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Three-dimensional spiral structure of tropical cyclone under four-force balance*

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The steady axis-symmetrical atmosphere dynamical equations are used for describing spiral structure of tropical cyclones under four-force (pressure gradient force, Coriolis force, centrifugal force, and friction force) balance, and the dynamical systems of three-dimensional (3D) velocity field are introduced. The qualitative analysis of the dynamical system shows that there are down 3D spiral structures in eye of tropical cyclone and tropical cyclone is 3D counterclockwise up spiral structure. These results are consistent with the observed tropical cyclone on the weather map.

Keywords: tropical cyclones, spiral structure, dynamical systems, four-force balance

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1. Introduction

Many scientists have studied the structure of tropical cyclones, either by numerical simulation or by modeling.[1–4] However, up to now, there has been no theoretical picture of three-dimensional (3D) structure for the tropical cyclone and its eye. Many models explain the structure of tropical cyclone in terms of gradient wind balance.[5] But, if the friction force is not considered, the spiral structure of the tropical cyclone cannot appear. So, the friction force must be added into the dynamical equations, which represent four-force (pressure gradient force, Coriolis force, centrifugal force, and friction force) balance. In addition, the Coriolis force is necessary to maintain low pressure system of the tropical cyclone, but it may be ignored for the eye of tropical cyclone because the scale of the eye is very small. Although the atmosphere dynamical equations are steady, we can find the dynamical system which represents the 3D velocity field of the tropical cyclone. The dynamical system uses 3D ordinary differential equations that can be easily analyzed qualitatively. From qualitative analysis, the 3D structure of the tropical cyclone and its eye can be obtained.

2. Four-force balance model of tropical cyclone

In the steady axis-symmetrical cylindrical coordinate (r,θ,z), the dynamical equations of atmospheric motions are as follows:[6,7]

$$-\frac{v_\theta^2}{r}-fv_\theta=-\frac{1}{\rho}\frac{\partial p}{\partial r}-kv_r, \tag{1}$$

$$-\frac{v_\theta v_r}{r}+fv_r=-kv_\theta, \tag{2}$$

$$\frac{1}{r}\frac{\partial}{\partial x}(rv_r)+\frac{\partial v_z}{\partial z}=0. \tag{3}$$

Equation (1) denotes four forces balance: the first and the second terms on the left-hand side are centrifugal force and Coriolis force, and the first and the second terms on the right-hand side are pressure gradient force and friction force, respectively. Equation (3) is a continuum equation. $k>0$ is friction factor and v_r, v_θ, v_z are radial, tangential, and vertical velocities, respectively.

The three velocity components can be written as[8]

$$v_r=\frac{\mathrm{d}r}{\mathrm{d}t}=\dot{r},\quad v_\theta=r\frac{\mathrm{d}\theta}{\mathrm{d}t}=r\dot{\theta},\quad v_z=\frac{\mathrm{d}z}{\mathrm{d}t}=\dot{z}. \tag{4}$$

The observation of tropical cyclone shows that it has a counterclockwise spiral behavior and horizontal convergence, so $v_\theta>0$, $v_r<0$. In Eq. (1), the terms on the left-hand side are both negative, the first term on the right-hand side (pressure gradient force) is also negative, and its absolute value is very large, it exceeds the positive second term on the right-hand side, so both sides of Eq. (1) are negative, this is reasonable for tropical cyclone. In continuum equation (3), because $v_r<0$, the updraft motion must occur, which is as it should be.

For the eye of the tropical cyclone, the sinking motion is observed. In Eq. (3) there is $v_r>0$, which implies horizontal divergence. The center of the eye has a very weak high-pressure, that is pressure gradient $\partial p/\partial r<0$. Although the first term on the right-hand side of Eq. (1) is positive, its value is quite small. So the value on the right-hand side of Eq. (1) is negative. Because the scale of eye is very small, the Coriolis force can be ignored. Then, the value on the left-hand side of Eq. (1) is also negative. This shows that equation (1) is also reasonable for the eye of tropical cyclone.

In summary, the steady dynamical equations (1), (2), and

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(3) are suitable to describe the behaviors of a tropical cyclone and its eye.

3. Pressure distribution of tropical cyclone

It is well know that the pressure gradient of tropical cyclone is very large, but it will be very small when approaching to the center of tropical cyclone. To obtain the velocity field of tropical cyclone, we assume the pressure distribution of tropical cyclone as follows:

$$p - p_* = \frac{f\rho}{2}\omega(r - r_*)^2, \tag{5}$$

$$\frac{\partial p}{\partial r} = f\rho\omega(r - r_*), \tag{6}$$

where ω is a positive constant, r_* is the radius of tropical cyclone eye, p is the pressure of eye wall. Figs. 1(a) and 1(b) show the plots of p versus r and p' versus r.

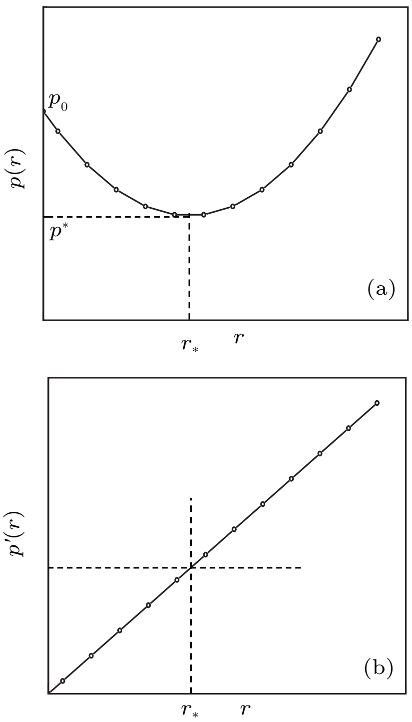

Fig. 1. Schematic representation of (a) the pressure p and (b) pressure gradient $\partial p/\partial r$ versus r.

From Eqs. (5) and (6) we see that this pressure distribution has the following characteristics:

(i) For the center of tropical cyclone $r = 0$, the pressure p_0 and pressure gradient $\partial p/\partial r$ are

$$p_0 - p_* = \frac{f\rho}{2}\omega r_*^2, \quad \frac{\partial p}{\partial r} = -f\rho\omega r_*. \tag{7}$$

This implies that the pressure p_0 of the center is larger more than the pressure p_* of eye wall and the pressure gradient $\partial p/\partial r < 0$. That is to say, the center of tropical cyclone has a quite small weak high pressure.

(ii) In eye wall of tropical cyclone $r = r_*$, the pressure is p_* and pressure gradient is 0.

(iii) In the outer region of tropical cyclone $r = r_* > 0$, the pressure p is larger than the pressure p_* of the eye wall and center pressure of the tropical cyclone, the pressure gradient is proportional to radial distance r, and it is very large.

These characteristics are much closer to the observed results of a tropical cyclone.

4. Dynamical system of velocity field for eye of tropical cyclone

For the eye of tropical cyclone, because pressure gradient is very small, the pressure gradient force may be ignored in four-force balance equation (1). At the same time, because of the smaller scale of the tropical cyclone eye, the Coriolis force may also be ignored in four-force balance equation (1). Then from Eq. (1) we have

$$\frac{v_\theta^2}{r} = kv_r. \tag{8}$$

From Eq. (8), the relation between v_θ and v_r is

$$v_\theta = -\sqrt{kv_r r}. \tag{9}$$

Substituting Eq. (8) into Eq. (2), where Coriolis force is ignored, we obtain

$$v_\theta^3 - k^2 r^2 v_\theta = 0. \tag{10}$$

Equation (10) is the 3rd power of tangential velocity v_θ, we remove a zero tangential velocity and another positive tangential velocity and obtain

$$v_\theta = -kr. \tag{11}$$

Substituting Eq. (11) into Eq. (8), we obtain the radial velocity as

$$v_r = kr. \tag{12}$$

Substituting Eq. (12) into Eq. (3) we obtain the vertical velocity as

$$v_z = -2kz. \tag{13}$$

Equations (11), (12), and (13) constitute the dynamical system of 3D velocity field[9] describing the tropical cyclone eye motion. The 3D ordinary differential equations are as follows:

$$\begin{cases} \dot{r} = v_r = kr, \\ r\dot{\theta} = v_\theta = -kr, \\ \dot{z} = v_z = -2kz, \end{cases} \tag{14}$$

where sign '·' represents the derivative with respect to time t.

By dividing the first expression of Eq. (14) by the third expression of Eq. (14), we obtain

$$\frac{\mathrm{d}r}{\mathrm{d}z} = -\frac{r}{2z}. \tag{15}$$

Equation (15) shows that the radius of the tropical cyclone eye decreases with the increase of z. The reason is that the lower level of the eye is of horizontal divergence, which is induced by the sinking motion, but the upper level of the eye must be of horizontal convergence of air.

Integrating Eq. (15) we have

$$r^2 z = \text{constant}. \tag{16}$$

Equation (16) is a hyperbola equation. It shows that the eye wall of tropical cyclone is of hyperbola, and it is shown in Fig. 2.

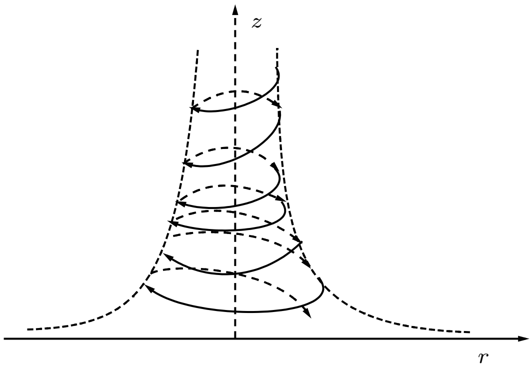

Fig. 2. Schematic representation of the eye wall curve for tropical cyclone.

Equations (14) can be converted into Cartesian coordinates (x, y, z), which are

$$\begin{cases} \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta = kx + ky, \\ \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta = -kx + ky, \\ \dot{z} = -2kz. \end{cases} \tag{17}$$

The singular point of Eq. (17) is

$$(x, y, z) = (0, 0, 0). \tag{18}$$

This is a no-wind point, which is the center point of the tropical cyclone eye at a surface level.

The Jacobian matrix of Eq. (17) is

$$J = \begin{pmatrix} k & k & 0 \\ -k & k & 0 \\ 0 & 0 & -2k \end{pmatrix}. \tag{19}$$

Its eigenvalues are

$$\lambda_{1,2} = k \pm \mathrm{i}k, \quad \lambda_3 = -2k. \tag{20}$$

This implies that the surface center of the tropical cyclone eye is a saddle-focus point.[10] That is to say, near the eye center, the horizontal motions spiral outward and the upper air sinks. This is shown in Fig. 3.

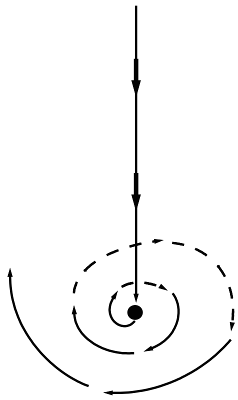

Fig. 3. Schematic representation of the air motion near eye of saddle-focus point.

This is the first time that the behavior of the tropical cyclone eye by two force (centrifugal force and friction force) balance has been explained. Obviously, the friction force is necessary to form the spiral structure of the tropical cyclone eye, although the motions of the eye are quite weak.

5. Dynamical system of perfect tropical cyclone

The scale of the eye is much smaller, while the scale of the perfect tropical cyclone is on the order of hundred kilometers, which are necessary for the four-force balance.

From four-force balance equation (1), we obtain

$$v_\theta^2 + frv_\theta - \frac{r}{\rho}\frac{\partial p}{\partial r} - krv_r = 0. \tag{21}$$

The solution of Eq. (21) is

$$v_\theta = \frac{1}{2}\left[-fr \pm \sqrt{f^2 r^2 + 4\left(\frac{r}{\rho}\frac{\partial p}{\partial r} + krv_r\right)}\right]. \tag{22}$$

In the northern hemisphere, the perfect tropical cyclone rotates counterclockwise and spirals inward, so the radial velocity is negative (i.e., $v_r < 0$), the tangential velocity is positive (i.e., $v_\theta > 0$), and the pressure gradient is positive (i.e., $\partial p/\partial r > 0$).

To be sure that tangential velocity v_θ is positive, we take the positive sign before root of solution (22), then we have

$$v_\theta = \frac{1}{2}\left[-fr + fr\sqrt{1 + \frac{4}{f^2 r^2}\left(\frac{r}{\rho}\frac{\partial p}{\partial r} - kr|v_r|\right)}\right]. \tag{23}$$

Because $v_r < 0$ and from Eq. (6) $\partial p/\partial r$ is proportional to r, the numerator is much less than denominator in fractional root. Equation (23) can be approximated to be

$$v_\theta = \frac{1}{f\rho}\frac{\partial p}{\partial r} - \frac{k}{f}|v_r|. \tag{24}$$

Equation (24) describes the relationship between tangential velocity v_θ and radial velocity v_r.

By substituting Eq. (24) into Eq. (2) the radial velocity v_r is derived as

$$v_r = -\frac{\dfrac{1}{\rho}\dfrac{\partial p}{\partial r} r}{\dfrac{1}{\rho k}\dfrac{\partial p}{\partial r} + \dfrac{f^2}{k} r + kr}. \tag{25}$$

By substituting Eq. (6) into Eq. (25), we obtain

$$v_r = -\frac{fk\omega(r-r_*)r}{fk\omega(r-r_*)+f^2r+k^2r}, \tag{26}$$

which can be simplified into

$$v_r = -a(r-r_*), \tag{27}$$

with

$$a = \frac{fk\omega}{fk\omega+f^2+k^2}. \tag{28}$$

By substituting Eq. (27) into Eq. (24) yields the tangential velocity v_θ, as follows:

$$v_\theta = b(r-r_*), \tag{29}$$

where

$$b = \omega - \frac{ak}{f} > 0. \tag{30}$$

Substituting Eq. (27) into Eq. (3) leads to the vertical velocity

$$v_z = 2az. \tag{31}$$

Equations (27), (29), and (31) describe the 3D velocity field for the perfect tropical cyclone, and are given as

$$\begin{cases} v_r = \dot{r} = -a(r-r_*), \\ v_\theta = r\dot{\theta} = b(r-r_*), \\ v_z = \dot{z} = 2az. \end{cases} \tag{32}$$

Obviously, the radius r_* of the eye wall is the singular point of horizontal velocity field, and $z=0$ is the singular point of vertical velocity. Let

$$x = (r-r_*)\cos\theta, \quad y = (r-r_*)\sin\theta. \tag{33}$$

Then equations can be transferred into Cartesian coordinates (x,y,z) form:

$$\begin{cases} \dot{x} = u = -ax-by, \\ \dot{y} = v = bx-ay, \\ \dot{z} = w = 2az. \end{cases} \tag{34}$$

The singular point of Eq. (34) is $(x,y,z)=(0,0,0)$, where $x=0$ and $y=0$ correspond to $r=r_*$ from Eq. (33).

The Jacobian matrix of Eq. (34) is

$$J = \begin{pmatrix} -a & -b & 0 \\ b & -a & 0 \\ 0 & 0 & 2a \end{pmatrix}. \tag{35}$$

Its eigenvalues are

$$\lambda_{1,2} = -a \pm \mathrm{i}b \ \text{ and } \ \lambda_3 = 2a. \tag{36}$$

It is shown that the singular point $(0,0,0)$ is also a saddle-focus point, as shown in Fig. 4.

From Fig. 4, we see that tropical cyclone has violent horizontal convergence and rotation, thereby inducing violent motion to the tropopause along the eye wall, the upper air then sinks to the surface.

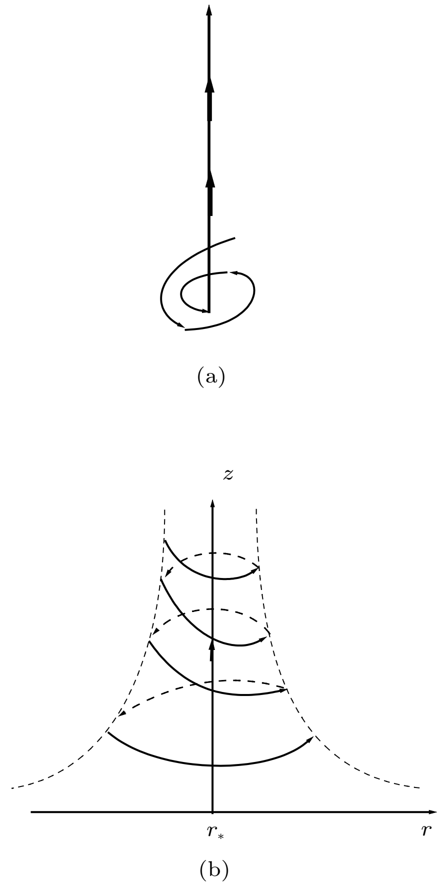

Fig. 4. Schematic representation of saddle focus point for tropical cyclone.

By dividing the first expression of Eq. (32) by the third expression of Eq. (32), we obtain

$$\frac{\mathrm{d}r}{\mathrm{d}z} = -\frac{r-r_*}{2z}. \tag{37}$$

Equation (37) shows that the outer behavior of tropical cyclone is also a hyperbola because r is much larger. It also shows that the radius of a tropical cyclone decreases very rapidly near the surface (see Fig. 4).

From velocity field equation (32) of tropical cyclone we can see the following points.

(i) In the four-force balance, centrifugal force and Coriolis force are necessary to maintain the rotation of a tropical cyclone. A huge pressure gradient is necessary for huge horizontal convergence, which can induce strong arising motions.

The friction force is necessary to form the spiral structure of tropical cyclone.

(ii) The eye center and eye wall are both singular points, near the eye there appears to be a sinking motion, and near the outer eye wall there is a strong arising motion.

(iii) Because of huge horizontal convergence in the surface, the tropopause is necessary for the horizontal divergence.

6. Conclusion

The 3D spiral structure of the eye and the perfect tropical cyclone can be derived by four-force balance relationship. The Coriolis force and centrifugal force are necessary for rotation motion of tropical cyclone, the friction force is necessary for forming spiral structure. The large pressure gradient force makes sure huge horizontal convergence.

The qualitative analysis of dynamical system for 3D velocity field is a key step. It is an easy, but very important aid to understand the trajectory of the atmosphere flow field. It is also a complement for the numerical simulation and diagnostic analysis of tropical cyclone and its effect.[11,12]

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