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Solitary Wave in Linear ODE with Variable Coefficients*

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Abstract *In this paper, the linear ordinary differential equations with variable coefficients are obtained from the controlling equations satisfied by wavelet transform or atmospheric internal gravity waves, and these linear equations can be further transformed into Weber equations. From Weber equations, the homoclinic orbit solutions can be derived, so the solitary wave solutions to linear equations with variable coefficients are obtained.*

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1 Introduction

Since the solitary wave solution in nonlinear KdV equation was found,[1−3] it has been generally shown that the solitary wave exists only in the conservative systems, and the solitary wave solutions appear to be a result of a balance between nonlinearity and dispersion or dissipation. But, for a given nonlinear partial differential equation (PDE), the assumption that $u(x,t) = u(\xi)$ (where $\xi = x - ct$) can reduce the nonlinear PDE to an ordinary differential equation (ODE). We[3−6] have proposed that the solitary wave solutions and wave front solutions in the PDE correspond to homoclinic and heteroclinic orbits in the ODE.[7] Hence, the solitary wave solutions exist also in nonlinear dissipative PDE. The solitary wave solutions in dissipative PDE are the result of balance between gain and loss of energy. So in order to form solitary waves in dissipative systems, there must be regions where energy is pumped from an external source, as well as regions where energy is dissipated outside the environment.

Mother wavelet in the wavelet transforms is also a solitary wave,[8,9] but mother wavelet satisfies just an ODE with a variable coefficients. Then here comes a question: Can solitary wave exist in the ODE with variable coefficients?

2 Wavelet Solitary Wave

The wavelet transform of a function $f(x)$ can be written as,[10−13]

$$T_g(a,b) = \frac{1}{a}\int_{-\infty}^{+\infty} f(x) g\Big(\frac{x-b}{a}\Big)\,\mathrm{d}x\,, \tag{1}$$

where the mother wavelet $g(x)$ takes the following form

$$g_1(x) = -\,\mathrm{e}^{-x^2/2}\,, \tag{2a}$$

$$g_2(x) = x\,\mathrm{e}^{-x^2/2}\,, \tag{2b}$$

$$g_3(x) = (1-x^2)\,\mathrm{e}^{-x^2/2}\,. \tag{2c}$$

Equation (2a) is a Gaussian function, and equations (2b) and (2c) are the first- and second-order derivatives of Gaussian function (2a), respectively. Equation (2c) is also called Mexican Cap wavelet.

Because the argument of the variable $g[(x-b)/a]$ is $(x-b)/a$, then $g[(x-b)/a]$ can be written as a rightward travelling wave, $g(\xi) = g(x - ct)$. From Eq. (2), it is obvious that $g(x) \to 0$, when $x \to \pm\infty$. Therefore the point of $g(x) = 0$ is a homoclinic point of homoclinic orbit, i.e., $g(\xi) \to 0$ as $\xi \to \pm\infty$, so $g(\xi)$ is a solitary wave.

3 Linear Weber Equation for Gaussian-Kind Wavelet

Assuming $g(\xi) = \mathrm{e}^{-\xi^2/2}$, the n-th order derivative of $g(\xi)$ satisfies the following linear ODE with variable coefficients,[13]

$$g^{(n+2)}(\xi) + \xi g^{(n+1)}(\xi) + (n+1) g^{(n)}(\xi) = 0\,, \tag{3}$$

where the super-primes of $g(\xi)$ denote the n-th order derivative.

The three wavelets (2) satisfy the following ODE with variable coefficients

$$\begin{aligned} &g_0''(\xi) + \xi g_0'(\xi) + g_0(\xi) = 0 \quad (n=0)\,,\\ &g_1''(\xi) + \xi g_1'(\xi) + 2g_1(\xi) = 0 \quad (n=1)\,,\\ &g_2''(\xi) + \xi g_2'(\xi) + 3g_2(\xi) = 0 \quad (n=2)\,, \end{aligned} \tag{4}$$

respectively.

Generally, they can be written as

$$g_n''(\xi) + \xi g_n'(\xi) + (n+1) g_n(\xi) = 0\,. \tag{5}$$

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Let

$$g_n(\xi) = \mathrm{e}^{-\xi^2/4} y(\xi)\,, \tag{6}$$

then

$$g_n'(\xi) = \mathrm{e}^{-\xi^2/4}\Big[y'(\xi) - \frac{\xi}{2}y(\xi)\Big], \tag{7}$$

$$g_n''(\xi) = \mathrm{e}^{-\xi^2/4}\Big[y''(\xi) - \xi y'(\xi) + \frac{\xi^2}{4}y(\xi) - \frac{1}{2}y(\xi)\Big]. \tag{8}$$

Substituting Eqs. (6) $\sim$ (8) into Eq. (5), we obtain

$$y''(\xi) + \frac{1}{2}\Big[(2n+1) - \frac{\xi^2}{2}\Big]y(\xi) = 0\,, \tag{9}$$

which is the well-known second kind of Weber equation.

Re-scaling the argument and taking $\eta = \xi/\sqrt{2}$, then equation (9) is reduced to

$$y''(\eta) + [(2n+1) - \eta^2]y(\eta) = 0\,, \tag{10}$$

which is the well-known first kind of Weber equation.

Equation (10) satisfying a bounded condition is an eigenvalue problem, with its eigenvalue being $(2n+1)$ and eigenfunction being

$$y(\eta) = \mathrm{e}^{-\eta^2/2} H_n(\eta)\,, \tag{11}$$

then the bounded solution of Eq. (9) is

$$y(\xi) = 2^{-n/2}\,\mathrm{e}^{-\xi^2/4} H_n\Big(\frac{\xi}{\sqrt{2}}\Big), \tag{12}$$

where H_n is Hermite polynomials. For $n = 0$, $n = 1$, and $n = 2$, Hermite polynomials are

$$H_0(\xi) = 1\,, \qquad H_1(\xi) = 2\xi\,,$$

$$H_2(\xi) = 2(2\xi^2 - 1)\,, \tag{13}$$

respectively.

Substituting Eq. (12) into Eq. (6), it is obvious that equation (6) is mother wavelet function (2) and takes the solitary form, see Fig. 1.

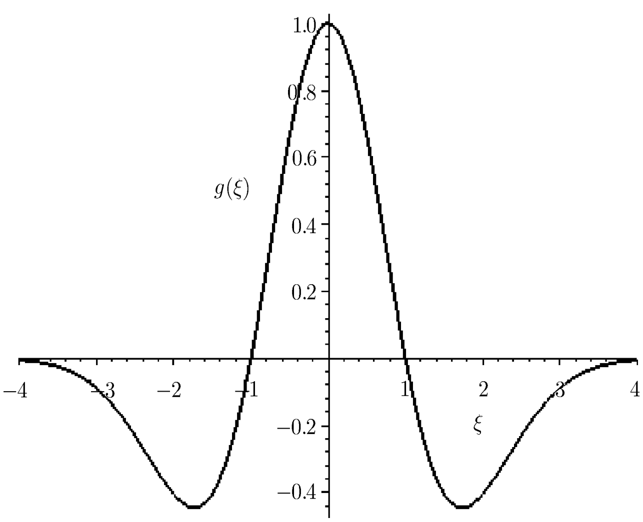

Fig. 1 The form of solitary waves for $g(\xi)$.

4 Homoclinic Orbit of Weber Equation

Let $y' = z$, the first kind of Weber equation (10) can be rewritten as

$$y' = z\,, \qquad z' = -[(2n+1) - \eta^2]y\,. \tag{14}$$

Taking $n = 2$, from Eq. (10) the initial condition is $y = -2$, $z = 0$, when $\eta = 0$. The orbit in phase plane (y, z) is shown in Fig. 2(a).

In Fig. 2(a), the point A is initial place, ABO is orbit when $\xi \to +\infty$, and OBA is orbit when $\xi \to -\infty$. The orbit in phase space (y, z, η) is shown in Fig. 2(b). From Fig. 2, we see that all orbits approach to point $(y = 0, z = 0)$, so the orbits are called homoclinic orbit.

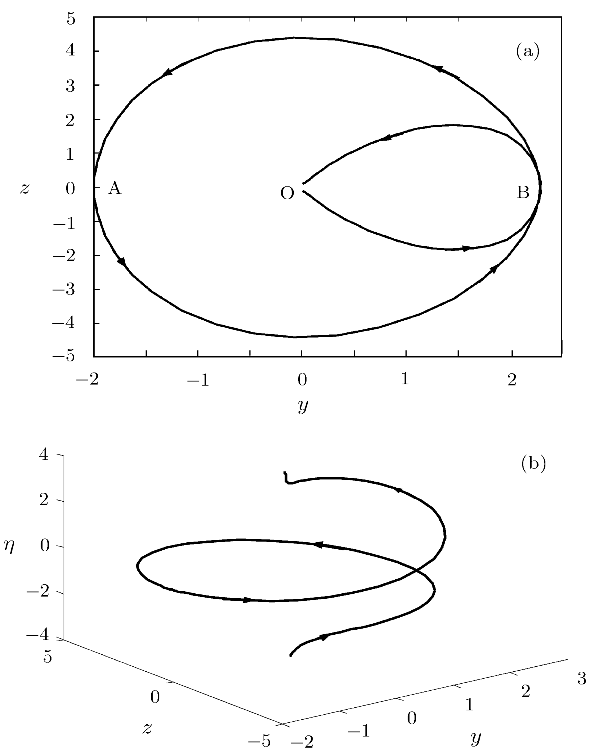

Fig. 2 The orbit in phase plane for Weber equation. (a) 2-D phase plane (y, z); (b) 3-D phase plane (y, z, η).

In fact, in Eq. (14) the point $(0, 0)$ is a saddle point when $|\eta| \to \infty$, equation (10) is approximated to

$$y''(\eta) - \eta^2 y(\eta) = 0\,. \tag{15}$$

Let $\zeta = \eta^2$, then one has

$$y' = 2\eta\frac{\mathrm{d}y}{\mathrm{d}\zeta}\,, \tag{16}$$

$$y'' = 4\zeta\frac{\mathrm{d}^2y}{\mathrm{d}\zeta^2} + 2\frac{\mathrm{d}y}{\mathrm{d}\zeta}\,. \tag{17}$$

So equation (15) is reduced to

$$4\zeta\frac{\mathrm{d}^2y}{\mathrm{d}\zeta^2} + 2\frac{\mathrm{d}y}{\mathrm{d}\zeta} - \zeta y = 0 \tag{18}$$

or

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\zeta^2}+\frac{1}{2\zeta}\frac{\mathrm{d}y}{\mathrm{d}\zeta}-\frac{1}{4}y=0\,. \tag{19}$$

When ζ is large enough (i.e., η is very large), one has

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\zeta^2}-\frac{1}{4}y=0\,. \tag{20}$$

From the physical view point, equation (20) is an ODE with negative restoring force, and obviously $(0,0)$ is a saddle point in phase plane (y,z).

From Eq. (10), we see that the negative restoring force in $y\to\pm\infty$ combining with positive restoring force $(2n+1)y$ makes the orbit depart from unstable manifold in saddle point, then rotate around η near zero due to positive restoring force, at last comes back to stable manifold in saddle point and this forms a homoclinic orbit. It is impossible for linear ODE with constant coefficients.

5 Internal Gravity Wave in Atmosphere

In the former sections, we have shown that there exists homoclinic orbit in the Weber equation, and Mexican Cap wavelet is just the solitary wave satisfying Weber equation. From the physical point of view, this is the result of balance between positive and negative restoring forces. Actually, the internal gravity wave in atmosphere is resulted from the atmospheric stratification, where the daily atmospheric boundary layer is often unstably stratified (correspond to negative restoring force) and the nocturnal atmospheric boundary layer is often stably stratified (correspond to positive restoring force).

When Boussinesq approximation is taken, the non-dimensional equations used to describe atmospheric internal gravity wave resulted from atmospheric stratification are

$$\begin{aligned}&\frac{\partial u}{\partial t}=-\frac{\partial p}{\partial x}\,, &&\frac{\partial w}{\partial t}=-\frac{\partial p}{\partial z}+\theta\,,\\ &\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0\,, &&\frac{\partial \theta}{\partial t}=-R_i w\,,\end{aligned} \tag{21}$$

where t is time and (u,w) are non-dimensional velocities in the plane (x,z); P,θ are non-dimensional pressure and potential temperature, respectively. R_i is Richardson number, and usually it varies with z.

From Eq. (21), elimination yields

$$\frac{\partial^2}{\partial t^2}\Big(\frac{\partial^2 w}{\partial z^2}\Big)+\frac{\partial^2}{\partial t^2}\Big(\frac{\partial^2 w}{\partial x^2}\Big)+R_i\frac{\partial^2 w}{\partial x^2}=0\,. \tag{22}$$

Assuming equation (22) to admit the following travelling wave solution

$$w=W(z)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,, \tag{23}$$

then substituting Eq. (23) into Eq. (22) leads to the equation satisfied by amplitude, i.e.,

$$\frac{\mathrm{d}^2 W}{\mathrm{d}z^2}+\Big(\frac{k^2}{\omega^2}R_i-k^2\Big)W=0\,. \tag{24}$$

If the variation of R_i with z is supposed to take the following form

$$R_i=a-z^2\quad(a>0)\,. \tag{25}$$

This indicates that R_i decreases with the increase of height.

Then equation (24) is rewritten as

$$\frac{\mathrm{d}^2 W}{\mathrm{d}z_1^2}+(\lambda-z_1^2)W=0 \tag{26}$$

with

$$\lambda=\frac{k}{\omega}a-\omega k\,,\qquad z_1=\sqrt{\frac{k}{\omega}}\,z\,. \tag{27}$$

Equation (26) is the first kind of Weber equation taking the form of Eq. (10). It has eigenvalue $\lambda=2n+1$ ($n=0,1,2,\ldots$) when $W\to 0$, λ ($z_1\to\pm\infty$), and the corresponding solution is

$$W(z_1)=\mathrm{e}^{-z_1^2/2}H_n(z_1)\quad(n=0,1,2,\ldots)\,, \tag{28}$$

so the amplitude of internal gravity $W(z_1)$ takes the form of solitary waves.

6 Conclusion

In this paper, from the characteristics of mother wavelet and the controlling equations of atmospheric internal gravity, the linear differential equations with variable coefficients are derived, and these linear differential equations can be transformed into well-known Weber equation. Then the homoclinic orbit solution to Weber equation is obtained, so the solitary wave solutions to the linear differential equations with variable coefficients are obtained.

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