New Exact Solutions to NLS Equation and Coupled NLS Equations*

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Abstract A transformation is introduced on the basis of the projective Riccati equations, and it is applied as an intermediate in expansion method to solve nonlinear Schrödinger (NLS) equation and coupled NLS equations. Many kinds of envelope travelling wave solutions including envelope solitary wave solution are obtained, in which some are found for the first time.

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1 Introduction
A number of problems are described in terms of suitable nonlinear models, such as nonlinear Schrödinger equations (NLS) in plasma physics,[1] KdV equation in shallow water model,[2] and so on, in branches of physics, mathematics, and other interdisciplinary sciences. It is an interesting topic to seek exact solutions to these nonlinear models. Here we will consider two nonlinear models, i.e. NLS equation and coupled NLS equations. The NLS equation reads

\[ iu_t + \alpha u_{xx} + \beta |u|^2 u = 0, \tag{1} \]

and the coupled NLS equations read

\[
\begin{align*}
    iu_t + \alpha u_{xx} &+ (\beta_1 |u|^2 + \beta_2 |v|^2)u = 0, \\
    iv_t + \alpha v_{xx} &+ (\beta_1 |v|^2 + \beta_2 |u|^2)v = 0.
\end{align*}
\tag{2}
\]

In Ref. [3], on the base of Lamé equation and Lamé functions, we obtained the perturbed solutions to the NLS equation and coupled NLS equations. In this paper, we will reconsider these two nonlinear evolution systems. A transformation is obtained from the well-known projective Riccati equations,[4–6] and then this transformation is taken as an intermediate to solve these two systems, where many kinds of envelope travelling wave solutions including envelope solitary wave solutions are derived, among which some are found for the first time.

2 Exact Solutions to NLS Equation
In order to solve Eq. (1), we take the following transformation

\[ u = \phi(\xi) e^{i(kx-\omega t)}, \quad \xi = x - c_g t, \tag{3} \]

where \( k \) is the wave number, \( \omega \) is the angular frequency, \( c_g \) is the group velocity, and \( \phi(\xi) \) is a real function.

Substituting Eq. (3) into Eq. (1) results in the following ordinary differential equation,

\[ \alpha \frac{d^2 \phi}{d\xi^2} + i(2\alpha k - \gamma) \frac{d\phi}{d\xi} + (\omega - \alpha k^2)\phi + \beta \phi^3 = 0. \tag{4} \]

Separating the real part and imaginary part yields

\[ c_g = 2\alpha k, \tag{5} \]

and if we suppose

\[ \omega - \alpha k^2 = -\gamma, \tag{6} \]

then equation (4) is rewritten as

\[ \alpha \frac{d^2 \phi}{d\xi^2} - \gamma \phi + \beta \phi^3 = 0. \tag{7} \]

In order to solve Eq. (7), we introduce the following crucial ansatz,

\[ \phi(\xi) = \sum_{i=1}^{n} f^{-1}(\xi)[A_i f(\xi) + B_i g(\xi)] + A_0, \]

where \( n \) can be determined by balancing the highest order derivative term with the high degree nonlinear term in Eq. (7). And \( f \) and \( g \) are solutions to the well-known projective Riccati equations,[4–6]

\[ f'(\xi) = pf(\xi)g(\xi), \]

\[ g'(\xi) = q + pg^2(\xi) - rf(\xi), \tag{9} \]

where \( p \neq 0 \) is a real constant, and \( q \) and \( r \) are two real constants. When \( p = -1 \) and \( q = 1 \), equations (9) reduce to the coupled equations given in Refs. [4] and [5], and when \( p = \pm 1 \) and \( q \geq 0 \), equations (9) reduce to the coupled equations given in Ref. [6]. There is a relation between \( f \) and \( g \),

\[ g^2 = \frac{1}{p} \left[ q - 2rf + \frac{r^2 + \delta}{q} f^2 \right]. \tag{10} \]
where $\delta = \pm 1$.

Applying the above expansion method, if we take the expansion order of $\phi$ as $O(\phi) = n$ and considering the relations (9), then $O(d\phi/d\xi) = n + 1$, so partial balance between the highest degree nonlinear term and the highest order derivative term leads to $n = 1$. Obviously, the formal solution can be written as

$$\phi = A_0 + A_1 f(\xi) + B_1 g(\xi), \quad A_1^2 + B_1^2 \neq 0. \quad (11)$$

Considering the relation (10), from Eq. (11) one can have

$$\phi^3 = \left( A_0^3 - \frac{3q}{p} A_0 B_1^2 \right) + \left( 3A_0^2 B_1 - \frac{q}{p} B_1^3 \right) g + \left( 3A_0^2 A_1 - \frac{3q}{p} A_1 B_1^2 + \frac{6r}{p} A_0 B_1^2 \right) f + \left( 6A_0 A_1 B_1 + \frac{2r}{p} B_1^3 \right) f g$$

$$+ \left[ 3A_0 A_1^2 + \frac{6r}{p} A_1 B_1^2 - \frac{3(r^2 + \delta)}{pq} A_0 B_1^2 \right] f^2 + \left[ 3A_1^2 B_1 - \frac{(r^2 + \delta)}{pq} B_1^3 \right] f^2 g + \left[ A_1^3 - \frac{3(r^2 + \delta)}{pq} A_1 B_1^2 \right] f^3$$

and

$$\frac{d^2\phi}{d\xi^2} = -pq A_1 f + pr B_1 f g + 3pr A_1 f^2 - \frac{2p(r^2 + \delta)}{q} B_1 f^2 g - \frac{2p(r^2 + \delta)}{q} A_1 f^3. \quad (13)$$

Substituting Eqs (11), (12), and (13) into Eq. (7) results in the following algebraic equations,

$$-\gamma A_0 + \beta \left( A_0^3 - \frac{3q}{p} A_0 B_1^2 \right) = 0, \quad \gamma A_1 + \beta \left( -\frac{3q}{p} A_1 B_1^2 + \frac{6r}{p} A_0 B_1^2 + 3A_0^2 A_1 \right) - \alpha pq A_1 = 0, \quad \gamma B_1 + \beta \left( -\frac{q}{p} B_1^3 + 3A_0^2 B_1 \right) = 0,$$

$$\beta \left( \frac{6r}{p} A_1 B_1^2 - \frac{3(r^2 + \delta)}{pq} A_0 B_1^2 + 3A_0 A_1^2 \right) + 3\alpha pr A_1 = 0,$$

$$\beta \left[ 3A_1^2 B_1 - \frac{(r^2 + \delta)}{pq} B_1^3 \right] - \frac{2\alpha p(r^2 + \delta)}{q} B_1 = 0,$$

$$\beta \left[ A_1^3 - \frac{3(r^2 + \delta)}{pq} A_1 B_1^2 \right] - \frac{2\alpha p(r^2 + \delta)}{q} A_1 = 0, \quad (14)$$

for the arbitrariness of the argument $\xi$, from which the parameters can be determined. For example, for $\delta = -1$, there are the following solutions:

**Case 1** If $A_1 = 0$, $A_0 = 0$, $r = 0$, then

$$B_1 = \pm \sqrt{-\frac{2\alpha \gamma}{\beta}}, \quad pq = \frac{\gamma}{2\alpha}. \quad (15)$$

Obviously, there is the constraint $\alpha \beta < 0$.

**Case 2** If $A_1 = 0$, $A_0 = 0$, $r \neq 0$, then

$$B_1 = \pm \sqrt{\frac{\alpha p^2}{2\beta}}, \quad pq = \frac{2\gamma}{\alpha}, \quad r = \pm 1. \quad (16)$$

Obviously, there is the constraint $\alpha \beta < 0$, too.

**Case 3** If $B_1 = 0$, $A_0 = 0$, then

$$A_1 = \pm \sqrt{\frac{2\alpha^2 p^2}{\beta \gamma}}, \quad pq = -\frac{\gamma}{\alpha}, \quad r = 0. \quad (17)$$

**Case 4** If $B_1 = 0$, $A_0 \neq 0$, then

$$A_1 = \pm \sqrt{\frac{\alpha^2 p^2 (r^2 + 2)}{\beta \gamma}}, \quad A_0 = \pm \sqrt{\frac{\gamma r^2}{\beta (r^2 + 2)}}, \quad pq = \frac{2(r^2 - 1)\gamma}{\alpha (r^2 + 2)}. \quad (18)$$

There is the constraint $r \neq 0$ and $r^2 \neq 1$. 
**Case 5** If \( A_0 = 0, A_1 \neq 0, B_1 \neq 0 \), then

\[
A_1 = \pm \sqrt{\frac{\alpha^2 \rho^2 (r^2 - 1)}{4\beta \gamma}}, \quad B_1 = \pm \sqrt{\frac{\alpha \rho}{2\beta}}, \quad pq = \frac{2\gamma}{\alpha}
\]

with the constraint \( r^2 \neq 1 \).

For \( \delta = 1 \), there are the following solutions:

**Case 1** If \( A_1 = 0, A_0 = 0, r = 0 \), then

\[
B_1 = \pm \sqrt{-\frac{2\alpha \rho^2}{\beta}}, \quad pq = \frac{\gamma}{2\alpha}.
\]

Obviously, there is the constraint that \( \alpha \beta < 0 \).

**Case 2** If \( B_1 = 0, A_0 = 0, \) then

\[
A_1 = \pm \sqrt{-\frac{2\alpha \rho^2}{\beta \gamma}}, \quad pq = \frac{\gamma}{\alpha}, \quad r = 0.
\]

**Case 3** If \( B_1 = 0, A_0 \neq 0, \) then

\[
A_1 = \pm \sqrt{\frac{\alpha^2 \rho^2 (r^2 - 2)}{\beta \gamma}}, \quad A_0 = \pm \sqrt{\frac{\gamma r^2}{\beta (r^2 - 2)}}, \quad pq = \frac{2(r^2 + 1)\gamma}{\alpha(r^2 - 2)}.
\]

There is the constraint \( r \neq 0 \) and \( r^2 \neq 2 \).

**Case 4** If \( A_0 = 0, A_1 \neq 0, B_1 \neq 0, \) then

\[
A_1 = \pm \sqrt{\frac{\alpha^2 \rho^2 (r^2 + 1)}{4\beta \gamma}}, \quad B_1 = \pm \sqrt{\frac{\alpha \rho}{2\beta}}, \quad pq = \frac{2\gamma}{\alpha}.
\]

For the projective Riccati equations (9), when \( pq < 0 \) and \( \delta = 1 \), its solution is

\[
f_1 = \frac{q}{r + \sinh(\sqrt{-pq} \xi)},
\]

\[
g_1 = -\frac{\sqrt{-pq}}{p} \cosh(\sqrt{-pq} \xi),
\]

and when \( pq < 0 \) and \( \delta = -1 \), its solution is

\[
f_2 = \frac{q}{r + \cosh(\sqrt{-pq} \xi)},
\]

\[
g_2 = -\frac{\sqrt{-pq}}{p} \sinh(\sqrt{-pq} \xi).
\]

When \( pq > 0 \) and \( \delta = -1 \), its solutions are

\[
f_3 = \frac{q}{r + \sin(\sqrt{pq} \xi)},
\]

\[
g_3 = -\frac{\sqrt{pq}}{p} \cos(\sqrt{pq} \xi),
\]

and

\[
f_4 = \frac{q}{r + \cos(\sqrt{pq} \xi)},
\]

\[
g_4 = \frac{\sqrt{pq}}{p} \sin(\sqrt{pq} \xi).
\]

Combining Eqs. (3), (11), and the results from Eq. (15) ~ (31), we can derive various envelope travelling solutions including envelope solitary wave solutions to NLS equation (1), for example,

**Type 1** For \( \delta = -1 \), if \( \alpha \beta < 0 \) and \( \alpha \gamma < 0 \), then the solution to NLS equation (1) is

\[
u_1 = B_1 g_2 e^{i(kx - \omega t)} = \mp \sqrt{\frac{\gamma}{\beta}} \tanh \left( \sqrt{-\frac{\gamma}{2\alpha} \xi} \right) e^{i(kx - \omega t)}.
\]
Type 2 For $\delta = -1$, if $\alpha \beta < 0$ and $\alpha \gamma > 0$, then the solution to NLS equation (1) is
\[ u_2 = B_1 g_3 e^{i(kx-\omega t)} = \pm \sqrt{\frac{-\gamma}{\beta}} \cot\left(\sqrt{\frac{-\gamma}{2\alpha}} \xi\right) e^{i(kx-\omega t)}, \] (33)
and
\[ u_3 = B_1 g_4 e^{i(kx-\omega t)} = \pm \sqrt{\frac{-\gamma}{\beta}} \tan\left(\sqrt{\frac{-\gamma}{2\alpha}} \xi\right) e^{i(kx-\omega t)}. \] (34)

Type 3 For $\delta = -1$, if $\alpha \beta < 0$ and $\alpha \gamma < 0$, then the solution to NLS equation (1) is
\[ u_4 = B_1 g_2 e^{i(kx-\omega t)} = \pm \sqrt{\frac{\gamma}{\beta}} \sinh\left(\sqrt{-\frac{2\gamma}{\alpha}} \xi\right) e^{i(kx-\omega t)}. \] (35)

Type 4 For $\delta = -1$, if $\alpha \beta < 0$ and $\alpha \gamma > 0$, then the solution to NLS equation (1) is
\[ u_5 = B_1 g_3 e^{i(kx-\omega t)} = \pm \sqrt{\frac{-\gamma}{\beta}} \cos\left(\sqrt{\frac{2\gamma}{\alpha}} \xi\right) e^{i(kx-\omega t)}, \] (36)
and
\[ u_6 = B_1 g_4 e^{i(kx-\omega t)} = \pm \sqrt{\frac{-\gamma}{\beta}} \sin\left(\sqrt{\frac{2\gamma}{\alpha}} \xi\right) e^{i(kx-\omega t)}. \] (37)

Type 5 For $\delta = -1$, if $\beta \gamma > 0$ and $\alpha \gamma > 0$, then the solution to NLS equation (1) is
\[ u_7 = A_1 f_2 e^{i(kx-\omega t)} = \pm \sqrt{\frac{2\gamma}{\beta}} \sech\left(\sqrt{\frac{-\gamma}{\alpha}} \xi\right) e^{i(kx-\omega t)}. \] (38)

Type 6 For $\delta = -1$, if $\beta \gamma > 0$ and $\alpha \gamma < 0$, then the solution to NLS equation (1) is
\[ u_8 = A_1 f_3 e^{i(kx-\omega t)} = \pm \sqrt{\frac{2\gamma}{\beta}} \csc\left(\sqrt{-\frac{\gamma}{\alpha}} \xi\right) e^{i(kx-\omega t)}, \] (39)
and
\[ u_9 = A_1 f_4 e^{i(kx-\omega t)} = \pm \sqrt{\frac{2\gamma}{\beta}} \sec\left(\sqrt{-\frac{\gamma}{\alpha}} \xi\right) e^{i(kx-\omega t)}. \] (40)

Type 7 For $\delta = -1$, if $\beta \gamma > 0$ and $(r^2 - 1)\alpha \gamma < 0$, then the solution to NLS equation (1) is
\[ u_{10} = (A_0 + A_1 f_2) e^{i(kx-\omega t)} \]
\[ = \left[ \pm \sqrt{\frac{\gamma r^2}{\beta(r^2 + 2)}} \pm \frac{2(r^2 - 1)\gamma}{\rho \alpha \beta(r^2 + 2)} \sqrt{\frac{\alpha^2 p^2(r^2 + 2)}{\beta \gamma}} \frac{1}{\sinh(\sqrt{2(1 - r^2)\gamma/\alpha(r^2 + 2)} \xi + \rho)} \right] e^{i(kx-\omega t)} \] (41)
with the constraint that $r \neq 0$ and $r^2 \neq 1$.

Type 8 For $\delta = -1$, if $\beta \gamma > 0$ and $(r^2 - 1)\alpha \gamma > 0$, then the solution to NLS equation (1) is
\[ u_{11} = (A_0 + A_1 f_3) e^{i(kx-\omega t)} \]
\[ = \left[ \pm \sqrt{\frac{\gamma r^2}{\beta(r^2 + 2)}} \pm \frac{2(r^2 - 1)\gamma}{\rho \alpha \beta(r^2 + 2)} \sqrt{\frac{\alpha^2 p^2(r^2 + 2)}{\beta \gamma}} \frac{1}{\sin(\sqrt{2(1 - r^2)\gamma/\alpha(r^2 + 2)} \xi + \rho)} \right] e^{i(kx-\omega t)}, \] (42)
and
\[ u_{12} = (A_0 + A_1 f_4) e^{i(kx-\omega t)} \]
\[ = \left[ \pm \sqrt{\frac{\gamma r^2}{\beta(r^2 + 2)}} \pm \frac{2(r^2 - 1)\gamma}{\rho \alpha \beta(r^2 + 2)} \sqrt{\frac{\alpha^2 p^2(r^2 + 2)}{\beta \gamma}} \frac{1}{\cos(\sqrt{2(1 - r^2)\gamma/\alpha(r^2 + 2)} \xi + \rho)} \right] e^{i(kx-\omega t)} \] (43)
with the constraint that $r \neq 0$ and $r^2 \neq 1$.

Type 9 For $\delta = -1$, if $\alpha \beta < 0$, $\alpha \gamma < 0$, and $r^2 > 1$, then the solution to NLS equation (1) is
\[ u_{13} = (A_1 f_2 + B_1 g_2) e^{i(kx-\omega t)} \]
\[ = \left[ \pm \frac{2\gamma}{\alpha p} \sqrt{\frac{\alpha^2 p^2(r^2 - 1)}{4\beta \gamma}} \frac{1}{\cosh(\sqrt{2\gamma/\alpha} \xi + \rho)} \right. \pm \frac{\gamma}{\beta} \frac{\sinh(\sqrt{-2\gamma/\alpha} \xi)}{\cosh(\sqrt{-2\gamma/\alpha} \xi + \rho)} \right] e^{i(kx-\omega t)} \] (44)
with the constraint $r^2 \neq 1$.

**Type 10** For $\delta = -1$, if $\alpha \beta < 0$, $\alpha \gamma > 0$ and $r^2 < 1$, then the solution to NLS equation (1) is

$$u_{14} = (A_1 f_3 + B_1 g_3) e^{i(kx - \omega t)} = \left[ \pm \frac{2\gamma}{\alpha p} \sqrt{\frac{\alpha^2 p^2 (r^2 - 1)}{4\beta^2}} \frac{1}{\sin(\sqrt{2}\gamma/\alpha \xi) + r} \mp \sqrt{-\frac{\gamma}{\beta}} \cos(\sqrt{2}\gamma/\alpha \xi) + r \right] e^{i(kx - \omega t)}$$

and

$$u_{15} = (A_1 f_4 + B_1 g_4) e^{i(kx - \omega t)} = \left[ \pm \frac{2\gamma}{\alpha p} \sqrt{\frac{\alpha^2 p^2 (r^2 - 1)}{4\beta^2}} \frac{1}{\cos(\sqrt{2}\gamma/\alpha \xi) + r} \mp \sqrt{-\frac{\gamma}{\beta}} \sin(\sqrt{2}\gamma/\alpha \xi) + r \right] e^{i(kx - \omega t)}$$

with the constraint $r^2 \neq 1$.

**Type 11** For $\delta = 1$, if $\alpha \beta < 0$ and $\alpha \gamma < 0$, then the solution to NLS equation (1) is

$$u_{16} = B_1 g_1 e^{i(kx - \omega t)} = \mp \sqrt{-\frac{\gamma}{\beta}} \coth \left( \sqrt{-\frac{\gamma}{2\alpha}} \xi \right) e^{i(kx - \omega t)}.$$  

**Type 12** For $\delta = 1$, if $\beta \gamma < 0$ and $\alpha \gamma > 0$, then the solution to NLS equation (1) is

$$u_{17} = A_1 f_1 e^{i(kx - \omega t)} = \mp \frac{2\gamma}{\beta} \csc \left( \sqrt{\frac{\gamma}{\alpha}} \xi \right) e^{i(kx - \omega t)}.$$  

**Type 13** For $\delta = 1$, if $\beta \gamma(r^2 - 2) > 0$ and $(r^2 - 2)\alpha \gamma < 0$, then the solution to NLS equation (1) is

$$u_{18} = (A_0 + A_1 f_1) e^{i(kx - \omega t)}$$

$$= \left[ \pm \frac{\sqrt{\gamma^3}}{\beta(r^2 - 2)} \frac{1}{\frac{\gamma}{\beta} \frac{1}{\cos(\sqrt{2(1 + r^2)/\gamma}(\gamma/\alpha(2 - r^2))) + r} \mp \sqrt{\frac{\gamma}{\beta}} \sin(\sqrt{2}\gamma/\alpha \xi) + r} \right] e^{i(kx - \omega t)}$$

with the constraint that $r \neq 0$ and $r^2 \neq 2$.

**Type 14** For $\delta = 1$, if $\alpha \beta < 0$ and $\alpha \gamma < 0$, then the solution to NLS equation (1) is

$$u_{19} = (A_1 f_1 + B_1 g_1) e^{i(kx - \omega t)}$$

$$= \left[ \pm \frac{2\gamma}{\alpha p} \sqrt{\frac{\alpha^2 p^2 (r^2 + 1)}{4\beta^2}} \frac{1}{\sinh(\sqrt{2}\gamma/\alpha \xi) + r} \mp \sqrt{\frac{\gamma}{\beta}} \sinh(\sqrt{-2\gamma/\alpha \xi}) + r \right] e^{i(kx - \omega t)}.$$  

3 Exact Solutions to Coupled NLS Equation

In order to solve Eq. (2), we take the following transformation,

$$u = \phi(\xi) e^{i(kx - \omega t)}, \quad v = \psi(\xi) e^{i(kx - \omega t)}, \quad \xi = x - c_\gamma t.$$  

If equations (5) and (6) are considered, then equation (2) can be rewritten as

$$\alpha \frac{d^2 \phi}{d\xi^2} - \gamma \phi + (\beta_1 \phi^3 + \beta_2 \phi^2) = 0,$$

$$\alpha \frac{d^2 \psi}{d\xi^2} - \gamma \psi + (\beta_1 \psi^3 + \beta_2 \psi^2) = 0.$$  

If $\phi = \psi$ is taken, then

$$\alpha \frac{d^2 \phi}{d\xi^2} - \gamma \phi + (\beta_1 + \beta_2) \phi^3 = 0.$$  

Comparing Eq. (53) with Eq. (7), one can see that the difference between these two equations is that $\beta$ in Eq. (7) is replaced by $\beta_1 + \beta_2$ in Eq. (53), so the solutions to Eq. (2) can be easily obtained from the solutions to Eq. (1), here we omit these details.
4 Conclusion

In this paper, we introduce an intermediate transformation from solutions to the projective Riccati equations and apply it to solve the NLS equation and the coupled NLS equations. Many solutions are obtained for these nonlinear systems, such as envelope solitary wave solutions constructed in terms of hyperbolic functions and envelope periodic solutions expressed in terms of sine and cosine functions. Some of which are not given in literatures to our knowledge. Of course, this transformation can be also applied to other nonlinear wave equations.

References