, m)}\,. \tag{72}$$

(ix) If $a_0 = 1 - m^2$, $a_1 = 2 - m^2$, and $a_2 = 1$, then $a_1^2 - 3a_0a_2 = 1 - m^2 + m^4 > 0$, and the solution is

$$z_9 = \mathrm{cs}(\xi, m) \equiv \frac{\mathrm{cn}(\xi, m)}{\mathrm{sn}(\xi, m)}\,, \tag{73}$$

$$\phi_{9a} = \frac{6\eta z^2(\xi)}{12a_0\alpha s^2 + (4a_1\alpha s^2 - \gamma)z^2(\xi)} = \frac{6\eta\, \mathrm{cs}^2(\xi, m)}{12a_0\alpha s^2 + (4a_1\alpha s^2 - \gamma)\mathrm{cs}^2(\xi, m)}\,. \tag{74}$$

(x) If $a_0 = 1$, $a_1 = -(1+m^2)$, and $a_2 = m^2$, then $a_1^2 - 3a_0a_2 = 1 - m^2 + m^4 > 0$, and the solution is

$$z_{10} = \mathrm{cd}(\xi, m) \equiv \frac{\mathrm{cn}(\xi, m)}{\mathrm{dn}(\xi, m)}\,, \tag{75}$$

$$\phi_{10a} = \frac{6\eta z^2(\xi)}{12a_0\alpha s^2 + (4a_1\alpha s^2 - \gamma)z^2(\xi)} = \frac{6\eta\,\mathrm{cd}^2(\xi, m)}{12a_0\alpha s^2 + (4a_1\alpha s^2 - \gamma)\mathrm{cd}^2(\xi, m)}\,. \tag{76}$$

(xi) If $a_0 = m^2(m^2-1)$, $a_1 = 2m^2 - 1$, and $a_2 = 1$, then $a_1^2 - 3a_0a_2 = 1 - m^2 + m^4 > 0$, and the solution is

$$z_{11} = \mathrm{ds}(\xi, m) \equiv \frac{\mathrm{dn}(\xi, m)}{\mathrm{sn}(\xi, m)}\,, \tag{77}$$

$$\phi_{11a} = \frac{6\eta z^2(\xi)}{12a_0\alpha s^2 + (4a_1\alpha s^2 - \gamma)z^2(\xi)} = \frac{6\eta\,\mathrm{ds}^2(\xi, m)}{12a_0\alpha s^2 + (4a_1\alpha s^2 - \gamma)\mathrm{ds}^2(\xi, m)}\,. \tag{78}$$

(xii) If $a_0 = m^2$, $a_1 = -(1+m^2)$, and $a_2 = 1$, then $a_1^2 - 3a_0a_2 = 1 - m^2 + m^4 > 0$, and the solution is

$$z_{12} = \mathrm{dc}(\xi, m) \equiv \frac{\mathrm{dn}(\xi, m)}{\mathrm{cn}(\xi, m)}\,, \tag{79}$$

$$\phi_{12a} = \frac{6\eta z^2(\xi)}{12a_0\alpha s^2 + (4a_1\alpha s^2 - \gamma)z^2(\xi)} = \frac{6\eta\,\mathrm{dc}^2(\xi, m)}{12a_0\alpha s^2 + (4a_1\alpha s^2 - \gamma)\,\mathrm{dc}^2(\xi, m)}\,. \tag{80}$$

Case 2 $b_0 \neq 0$, $b_1 = 0$, and $b_2 \neq 0$

In this case, we can derive

$$b_0 = \frac{\eta}{2a_2\alpha s^2}b_2\,, \quad b_2 = \frac{12a_2\alpha s^2}{4a_1\alpha s^2 - \gamma} \tag{81}$$

with constraints (54) and (55). Similarly, we can obtain solutions just similar to solutions from ϕ_{1a} to ϕ_{12a}. Here we omit the details.

5 Conclusion

A simple shallow-water model with influence of diabatic heating on a β-plane is applied to investigate the nonlinear equatorial Rossby waves in a shear flow. By the asymptotic method of multiple scales, the cubic nonlinear Schrödinger equation with an external heating source is derived for large amplitude equatorial envelope Rossby wave in a shear flow. And then various periodic structures for these equatorial envelope Rossby waves are obtained with the help of Jacobi elliptic functions and elliptic equation. It is shown that the results are different for equatorial envelope Rossby waves without a source and with a phase-locked diabatic heating source, they have different structures due to the phase-locked diabatic heating source, and the phase-locked diabatic heating source plays an important role in forming periodic structures of rational form. Of course, these periodic structures contain solitons, solitary waves, as also singular structures, and they also have their different practical applications in explaining atmospheric events. This needs further research. Moreover, in this paper, we only consider one special case of external heating and find some exact results. For more various heating sources, this effort provides a better starting point for the treatment of general external heating sources and their impacts on the equatorial Rossby waves and climate changes.

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