Periodic solutions for a class of coupled nonlinear partial differential equations

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Abstract

In this Letter, by applying the Jacobi elliptic function expansion method, the periodic solutions for three coupled nonlinear partial differential equations are obtained.

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1. Introduction

Recently, Hu presented a new method finding exact traveling wave solutions of coupled nonlinear differential equations [1,2]. This new ansatz method, in which a simple rational polynomial relation is assumed to exist between dependent variables in the coupled differential equations, was successfully applied to obtain some new solutions to three kinds of coupled differential equations of mathematical physics. However, there only some soliton-like solutions were derived and some conditions are coarse. In this Letter, by using the Jacobi elliptic function expansion method [3–5], we obtain the periodic solutions for a class of coupled nonlinear partial differential equations, which play an important role in modern physics.

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2. Periodic solutions for coupled nonlinear plasma system

The coupled nonlinear plasma system \([6,7]\) reads

\[
\begin{align*}
    u_{xx} &= \alpha_1 u + \alpha_2 uv, \\
    v_{xx} &= \beta_1 v + \beta_2 v^2 + \beta_3 u^2,
\end{align*}
\]

when \(\beta_3 = 0\), this implies that \(v\) is independent on \(u\). Hu \([1,2]\) obtained some new soliton-like solutions to Eqs. (1).

By using the Jacobi elliptic function expansion method \([3–5]\), \(u\) and \(v\) can be expressed as

\[
\begin{align*}
    u &= a_0 + a_1 \text{sn} \xi + a_2 \text{sn}^2 \xi, \\
    v &= b_0 + b_1 \text{sn} \xi + b_2 \text{sn}^2 \xi,
\end{align*}
\]

where \(\xi = kx\), \(\text{sn} \xi\) is the Jacobi elliptic sine function \([8–11]\).

Substituting Eqs. (2) into Eqs. (1) leads to a set of algebraic equations for \(\text{sn}^i \xi\) \((i = 0, 1, 2, 3, 4)\), from which one has

\[
\begin{align*}
    a_1 &= b_1 = 0, \quad b_2 = \frac{6m^2k^2}{\alpha_1}, \quad a_2 = \pm \sqrt{\frac{\alpha_2 - \beta_2}{\beta_3}b_2}, \\
    a_0 &= \mp \sqrt{\frac{\alpha_2 - \beta_2}{\beta_3}b_2} \left[ 2(1 + m^2)k^2 + \frac{2\alpha_2 \beta_1 - 2\alpha_1 \beta_2}{2\alpha_2(\alpha_2 - 2\beta_2)} \right], \\
    b_0 &= -\frac{2(1 + m^2)k^2}{\alpha_2} - \frac{2\alpha_1 \alpha_2 - 2\alpha_1 \beta_2 - \alpha_2 \beta_1}{2\alpha_2(\alpha_2 - 2\beta_2)}, \\
    k^2 &= \frac{(\alpha_1 + \alpha_2 b_0)a_0}{2a_2} = \frac{(\beta_1 + \beta_2 b_0)b_0 + \beta_3 a_0^2}{2b_2},
\end{align*}
\]

with \(m\) \((0 < m < 1)\) is the modulus.

So, the periodic solutions to coupled nonlinear plasma system (1) are

\[
\begin{align*}
    u &= a_0 + a_2 \text{sn}^2 \xi = (a_0 + a_2) - a_2 \text{cn}^2 \xi = a_0 + \frac{a_2}{m^2} - \frac{a_2}{m^2} \text{dn}^2 \xi, \\
    v &= b_0 + b_2 \text{sn}^2 \xi = (b_0 + b_2) - b_2 \text{cn}^2 \xi = b_0 + \frac{b_2}{m^2} - \frac{b_2}{m^2} \text{dn}^2 \xi,
\end{align*}
\]

where \(\text{cn} \xi\) and \(\text{dn} \xi\) are the Jacobi elliptic cosine function and Jacobi elliptic function of the third kind \([8–11]\).

When \(m \rightarrow 1\), Eqs. (4) reduce to the following solitary wave solutions

\[
\begin{align*}
    u &= (a_0 + a_2) - a_2 \text{sech}^2 \xi, \quad v = (b_0 + b_2) - b_2 \text{sech}^2 \xi.
\end{align*}
\]

3. Periodic solutions for coupled physical system

The coupled physical system \([12,13]\) reads

\[
\begin{align*}
    u_{xx} &= \alpha_1 u + \alpha_2 u^3 + \alpha_3 uv, \\
    v_{xx} &= \beta_1 v + \beta_2 v^3 + \beta_3 (u^2 - 1).
\end{align*}
\]

Similarly, we seek the periodic solutions of Eqs. (6) in the form

\[
\begin{align*}
    u &= a_0 + a_1 \text{sn} \xi, \quad v = b_0 + b_1 \text{sn} \xi, \quad \xi = kx.
\end{align*}
\]
Substituting (7) into Eqs. (6) leads to the following results

\[ a_0^2 = \frac{\alpha_3 (\beta_1 - \beta_3) - \alpha_1 \beta_2}{\alpha_2 \beta_2 - \alpha_3 \beta_3}, \quad b_0^2 = \frac{\alpha_1 \beta_3 - \alpha_2 (\beta_1 - \beta_3)}{\alpha_2 \beta_2 - \alpha_3 \beta_3}, \]

\[ b_1^2 = \frac{2k^2 m^2 (\alpha_2 - \beta_3)}{\alpha_2 \beta_2 - \alpha_3 \beta_3}, \quad a_1^2 = \frac{2k^2 m^2 (\beta_2 - \alpha_3)}{\alpha_2 \beta_2 - \alpha_3 \beta_3}, \]

\[ k^2 = -\left(\frac{\alpha_1 + 3\alpha_2 a_0^2 + \alpha_3 b_0^2}{1 + m^2}a_1 + 2\alpha_3 a_0 b_0 b_1\right) = -\left(\frac{\beta_1 - \beta_3 + 3\beta_2 b_0 + \beta_3 a_0^2}{1 + m^2}b_1 + 2\beta_3 a_0 b_0 a_1\right), \]

(8)

When \( m \to 1 \), (7) reduces to

\[ u = u_0 + a_1 \tanh \xi, \quad v = v_0 + b_1 \tanh \xi, \quad \xi = kx. \]

(9)

Similar to (7), we have

\[ u = u_0 + a_1 \cosh \xi, \quad v = v_0 + b_1 \cosh \xi, \quad \xi = kx, \]

(10)

with \( a_0 \) and \( b_0 \) same as (8), but

\[ b_1^2 = -\frac{2k^2 m^2 (\alpha_2 - \beta_3)}{\alpha_2 \beta_2 - \alpha_3 \beta_3}, \quad a_1^2 = -\frac{2k^2 m^2 (\beta_2 - \alpha_3)}{\alpha_2 \beta_2 - \alpha_3 \beta_3}, \]

\[ k^2 = -\left(\frac{\alpha_1 + 3\alpha_2 a_0^2 + \alpha_3 b_0^2}{2m^2 - 1}a_1 + 2\alpha_3 a_0 b_0 b_1\right) = -\left(\frac{\beta_1 - \beta_3 + 3\beta_2 b_0 + \beta_3 a_0^2}{2m^2 - 1}b_1 + 2\beta_3 a_0 b_0 a_1\right), \]

(11)

and

\[ u = u_0 + a_1 \sinh \xi, \quad v = v_0 + b_1 \sinh \xi, \quad \xi = kx, \]

(12)

with \( a_0 \) and \( b_0 \) same as (8), but

\[ b_1^2 = -\frac{2k^2 (\alpha_2 - \beta_3)}{\alpha_2 \beta_2 - \alpha_3 \beta_3}, \quad a_1^2 = -\frac{2k^2 (\beta_2 - \alpha_3)}{\alpha_2 \beta_2 - \alpha_3 \beta_3}, \]

\[ k^2 = -\left(\frac{\alpha_1 + 3\alpha_2 a_0^2 + \alpha_3 b_0^2}{2 - m^2}a_1 + 2\alpha_3 a_0 b_0 b_1\right) = -\left(\frac{\beta_1 - \beta_3 + 3\beta_2 b_0 + \beta_3 a_0^2}{2 - m^2}b_1 + 2\beta_3 a_0 b_0 a_1\right), \]

(13)

When \( m \to 1 \), (10) and (12) reduce to

\[ u = u_0 + a_1 \sech \xi, \quad v = v_0 + b_1 \sech \xi, \quad \xi = kx. \]

(14)

4. Periodic solutions for generalized DS equations

The generalized Drinfeld–Sokolov (DS for short) equations [14] can be written as

\[ u_t + \alpha_1 uu_x + \beta_1 u_{xxx} + \gamma (u^3)_x = 0, \quad (15a) \]

\[ v_t + \alpha_2 uu_x + \beta_2 u_{xxx} = 0. \quad (15b) \]

We seek the traveling wave solutions of Eqs. (15) in the form

\[ u = u(\xi), \quad v = v(\xi), \quad \xi = k(x - ct), \]

(16)

where \( k \) and \( c \) are wave number and wave speed, respectively.
is introduced, then Eqs. (17) can be rewritten as

\[ -c \frac{du}{d\xi} + a_1 u \frac{du}{d\xi} + \beta_1 k^2 d^3 u \frac{d^3 u}{d\xi^3} + \gamma \frac{d^2 u}{d\xi^2} = 0, \]  
\[ -c \frac{dv}{d\xi} + a_2 u \frac{dv}{d\xi} + \beta_2 k^2 d^3 v \frac{d^3 v}{d\xi^3} = 0. \]  

In order to solve Eqs. (17), the following transformation

\[ w = u^{1/2}, \]  

is introduced, then Eqs. (17) can be rewritten as

\[ -c \frac{du}{d\xi} + a_1 u \frac{du}{d\xi} + \beta_1 k^2 d^3 u \frac{d^3 u}{d\xi^3} + 2\gamma w \frac{dw}{d\xi} = 0, \]  
\[ -cw \frac{dw}{d\xi} + a_2 u w \frac{dw}{d\xi} + \beta_2 k^2 \left[ \left( \frac{2}{\delta} - 1 \right) \left( \frac{2}{\delta} - 2 \right) \left( \frac{d\xi}{d\xi} \right)^3 + 3 \left( \frac{2}{\delta} - 1 \right) \frac{d\xi}{d\xi} \frac{d^2 w}{d\xi^2} + w \frac{d^3 w}{d\xi^3} \right] = 0. \]  

Similarly, the formal solutions can be written as

\[ u = a_0 + a_1 \text{sn} \xi + a_2 \text{sn}^2 \xi, \]  
\[ v = b_0 + b_1 \text{sn} \xi + b_2 \text{sn}^2 \xi. \]  

Substituting (20) into (19), we have

\[ a_1 = b_1 = 0, \quad a_2 = -\frac{2(4 + \delta)(2 + \delta)m^2 \beta_2 k^2}{a_2 \delta^2}, \]  
\[ b_2 = \pm \frac{2m^2 k^2}{a_2 \delta^2} \sqrt{\frac{\beta_2 (4 + \delta)(2 + \delta)}{\gamma} \left[ 6\delta^2 \alpha_2 \beta_1 - (4 + \delta)(2 + \delta) \alpha_1 \beta_2 \right]}, \]  

and

\[ [-c + a_2 a_0 - 4(1 + m^2) \beta_2 k^2] b_0 + 6 \left( \frac{2}{\delta} - 1 \right) \beta_2 k^2 b_2 = 0, \]  
\[ 2 \left[ a_2 a_2 + 3m^2 \left( \frac{6}{\delta} + 1 \right) \beta_2 k^2 \right] b_0 + \left[ -c + a_2 a_0 - \frac{16}{\delta^2} (1 + m^2) \beta_2 k^2 \right] b_2 = 0, \]  
\[ [-c + a_2 a_0 - 4(1 + m^2) \beta_1 k^2] a_2 + 2\gamma b_0 b_2 = 0, \]  
\[ 2 \left[ [-c + a_2 a_0 - 4 \left( \frac{6}{\delta} - 1 \right)(1 + m^2) \beta_2 k^2 \right] b_0 b_2 + 2 \left( \frac{2}{\delta} - 1 \right) \left( \frac{4}{\delta} - 1 \right) \beta_2 k^2 b_2^2 + (a_2 a_2 + 12m^2 \beta_2 k^2) b_0^2 = 0. \]  

From which \( a_0, b_0, k \) and \( c \) can be determined.

Thus, the periodic solution to the generalized DS equations are

\[ u = a_0 + a_2 \text{sn}^2 \xi = (a_0 + a_2) - a_2 \text{cn}^2 \xi = a_0 + \frac{a_2}{m^2} - \frac{a_2}{m^2} \text{dn}^2 \xi, \]  
\[ v = w^{1/\delta} = (b_0 + b_2 \text{sn}^2 \xi)^{1/\delta} = \left[ (b_0 + b_2) - b_2 \text{cn}^2 \xi \right]^{1/\delta} = \left[ b_0 + \frac{b_2}{m^2} - \frac{b_2}{m^2} \text{dn}^2 \xi \right]^{1/\delta}. \]  

When \( m \to 1 \), Eqs. (23) reduce to

\[ u = (a_0 + a_2) - a_2 \text{sech}^2 \xi, \quad v = \left[ (b_0 + b_2) - b_2 \text{sech}^2 \xi \right]^{2/\delta}. \]
5. Conclusion

In this Letter, we apply the Jacobi elliptic function expansion to solve three coupled nonlinear systems, there many periodic wave solutions and shock wave or solitary wave solutions are derived. These solutions are helpful in understanding the problems in modern physics.

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