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Exact Periodic-Wave Solutions to Nizhnik–Novikov–Veselov Equation*

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Abstract *Exact periodic-wave solutions to the generalized Nizhnik–Novikov–Veselov (NNV) equation are obtained by using the extended Jacobi elliptic-function method, and in the limit case, the solitary wave solution to NNV equation are also obtained.*

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1 Introduction

In this letter we consider the generalized Nizhnik–Novikov–Veselov (NNV) equation,[1−3]

$$u_t + au_{xxx} + bu_{yyy} - 3a(uv)_x - 3b(uw)_y = 0\,,$$
$$u_x = v_y\,, \quad u_y = w_x\,, \tag{1}$$

which is a (2+1)-dimensional integrable system known as isotropic Lax extension of the well-known one-dimensional KdV equation. Equations (1) have been payed much attention by many authors[4−11] and some types of the soliton, multisoliton, chaotic and fractal solutions have been obtained by means of the inverse scattering transform method, Bäcklund transformation, homogeneous balance method, and so on. The aim of this letter is to obtain the periodic-wave solutions to Eqs. (1) by making use of the extended Jacobi elliptic-function method, which can be thought of as an over-all generalization of tanh-method,[12] sech-method,[13] and the Jacobi elliptic-function method proposed and developed in Refs. [14] ∼ [16]. In Sec. 2 we describe briefly the method. In Sec. 3 we apply the method described in Sec. 2 to Eqs. (1) and bring out exact solutions. Conclusions will be presented in Sec. 4.

2 Extended Jacobi Elliptic-Function Method

Consider a given PDE of the form

$$P(u, u_t, u_x, u_y, u_z, u_{xx}, u_{xy}, u_{xz}, u_{yz}, u_{xt}, \ldots) = 0\,, \tag{2}$$

where $u_t = \partial u/\partial t$, $u_x = \partial u/\partial x$, etc. In this paper, we seek the following formal travelling wave solutions to Eq. (2)

$$u(x, y, z, t) = u(\xi)\,, \quad \xi = kx + ly + rz - \omega t\,, \tag{3}$$

where $(k,\ l,\ r)$ are the components of the wave-number vector in the x, y, and z directions, respectively, and ω is the angular frequency. Substituting Eq. (3) into Eq. (2) yields an ordinary differential equation (ODE) for $u(\xi)$ with constant coefficients

$$N\Big(u, \frac{\mathrm{d}u}{\mathrm{d}\xi}, \frac{\mathrm{d}^2u}{\mathrm{d}\xi^2}, \frac{\mathrm{d}^3u}{\mathrm{d}\xi^3}, \cdots\Big) = 0\,. \tag{4}$$

We assume the degree of $u(\xi)$ as $O(u(\xi)) = n$, which leads to the degrees of other expressions in Eq. (4) as

$$O\Big(\frac{\mathrm{d}^pu}{\mathrm{d}\xi^p}\Big) = n + p\,,$$
$$O\Big(u^q\frac{\mathrm{d}^pu}{\mathrm{d}\xi^p}\Big) = (q+1)n + p\,,$$
$$q = 0, 1, 2, \ldots\,, \qquad p = 1, 2, 3, \ldots \tag{5}$$

The next crucial step is that $u(\xi)$ is expanded into a finite power series of $E(\xi)$,

$$u(\xi) = \sum_{j=0}^{n} A_j E^j(\xi)\,, \qquad A_n \neq 0\,, \tag{6}$$

where A_j are constants to be determined later, n will be fixed by balancing the highest order of derivative term and the nonlinear term in the ODE (2) by using Eq. (5), and $E(\xi)$ satisfies the following ODE,

$$E'' + \alpha E + \beta E^3 = 0\,, \quad (E')^2 + \alpha E^2 + \frac{1}{2}\beta E^4 = \gamma\,, \tag{7}$$

where α, β, and γ are constants to be determined. Because of the entrance of three parameters α, β, and γ, equation (7) has rich structures of solutions. For example, as $\alpha = 2$, $\beta = -2$, and $\gamma = 1$, the solutions of Eq. (7) read $E(\xi) = \tanh\xi$, and the method is called tanh-function method. When $\alpha = -1$, $\beta = 2$, and $\gamma = 0$, equation (7) has the solution $E(\xi) = \mathrm{sech}\,\xi$, and the method is named sech-function method. Above all, equation (7) has many Jacobi elliptic-function solutions for different values of α, β, and γ. In Table 1 we present the constants α, β, γ for the twelve Jacobi elliptic functions.

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Table 1 Jacobi elliptic functions and parameters of Eq. (7).

N	$E(\xi)=E(\xi,m)$	α	β	γ
1	$\mathrm{sn}\,\xi$	$1+m^2$	$-2m^2$	1
2	$\mathrm{cn}\,\xi$	$1-2m^2$	$2m^2$	$1-m^2$
3	$\mathrm{dn}\,\xi$	$-(2-m^2)$	2	$-(1-m^2)$
4	$\mathrm{ns}\,\xi\equiv 1/\mathrm{sn}\,\xi$	$1+m^2$	-2	m^2
5	$\mathrm{nc}\,\xi\equiv 1/\mathrm{cn}\,\xi$	$1-2m^2$	$-2(1-m^2)$	$-m^2$
6	$\mathrm{nd}\,\xi\equiv 1/\mathrm{dn}\,\xi$	$-(2-m^2)$	$2(1-m^2)$	-1
7	$\mathrm{sc}\,\xi\equiv \mathrm{sn}\,\xi/\mathrm{cn}\,\xi$	$-(2-m^2)$	$-2(1-m^2)$	1
8	$\mathrm{sd}\,\xi\equiv \mathrm{sn}\,\xi/\mathrm{dn}\,\xi$	$1-2m^2$	$2m^2(1-m^2)$	1
9	$\mathrm{cs}\,\xi\equiv \mathrm{cn}\,\xi/\mathrm{sn}\,\xi$	$-(2-m^2)$	-2	$1-m^2$
10	$\mathrm{cd}\,\xi\equiv \mathrm{cn}\,\xi/\mathrm{dn}\,\xi$	$1+m^2$	$-2m^2$	1
11	$\mathrm{ds}\,\xi\equiv \mathrm{dn}\,\xi/\mathrm{sn}\,\xi$	$1-2m^2$	-2	$-m^2(1-m^2)$
12	$\mathrm{dc}\,\xi\equiv \mathrm{dn}\,\xi/\mathrm{cn}\,\xi$	$1+m^2$	-2	m^2

In this article, for Jacobi elliptic functions, we use the notation $E(\xi)$ instead of $E(\xi,m)$ (ξ is the variable argument, m is the modulus parameter). The three basic Jacobi elliptic functions are determined as

$$\mathrm{sn}\,\xi=\sin\varphi,\quad \mathrm{cn}\,\xi=\cos\varphi,\quad \mathrm{dn}\,\xi=\sqrt{1-m^2\sin^2\varphi}\,,\tag{8}$$

where φ is implicity defined by the elliptic integral of the first kind,

$$\xi=\int_0^{\varphi}\frac{\mathrm{d}\tau}{\sqrt{1-m^2\sin^2\tau}}\,,\tag{9}$$

and

$$\mathrm{cn}^2\xi=1-\mathrm{sn}^2\xi\,,\quad \mathrm{dn}^2\xi=1-m^2\,\mathrm{sn}^2\xi\,,\tag{10}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{sn}\,\xi=\mathrm{cn}\,\xi\;\mathrm{dn}\,\xi\,,\quad \frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{cn}\,\xi=-\,\mathrm{sn}\,\xi\;\mathrm{dn}\,\xi\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{dn}\,\xi=-m^2\,\mathrm{sn}\,\xi\;\mathrm{cn}\,\xi\,,\tag{11}$$

where $\mathrm{sn}\,\xi$, $\mathrm{cn}\,\xi$, and $\mathrm{dn}\,\xi$ are Jacobi elliptic sine, cosine functions, and the Jacobi elliptic function of the third kind, respectively. The rest of the nine Jacobi elliptic functions are reciprocals of these three functions, and the quotients of any two of them. The Jacobi elliptic functions are doubly periodic functions of the complex argument ξ. If restricting ξ to real values we see that $\mathrm{sn}\,\xi$, $\mathrm{cn}\,\xi$, and $\mathrm{dn}\,\xi$ have periods $4K$, $4K$, $2K$, respectively, where

$$K=K(m)=\int_0^{\pi/2}\frac{\mathrm{d}\tau}{\sqrt{1-m^2\sin^2\tau}}\,.\tag{12}$$

Detailed explanations about the Jacobi elliptic functions can be found in Refs. [17] and [18],

Substituting Eq. (6) (with fixed value of n) into the reduced nonlinear ODE (4) and equating the coefficients of various powers of $E(\xi)$ to zero we get a set of algebraic equations for A_j, k, l, r, and ω. Solving them consistently we obtain relations among the parameters A_j, k, l, r, and ω, if any of the parameters is left unspecified, it is regarded as arbitrary constants. Making use of these relations we can find a final expression for $u(\xi)$, which leads to an expression for the travelling wave solutions for Eq. (2). Therefore, equation (6) establishes an algebraic mapping relation between the solution to Eq. (7) and that to Eq. (2). Obviously, the extended Jacobi elliptic-function method is a unified approach, including tanh-, sech-, and the Jacobi elliptic-function methods in Refs. [14] $\sim$ [16] as special cases.

3 Periodic-Wave and Solitary Wave Solutions

We perform a travelling wave solutions $u(x,y,t)=u(\xi)$, $v(x,y,t)=v(\xi)$, $w(x,y,t)=w(\xi)$, $\xi=kx+ly-\omega t$. Substituting $u(\xi)$, $v(\xi)$, and $w(\xi)$ into Eqs. (1) yields

$$-\omega\frac{\mathrm{d}u}{\mathrm{d}\xi}+(ak^3+bl^3)\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3}-3ak\frac{\mathrm{d}(uv)}{\mathrm{d}\xi}-3bl\frac{\mathrm{d}(uw)}{\mathrm{d}\xi}=0\,,\tag{13a}$$

$$k\frac{\mathrm{d}u}{\mathrm{d}\xi}=l\frac{\mathrm{d}v}{\mathrm{d}\xi}\,,\quad l\frac{\mathrm{d}u}{\mathrm{d}\xi}=k\frac{\mathrm{d}w}{\mathrm{d}\xi}\,.\tag{13b}$$

Integrating Eq. (13b) with respect to ξ and taking the integration constants to be zero yields

$$v=\frac{k}{l}u\,,\qquad w=\frac{l}{k}u\,.\tag{14}$$

Substituting Eq. (14) into Eq. (13a) yields

$$\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3}-\frac{6}{kl}u\frac{\mathrm{d}u}{\mathrm{d}\xi}-\frac{\omega}{ak^3+bl^3}\frac{\mathrm{d}u}{\mathrm{d}\xi}=0\,,\tag{15}$$

where $ak^3+bl^3\neq 0$. Considering Eq. (5) to balance the highest derivative with the nonlinear terms in Eq. (15) we get $n=2$, that is

$$u(\xi)=A_0+A_1E(\xi)+A_2E^2(\xi)\,.\tag{16}$$

Substituting Eq. (16) into Eq. (15) and using Eq. (7) yields

$$\Big(\alpha-\frac{6A_0}{kl}-\frac{\omega}{ak^3+bl^3}\Big)A_1E'(\xi)+\Big[2A_2\Big(4\alpha-\frac{6A_0}{kl}-\frac{\omega}{ak^3+bl^3}\Big)-\frac{6A_1^2}{kl}\Big]E(\xi)E'(\xi)$$
$$+3A_1\Big(\beta-\frac{6A_2}{kl}\Big)E^2(\xi)E'(\xi)+12A_2\Big(\beta-\frac{A_2^2}{kl}\Big)E^3(\xi)E'(\xi)=0\,. \tag{17}$$

Cancelling $E'(\xi)$ and setting each coefficient of $E^n(\xi)$ $(n=0,1,2,3)$ to zero yields a set of equations for A_0, A_1, A_2, k, l, and ω. From the solution of these equations under condition $A_2\neq 0$, $k\neq 0$, $l\neq 0$, and $\omega\neq 0$, the coefficients are so determined that

$$A_0=\frac{2\alpha kl}{3}-\frac{kl\omega}{6(ak^3+bl^3)}\,,\qquad A_1=0,\qquad A_2=\pm\sqrt{\beta kl}\;. \tag{18}$$

Substituting Eqs. (18) into Eqs. (16) and (14) yields a general form solution to Eqs. (1),

$$u(x,y,t)=u(\xi)=\frac{2\alpha kl}{3}-\frac{kl\omega}{6(ak^3+bl^3)}\pm\sqrt{\beta kl}E^2(\xi)\;,$$
$$v(x,y,t)=v(\xi)=\frac{2\alpha k^2}{3}-\frac{k^2\omega}{6(ak^3+bl^3)}\pm\frac{k}{l}\sqrt{\beta kl}E^2(\xi)\;,$$
$$w(x,y,t)=w(\xi)=\frac{2\alpha l^2}{3}-\frac{l^2\omega}{6(ak^3+bl^3)}\pm\frac{l}{k}\sqrt{\beta kl}E^2(\xi)\;. \tag{19}$$

From Table 1, if we take $\alpha=1-2m^2$, $\beta=2m^2$, $\gamma=1-m^2$, then $E(\xi)=\operatorname{cn}\xi$, thus

$$u(x,y,t)=\frac{2(1-2m^2)kl}{3}-\frac{kl\omega}{6(ak^3+bl^3)}\pm m\sqrt{2kl}\operatorname{cn}^2(kx+ly-\omega t)\;,$$
$$v(x,y,t)=\frac{2(1-2m^2)k^2}{3}-\frac{k^2\omega}{6(ak^3+bl^3)}\pm\frac{km}{l}\sqrt{2kl}\operatorname{cn}^2(kx+ly-\omega t)\;,$$
$$w(x,y,t)=\frac{2(1-2m^2)l^2}{3}-\frac{l^2\omega}{6(ak^3+bl^3)}\pm\frac{lm}{k}\sqrt{2kl}\operatorname{cn}^2(kx+ly-\omega t)\;, \tag{20}$$

which are the exact periodic-wave solutions to Eqs. (1). Usually, they are known as the cnoidal wave solutions of the generalized NNV equation. In the limit case when $m\to 1$, then $\operatorname{cn}\xi\to\operatorname{sech}\xi$, and equation (20) becomes the solitary wave solution to Eqs. (1),

$$u(x,y,t)=\frac{-2kl}{3}-\frac{kl\omega}{6(ak^3+bl^3)}\pm\sqrt{2kl}\operatorname{sech}^2(kx+ly-\omega t)\,,$$
$$v(x,y,t)=\frac{-2k^2}{3}-\frac{k^2\omega}{6(ak^3+bl^3)}\pm\frac{k}{l}\sqrt{2kl}\operatorname{sech}^2(kx+ly-\omega t)\,,$$
$$w(x,y,t)=\frac{-2l^2}{3}-\frac{l^2\omega}{6(ak^3+bl^3)}\pm\frac{l}{k}\sqrt{2kl}\operatorname{sech}^2(kx+ly-\omega t)\,. \tag{21}$$

More exact periodic-wave solutions to Eqs. (1) may also be obtained in terms of other Jacobi elliptic functions in Table 1, which we omit here for simplicity.

4 Conclusions

Exact periodic-wave solutions to the generalized NNV equation are obtained by using the extended Jacobi elliptic-function method. It can be seen that the method has three obvious advantages.

(i) We may obtain multiple exact solutions to the equations under consideration in a unified way, and only some algebra is needed to find these solutions.

(ii) The Jacobi elliptic functions involved can be easily manipulated by the symbolic computation programs Mathematica or Maple, which allow us to perform complicated deducing and tedious algebraic calculation on a computer and output directly the required solutions.

(iii) The periodic-wave solutions for the equations we have studied can also be obtained by making appropriate linear superpositions of known periodic solutions.

This unusual procedure for generating solutions is successful as a consequence of some powerful, recently discovered, cyclic identities of the Jacobi elliptic functions. The interested reader is referred to Refs. [19] $\sim$ [21] for more details.

We also wish to mention that the method presented in Sec. 2 can be applied to other higher-dimensional nonlinear PDEs, too.

References

[1] L.P. Nizhnik, Sov. Phys. Dokl. **25** (1980) 706.

[2] A.P. Veselov and S.P. Novikov, Sov. Math. Dokl. **30** (1984) 588.

[3] S.P. Novikov and A.P. Veselov, Physica **D18** (1986) 267.

[4] M. Boiti, J.J-P. Leon, M. Manna, and F. Pempinelli, Inverse Problem **2** (1996) 116.

[5] Y. Tagami, Phys. Lett. **A141** (1989) 116.

[6] X.B. Hu and Y.S. Li, J. Phys. A: Math. Gen. **24** (1991) 1979.

[7] X.B. Hu, J. Phys. A: Math. Gen. **24** (1991) 1331.

[8] Y. Ohta, J. Phys. Soc. Jpn. **61** (1992) 3928.

[9] S.Y. Lou, Chin. Phys. Lett. **17** (2000) 781.

[10] J.F. Zhang and F.J. Chen, Commun. Theor. Phys. (Beijing, China) **38** (2002) 395.

[11] J.F. Zhang, Phys. Lett. **A313** (2003) 401.

[12] E.J. Parkes and B.R. Duffy, Phys. Lett. **A229** (1997) 217.

[13] B.R. Duffy and E.J. Parkes, Phys. Lett. **A214** (1996) 271.

[14] S.K. Liu, Z.T. Fu, S.D. Liu, and Q. Zhao, Phys. Lett. **A289** (2001) 69.

[15] Z.T. Fu, S.K. Liu, S.D. Liu, and Q. Zhao, Phys. Lett. **A290** (2001) 72.

[16] E.J. Parkes, B.R. Duffy, and P.C. Abbott, Phys. Lett. **A295** (2002) 280; Yan Zhen-Ya, Commun. Theor. Phys. (Beijing, China) **38** (2002) 143.

[17] F. Browman, *Introduction to Elliptic Functions with Application*, Universities, London (1959).

[18] V. Prasolov and Y. Solovyev, *Elliptic Functions and Elliptic Integrals*, American Mathematical Society, Providence (1997).

[19] F. Cooper, A. Khare, and U. Sukhatme, J. Phys. A: Math. Gen. **35** (2002) 10085.

[20] A. Khare and U. Sukhatme, J. Math. Phys. **43** (2002) 3798.

[21] A. Khare and U. Sukhatme, preprint math-ph/0204054.