Commun. Theor. Phys. (Beijing, China) **51** (2009) pp. 395–396

Vol. 51, No. 3, March 15, 2009

Exact Solutions to Short Pulse Equation*

FU Zun-Tao,† ZHENG Ming-Hua, and LIU Shi-Kuo

School of Physics & State Key Laboratory for Turbulence and Complex Systems, Peking University, Beijing 100871, China

(Received June 6, 2008)

Abstract *In this paper, dependent and independent variable transformations are introduced to solve the short pulse equation. It is shown that different kinds of solutions can be obtained to the short pulse equation.*

PACS numbers: 03.65.Ge
Key words: Jacobian elliptic function, the short pulse equation, transformation

1 Introduction

The short pulse equation (SPE for short)

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}, \tag{1}$$

was first introduced by Schäfer and Wayne[1] as a model equation to describe the propagation of ultra-short light pulses in silica optical fibres. Different from the celebrated nonlinear Schrödinger equation (NLSE for short), which is used to model the evolution of slowly varying wave trains, the SPE is proposed to describe the pulse whose spectrum is not narrowly localized around the carrier frequency. It has been proven that as the pulse length shortens, the NLSE approximation describing the optical pulses becomes steadily less accurate, while the SPE provides a better and better approximation.[2]

Contrary to the well studied NLSE, we know little about SPE. It has been proven that the SPE is an integrable equation possessing a Lax pair[3] of the Wadati–Konno–Ichikawa type[4] and the bi-Hamiltonain structure.[5] Usually, Eq. (1) is difficult to solve, some transformations have to be introduced. For example, Parkes[6] introduced a new dependent variable z

$$z = \frac{u-v}{|v|}, \tag{2}$$

and assumed that z is an implicit or explicit function of η, where

$$\eta = x - vt - x_0, \tag{3}$$

v and x_0 are arbitrary constants and $v \neq 0$. Through above transformations, he obtained periodic-hump solution, solitary loop solution and periodic loop solution, "figure-eight" solution and other type solution to Eq. (1).

The transformation between the SPE and the sine-Gordon equation was discovered in Ref. [3] and the derivation of this transformation was considerably simplified in Ref. [7], and later it was used in Ref. [8] to obtain exact loop and pulse solutions of the SPE from the well-known kink and breather solutions of the sine-Gordon equation. The recursion operator found in Ref. [3] was used to study the N-loop soliton solutions to SPE.[9]

The solutions found in above mentioned references have been shown to result from a delicate nonlocal balance between dispersion and nonlinearity, and their stable propagation is confirmed by numerical simulations.[10] Since SPE is a current research interest in nonlinear optical fibres theory, in this paper, based on the newly introduced transformations, we will show systematical results for the SPE (1) by using the knowledge of elliptic equation and Jacobian elliptic functions.[11−15]

2 Transformed Equation (1)

In order to solve the SPE, certain dependent or independent variable transformations must be introduced. Starting from Eq. (1), we define

$$x = w(y,\tau), \qquad t = \tau, \tag{4}$$

then we have

$$\frac{\partial}{\partial x} = \frac{1}{w_y}\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - \frac{w_\tau}{w_y}\frac{\partial}{\partial y}. \tag{5}$$

Substituting this transformation into Eq. (1) yields

$$[w_y^3 + w_y u_y^2]u = w_y^2 u_{\tau y} - w_\tau w_y u_{yy} - (w_{\tau y} w_y - w_\tau w_{yy})u_y - \frac{1}{2}w_y u^2 u_{yy} + \frac{1}{2}w_{yy} u^2 u_y. \tag{6}$$

If we set

$$u(x,t) = R(y,\tau), \tag{7}$$

then from Eq. (6) we have

$$[w_y^3 + w_y R_y^2]R = w_y^2 R_{\tau y} - w_\tau w_y R_{yy} - (w_{\tau y} w_y - w_\tau w_{yy}) \times R_y - \frac{1}{2}w_y R^2 R_{yy} + \frac{1}{2}w_{yy} R^2 R_y. \tag{8}$$

It is obvious that the key step to solve the Eq. (8) is to build the bridge between $w(y,\tau)$ and $R(y,\tau)$, in Ref. [3], the relation between $w(y,\tau)$ and $R(y,\tau)$ is

$$w_\tau = -\frac{1}{2}R^2 = -\frac{1}{2}X_\tau^2, \tag{9}$$

from which one can derive the well-known sine-Gordon equation

$$X_{\tau y} = \sin X(y,\tau). \tag{10}$$

Since the solutions to the sine-Gordon equation has been well studied, we can apply the above relation to derive the solutions to SPE easily. Of course, there still exist more relations between $w(y,\tau)$ and $R(y,\tau)$.

If we choose

$$R(y,\tau) = w_\tau(y,\tau), \tag{11}$$

for SPE, we have

$$w_y^3 w_\tau + w_y w_\tau w_{\tau y}^2 = w_y^2 w_{\tau\tau y} - w_\tau w_y w_{\tau yy} - w_y w_{\tau y}^2 + w_\tau w_{\tau y} w_{yy} - \frac{1}{2}w_y w_\tau^2 w_{\tau yy}$$

*The project supported by National Natural Science Foundation of China under Grant Nos. 40775040 and 90511009
†Corresponding author, E-mail: fuzt@pku.edu.cn

$$+\frac{1}{2}w_\tau^2 w_{yy}w_{\tau y}. \tag{12}$$

If we choose

$$R(y,\tau)=w_y(y,\tau), \tag{13}$$

for SPE, we have

$$w_y^4+w_y^2w_{yy}^2=w_y^2w_{\tau yy}-w_\tau w_y w_{yyy}-w_y w_{\tau y}w_{yy} + w_\tau w_{yy}^2-\frac{1}{2}w_y^3w_{yyy}+\frac{1}{2}w_y^2w_{yy}^2. \tag{14}$$

Above transformed equations can be easily solved to derive its traveling wave solutions, we will show this in the next section.

3 Exact Traveling Wave Solutions to SPE

In this section, we will try ro find the traveling wave solutions to transformed SPE (12) and (14). In doing so, we first take the transformation in the following frame

$$\xi=k(y-c\tau), \tag{15}$$

where k is wave number and c is wave speed.

Combining Eq. (15) with Eq. (12) leads to

$$w_\xi^2+\frac{1}{2}k^2c^2w_{\xi\xi}+\frac{1}{2}k^2c^2w_\xi w_{\xi\xi\xi}=0. \tag{16}$$

Combining Eq. (15) with Eq. (14) leads to

$$w_\xi^2+\frac{1}{2}k^2w_{\xi\xi}+\frac{1}{2}k^2w_\xi w_{\xi\xi\xi}=0. \tag{17}$$

It is obvious that Eq. (16) and Eq. (17) can be exchanged by a simple transformation $k^2c^2\leftrightarrow k'^2$, if we find solution to Eq. (16), then we can apply this transformation to find the solution to Eq. (17), vice versa.

In fact, Eq. (16) or Eq. (17) is still not easily to solved by the usual function expansion methods, such as Jacobi elliptic function expansion methods,[11,12] for the expansion rank of Eq. (16) is zero.

In order to solve Eq. (16) exactly, we set

$$V=\frac{w_{\xi\xi}}{w_\xi}, \tag{18}$$

then Eq. (16) can be rewritten as

$$V_\xi=-\frac{2}{k^2c^2}(k^2c^2V^2+1), \tag{19}$$

which can be integrated to derive

$$V=-\frac{1}{kc}\tan\Big[\frac{2}{kc}(\xi-\xi_0)\Big], \tag{20}$$

where ξ_0 is an integration constant.

Substituting Eq. (20) back to Eq. (18) yields

$$\frac{w_{\xi\xi}}{w_\xi}=-\frac{1}{kc}\tan\Big[\frac{2}{kc}(\xi-\xi_0)\Big], \tag{21}$$

from which we have

$$\ln|w_\xi|=\frac{1}{2}\ln\Big|\cos\Big[\frac{2}{kc}(\xi-\xi_0)\Big]\Big|+C_0, \tag{22}$$

where C_0 is an integration constant.

Equation (22) can be rewritten as

$$w_\xi=C'\sqrt{\Big|\cos\Big[\frac{2}{kc}(\xi-\xi_0)\Big]\Big|}, \tag{23}$$

where C' is a constant related to integration constant C_0.

So the solution to the SPE (1) is

$$u(x,t)=-kcC'\sqrt{\Big|\cos\Big[\frac{2}{kc}(\xi-\xi_0)\Big]\Big|}. \tag{24}$$

From Eq. (23), we have

$$x=w=\frac{C'}{\sqrt{2}kc}F\Big\{\frac{\sqrt{2}}{2},\arcsin\Big[\sqrt{2}\sin\Big(-\frac{\xi-\xi_0}{kc}\Big)\Big]\Big\} -\frac{\sqrt{2}C'}{kc}E\Big\{\frac{\sqrt{2}}{2},\arcsin\Big[\sqrt{2}\sin\Big(-\frac{\xi-\xi_0}{kc}\Big)\Big]\Big\}, \tag{25}$$

with

$$F(m,\varphi)=\int_0^\varphi\frac{\mathrm{d}\theta}{\sqrt{1-m^2\sin^2\theta}}, \tag{26}$$

called the normal elliptic integral of the first kind, and

$$E(m,\varphi)=\int_0^\varphi\sqrt{1-m^2\sin^2\theta}\,\mathrm{d}\theta, \tag{27}$$

called the normal elliptic integral of the second kind,[16] where $0\le m\le 1$ is called modulus of Jacobi elliptic functions.[13−15]

Obviously, the solution of SPE (24) and (25) has not been reported in the literature.

4 Conclusion

In this paper, we presented the process to find exact solutions for the SPE and obtained some new types of solutions, these solutions may be applied to describe and/or explain some phenomena found in the nonlinear optical fibres, since the model has been proposed to model short optical pulse. Due to the solutions presented in this paper are just some special solutions, more methods are still needed to find more types of solutions to SPE.

References

[1] T. Schafer and C.E. Wayne, Physica D **196** (2004) 105.
[2] Y. Chung, C.K.R.T. Jones, T. Schafer, and C.E. Wayne, Nonlinearity **18** (2005) 1351.
[3] A. Sakovich and S. Sakovich, J. Phys. Soc. Jpn. **74** (2005) 239.
[4] K.K. Victor, B.B. Thomas, and T.C. Kofane, J. Phys. A **39** (2007) 5585.
[5] J.C. Brunelli, Phys. Lett. A **353** (2006) 475.
[6] E.J. Parkes, Chaos, Solitons & Fractals **38** (2008) 154.
[7] A. Sakovich and S. Sakovich, SIGMA **3** (2007) 086.
[8] A. Sakovich and S. Sakovich, J. Phys. A **39** (2006) L361.
[9] V.K. Kuetche, T.B. Bouetou, and T.C. Kofane, Phys. Lett. A **372** (2008) 665.
[10] Y. Chung and T. Schafer, Phys. Lett. A **361** (2007) 63.
[11] Z.T. Fu, S.K. Liu, S.D. Liu, and Q. Zhao, Phys. Lett. A **290** (2001) 72.
[12] S.K. Liu, Z.T. Fu, S.D. Liu, and Q. Zhao, Phys. Lett. A **289** (2001) 69.
[13] S.K. Liu and S.D. Liu, *Nonlinear Equations in Physics*, Peking University Press, Beijing (2000) (in Chinese).
[14] Z.X. Wang and D.R. Guo, *Special Functions*, World Scientific, Singapore (1989).
[15] P.F. Byrd and M.D. Friedman, *Handbook of Elliptic Integrals for Engineers and Physics*, Springer-Verlag, Berlin (1954).
[16] Y.M. Jin, eds., *Applied Table for Integral*, University of Science and Technology of China Press, Hefei (1988) (in Chinese).