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# New solutions to mKdV equation

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**Abstract**

A transformation is introduced on the basis of the projective Riccati equations, and it is applied as an intermediate in expansion method to solve mKdV equation. Many kinds of travelling wave solutions including solitary wave solution are obtained, in which some are found for the first time.


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## 1. Introduction

A number of problems are described in terms of suitable nonlinear models, such as nonlinear Schrödinger equations in plasma physics [1], KdV equation in shallow water model [2] and so on, in branches of physics, mathematics and other interdisciplinary sciences. Recently, special attention has been devoted in literature for solving nonlinear evolution equations, and many methods have been proposed to construct exact solutions to nonlinear equations, such as the homogeneous balance method [3,4], the nonlinear transformation method [5,6], the trial function method [7,8] and so on. Among them the function transformation method [9,10], the hyperbolic function expansion method [11,12], the Jacobi elliptic function expansion method [13,14] and the sine–cosine method [15] can be taken as expansion methods, in which some basic functions or transformations from some famous equation(s) are needed. For example, the basic transformation in the function transformation method [10] is obtained from sine-Gordon equation [16], the bases in the hyperbolic function expansion method [11,12] are hyperbolic functions, the bases in the Jacobi elliptic function expansion method [13,14] are the Jacobi elliptic functions [16–19] and the bases in the sine–cosine method [15] are sine and cosine functions.

In this Letter, we will reconsider this case. A transformation is obtained from the well-known projective Riccati equations [20–22], and then this transformation is taken as an intermediate to solve mKdV equation. Many kinds

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of travelling wave solutions including solitary wave solutions are derived, among them some are found for the first time.

## 2. Analysis on the projective Riccati equations

The well-known projective Riccati equations [20–22] read

$$f'(\xi) = pf(\xi)g(\xi), \tag{1a}$$

$$g'(\xi) = q + pg^2(\xi) - rf(\xi), \tag{1b}$$

where $p \neq 0$ is a real constant, $q$ and $r$ are two real constants. When $p = -1$ and $q = 1$, Eqs. (1) reduce to the coupled equations given in the references [20,21], and when $p = \pm 1$ and $q \geqslant 0$, Eqs. (1) reduce to the coupled equations given in the Ref. [22].

Next, we will analyze the solutions to Eqs. (1). From Eq. (1a), one has

$$g = \frac{1}{p}\frac{f'}{f}. \tag{2}$$

Substituting Eq. (2) into Eq. (1b) leads to

$$f''f - 2f'^2 - pqf^2 + prf^3 = 0. \tag{3}$$

In order to solve Eq. (3), we introduce the following transformation

$$f = \frac{1}{w}, \tag{4}$$

then

$$\frac{f'}{f} = -\frac{w'}{w}, \qquad g = -\frac{1}{p}\frac{w'}{w} \tag{5}$$

and

$$w'' + pqw - pr = 0. \tag{6}$$

For the solutions to Eq. (6), two basic cases need to be considered. The first basic case is

**Case A**: $q \neq 0$
There are still two cases need to be considered. The first one is

**Case A1**: $pq < 0$
Then we can assume $k^2 = -pq$, the Eq. (6) can be rewritten as

$$w'' - k^2 w - pr = 0, \tag{7}$$

and the general solution to Eq. (7) is

$$w = a_0 + a_1 \sinh k\xi + a_2 \cosh k\xi, \tag{8}$$

where $a_0 = r/q$, i.e.,

$$w = \frac{r}{q} + a_1 \sinh(\sqrt{-pq}\,\xi) + a_2 \cosh(\sqrt{-pq}\,\xi). \tag{9}$$

Considering the relation in Eq. (5), here we select two special solutions from Eq. (9). The first one is

$$w = \frac{r}{q} + \frac{1}{q}\sinh(\sqrt{-pq}\,\xi), \tag{10}$$

then

$$f_1 = \frac{1}{w} = \frac{q}{r + \sinh(\sqrt{-pq}\,\xi)} \tag{11}$$

and

$$g_1 = -\frac{1}{p}\frac{w'}{w} = -\frac{1}{p}\frac{\sqrt{-pq}\cosh(\sqrt{-pq}\,\xi)}{r + \sinh(\sqrt{-pq}\,\xi)}. \tag{12}$$

From Eqs. (11) and (12) one can derive the relation between $f(\xi)$ and $g(\xi)$

$$g_1^2 = -\frac{1}{p}\left[q - 2rf_1 + \frac{r^2+1}{q}f_1^2\right]. \tag{13}$$

The second one is

$$w = \frac{r}{q} + \frac{1}{q}\cosh(\sqrt{-pq}\,\xi), \tag{14}$$

then

$$f_2 = \frac{1}{w} = \frac{q}{r + \cosh(\sqrt{-pq}\,\xi)} \tag{15}$$

and

$$g_2 = -\frac{1}{p}\frac{w'}{w} = -\frac{1}{p}\frac{\sqrt{-pq}\sinh(\sqrt{-pq}\,\xi)}{r + \cosh(\sqrt{-pq}\,\xi)}. \tag{16}$$

From Eqs. (15) and (16) one can derive the relation between $f(\xi)$ and $g(\xi)$

$$g_2^2 = -\frac{1}{p}\left[q - 2rf_2 + \frac{r^2-1}{q}f_2^2\right]. \tag{17}$$

**Case A2**: $pq > 0$

Then we can assume $k^2 = pq$, the Eq. (6) can be rewritten as

$$w'' + k^2 w - pr = 0, \tag{18}$$

and the general solution to Eq. (18) is

$$w = a_0 + a_1\sin k\xi + a_2\cos k\xi, \tag{19}$$

where $a_0 = r/q$, i.e.,

$$w = \frac{r}{q} + a_1\sin(\sqrt{pq}\,\xi) + a_2\cos(\sqrt{pq}\,\xi). \tag{20}$$

Considering the relation in Eq. (5), here we also select two special solutions from Eq. (20). The first one is

$$w = \frac{r}{q} + \frac{1}{q}\sin(\sqrt{pq}\,\xi), \tag{21}$$

then

$$f_3 = \frac{1}{w} = \frac{q}{r + \sin(\sqrt{pq}\xi)} \tag{22}$$

and

$$g_3 = -\frac{1}{p}\frac{w'}{w} = -\frac{1}{p}\frac{\sqrt{pq}\cos(\sqrt{pq}\xi)}{r + \sin(\sqrt{pq}\xi)}. \tag{23}$$

From Eqs. (22) and (23) one can derive the relation between $f(\xi)$ and $g(\xi)$ is just the same as (17).

The second one is

$$w = \frac{r}{q} + \frac{1}{q}\cos(\sqrt{pq}\xi), \tag{24}$$

then

$$f_4 = \frac{1}{w} = \frac{q}{r + \cos(\sqrt{pq}\xi)} \tag{25}$$

and

$$g_4 = -\frac{1}{p}\frac{w'}{w} = \frac{1}{p}\frac{\sqrt{pq}\sin(\sqrt{pq}\xi)}{r + \cos(\sqrt{pq}\xi)}. \tag{26}$$

From Eqs. (25) and (26) one can derive the relation between $f(\xi)$ and $g(\xi)$ is just the same as (17).

The second basic case is

**Case B**: $q = 0$

Then Eq. (6) can be rewritten as

$$w'' - pr = 0. \tag{27}$$

Its general solution is

$$w = \frac{pr}{2}\xi^2 + a_1\xi + a_0, \tag{28}$$

where $a_1$ and $a_0$ are two arbitrary real constants.

From Eq. (28), one has

$$f_5 = \frac{1}{w} = \frac{1}{\frac{pr}{2}\xi^2 + a_1\xi + a_0} \tag{29}$$

and

$$g_5 = -\frac{1}{p}\frac{w'}{w} = -\frac{1}{p}\frac{pr\xi + a_1}{\frac{pr}{2}\xi^2 + a_1\xi + a_0}. \tag{30}$$

**Remark 1.** In the Refs. [20,21], they considered the case for $q = 1$ and $p = -1$, so there is only a special case of Eqs. (15) and (16).

**Remark 2.** In the Ref. [22], they considered the case for $q \geqslant 0$ and $p = \pm 1$, so there are only some special cases of case A and case B. It is worthy noting that the solutions to $g$ given in the Ref. [22] are wrong (in Ref. [22], corresponding solutions are $\tau_1 \sim \tau_4$).

**Remark 3.** In the Ref. [22], the given relation between $f$ and $g$ is corresponding to Eq. (17) when $p = \pm 1$, but no relation (13), so it is wrong for the case (11) and (12).

## 3. Application of solutions from the projective Riccati equations

In the above section, we discuss the solutions to the projective Riccati equations under some conditions. In fact, the solutions to the projective Riccati equations combining the relations between the solutions can construct an intermediate transformation, and this transformation can be applied to solve nonlinear evolution equations. Here when $q \neq 0$, the solutions to $f$ and $g$ are taken as two bases in the expansion method, i.e.,

$$u(\xi) = \sum_{i=1}^{n} f^{i-1}(\xi)\big[A_i f(\xi) + B_i g(\xi)\big] + A_0, \qquad A_n^2 + B_n^2 \neq 0, \tag{31}$$

where $u(\xi)$ is a nonzero solution to any nonlinear evolution equation, $n$ can be determined by balancing the highest order derivative term with the high degree nonlinear term in the given nonlinear evolution equation. And $f$ and $g$ satisfy the projective Riccati equations (1), there is the relation between $f$ and $g$

$$g_j^2 = -\frac{1}{p}\left[q - 2rf_j + \frac{r^2+\delta}{q} f_j^2\right], \quad j = 1, 2, 3, 4, \tag{32}$$

where $\delta = \pm 1$, if $j = 1$, then $\delta = 1$, otherwise, $\delta = -1$.

Next, we take mKdV equation as an example to illustrate the application of the solutions from the projective Riccati equations. The mKdV equation reads

$$\frac{\partial u}{\partial t} + \alpha u^2 \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = 0, \tag{33}$$

where $u$ is a real function, and $\alpha$ and $\beta$ are real numbers. We seek its travelling wave solution, i.e.,

$$u = u(\xi), \quad \xi = x - ct, \tag{34}$$

where $c$ is wave speed. Substitution Eq. (34) into Eq. (33) yields

$$-c\frac{du}{d\xi} + \alpha u^2 \frac{du}{d\xi} + \beta \frac{d^3 u}{d\xi^3} = 0, \tag{35}$$

i.e.,

$$-cu + \frac{\alpha}{3}u^3 + \beta \frac{d^2 u}{d\xi^2} = c_0, \tag{36}$$

where $c_0$ is an integration constant.

Applying expansion method, if we take the expansion order of $u$ as $O(u) = n$ and considering the relations (1), then $O(\frac{du}{d\xi}) = n + 1$, so partial balance between the highest degree nonlinear term and the highest order derivative term leads to $n = 1$. Obviously, the formal solution can be written as

$$u = A_0 + A_1 f(\xi) + B_1 g(\xi), \qquad A_1^2 + B_1^2 \neq 0. \tag{37}$$

Considering the relation (32), from Eq. (37) one can has

$$\begin{aligned} u^3 = {} & \left[A_0^3 - \frac{3q}{p}A_0 B_1^2\right] + \left[3A_0^2 B_1 - \frac{q}{p}B_1^3\right]g + \left[3A_0^2 A_1 - \frac{3q}{p}A_1 B_1^2 + \frac{6r}{p}A_0 B_1^2\right]f \\ & + \left[6A_0 A_1 B_1 + \frac{2r}{p}B_1^3\right]fg + \left[3A_0 A_1^2 + \frac{6r}{p}A_1 B_1^2 - \frac{3(r^2+\delta)}{pq}A_0 B_1^2\right]f^2 \\ & + \left[3A_1^2 B_1 - \frac{(r^2+\delta)}{pq}B_1^3\right]f^2 g + \left[A_1^3 - \frac{3(r^2+\delta)}{pq}A_1 B_1^2\right]f^3 \end{aligned} \tag{38}$$

and

$$\frac{d^2u}{d\xi^2} = -pqA_1f + prB_1fg + 3prA_1f^2 - \frac{2p(r^2+\delta)}{q}B_1f^2g - \frac{2p(r^2+\delta)}{q}A_1f^3. \tag{39}$$

Substituting Eqs. (37), (38) and (39) into Eq. (36) yields

$$\begin{aligned}
&\left[-cA_0 + \frac{\alpha}{3}\left(A_0^3 - \frac{3q}{p}A_0B_1^2\right) - c_0\right] + \left[-cA_1 + \frac{\alpha}{3}\left(-\frac{3q}{p}A_1B_1^2 + \frac{6r}{p}A_0B_1^2 + 3A_0^2A_1\right) - \beta pqA_1\right]f \\
&\quad + \left[-cB_1 + \frac{\alpha}{3}\left(-\frac{q}{p}B_1^3 + 3A_0^2B_1\right)\right]g + \left[\frac{\alpha}{3}\left(\frac{2r}{p}B_1^3 + 6A_0A_1B_1\right) + \beta prB_1\right]fg \\
&\quad + \left[\frac{\alpha}{3}\left(\frac{6r}{p}A_1B_1^2 - \frac{3(r^2+\delta)}{pq}A_0B_1^2 + 3A_0A_1^2\right) + 3\beta prA_1\right]f^2 \\
&\quad + \left[\frac{\alpha}{3}\left(3A_1^2B_1 - \frac{(r^2+\delta)}{pq}B_1^3\right) - \frac{2\beta p(r^2+\delta)}{q}B_1\right]f^2g \\
&\quad + \left[\frac{\alpha}{3}\left(A_1^3 - \frac{3(r^2+\delta)}{pq}A_1B_1^2\right) - \frac{2\beta p(r^2+\delta)}{q}A_1\right]f^3 = 0.
\end{aligned} \tag{40}$$

The arbitrariness of the argument $\xi$ results in the following algebraic equations

$$-cA_0 + \frac{\alpha}{3}\left(A_0^3 - \frac{3q}{p}A_0B_1^2\right) - c_0 = 0, \tag{41a}$$

$$-cA_1 + \frac{\alpha}{3}\left(-\frac{3q}{p}A_1B_1^2 + \frac{6r}{p}A_0B_1^2 + 3A_0^2A_1\right) - \beta pqA_1 = 0, \tag{41b}$$

$$-cB_1 + \frac{\alpha}{3}\left(-\frac{q}{p}B_1^3 + 3A_0^2B_1\right) = 0, \tag{41c}$$

$$\frac{\alpha}{3}\left(\frac{2r}{p}B_1^3 + 6A_0A_1B_1\right) + \beta prB_1 = 0, \tag{41d}$$

$$\frac{\alpha}{3}\left(\frac{6r}{p}A_1B_1^2 - \frac{3(r^2+\delta)}{pq}A_0B_1^2 + 3A_0A_1^2\right) + 3\beta prA_1 = 0, \tag{41e}$$

$$\frac{\alpha}{3}\left(3A_1^2B_1 - \frac{(r^2+\delta)}{pq}B_1^3\right) - \frac{2\beta p(r^2+\delta)}{q}B_1 = 0, \tag{41f}$$

$$\frac{\alpha}{3}\left(A_1^3 - \frac{3(r^2+\delta)}{pq}A_1B_1^2\right) - \frac{2\beta p(r^2+\delta)}{q}A_1 = 0, \tag{41g}$$

from which the parameters can be determined. For example, for $\delta = -1$, there are the following solutions

**Case 1.** If $A_1 = 0$, $A_0 = 0$, $r = 0$, then

$$B_1 = \pm\sqrt{-\frac{6\beta p^2}{\alpha}}, \qquad pq = \frac{c}{2\beta}, \tag{42}$$

obviously, there is the constraint $\alpha\beta < 0$.

**Case 2.** If $A_1 = 0$, $A_0 = 0$, $r \neq 0$, then

$$B_1 = \pm\sqrt{-\frac{3\beta p^2}{2\alpha}}, \qquad pq = \frac{2c}{\beta}, \qquad r = \pm 1, \tag{43}$$

obviously, there is the constraint $\alpha\beta < 0$, too.

**Case 3.** If $B_1 = 0$, $A_0 = 0$, then

$$A_1 = \pm\sqrt{\frac{6\beta^2 p^2}{\alpha c}}, \qquad pq = -\frac{c}{\beta}, \qquad r = 0. \tag{44}$$

**Case 4.** If $B_1 = 0$, $A_0 \neq 0$, then

$$A_1 = \pm\sqrt{\frac{3\beta^2 p^2 (r^2+2)}{\alpha c}}, \qquad A_0 = \pm\sqrt{\frac{3cr^2}{\alpha(r^2+2)}}, \qquad pq = \frac{2(r^2-1)c}{\beta(r^2+2)}, \tag{45}$$

there is the constraint $r \neq 0$ and $r^2 \neq 1$.

**Case 5.** If $A_0 = 0$, $A_1 \neq 0$, $B_1 \neq 0$, then

$$A_1 = \pm\sqrt{\frac{3\beta^2 p^2 (r^2-1)}{4\alpha c}}, \qquad B_1 = \pm\sqrt{-\frac{3\beta p^2}{2\alpha}}, \qquad pq = \frac{2c}{\beta} \tag{46}$$

with the constraint $r^2 \neq 1$.

For $\delta = 1$, there are the following solutions

**Case 1.** If $A_1 = 0$, $A_0 = 0$, $r = 0$, then

$$B_1 = \pm\sqrt{-\frac{6\beta p^2}{\alpha}}, \qquad pq = \frac{c}{2\beta}, \tag{47}$$

obviously, there is the constraint that $\alpha\beta < 0$.

**Case 2.** If $B_1 = 0$, $A_0 = 0$, then

$$A_1 = \pm\sqrt{-\frac{6\beta^2 p^2}{\alpha c}}, \qquad pq = -\frac{c}{\beta}, \qquad r = 0. \tag{48}$$

**Case 3.** If $B_1 = 0$, $A_0 \neq 0$, then

$$A_1 = \pm\sqrt{\frac{3\beta^2 p^2 (r^2-2)}{\alpha c}}, \qquad A_0 = \pm\sqrt{\frac{3cr^2}{\alpha(r^2-2)}}, \qquad pq = \frac{2(r^2+1)c}{\beta(r^2-2)}, \tag{49}$$

there is the constraint $r \neq 0$ and $r^2 \neq 2$.

**Case 4.** If $A_0 = 0$, $A_1 \neq 0$, $B_1 \neq 0$, then

$$A_1 = \pm\sqrt{\frac{3\beta^2 p^2 (r^2+1)}{4\alpha c}}, \qquad B_1 = \pm\sqrt{-\frac{3\beta p^2}{2\alpha}}, \qquad pq = \frac{2c}{\beta}. \tag{50}$$

Combining the results (11), (12), (15), (16), (22), (23), (25) and (26) with (37) and from (42) to (50), we can obtain many kinds of travelling wave solutions to mKdV equation (33):

**Type 1.** For $\delta=-1$, if $\alpha\beta<0$ and $c\beta<0$, then the solution to mKdV equation (33) is

$$u_1=B_1 g=\mp\sqrt{\frac{3c}{\alpha}}\tanh\left(\sqrt{-\frac{c}{2\beta}}\xi\right). \tag{51}$$

**Type 2.** For $\delta=-1$, if $\alpha\beta<0$ and $c\beta>0$, then the solution to mKdV equation (33) is

$$u_2=B_1 g=\mp\sqrt{-\frac{3c}{\alpha}}\cot\left(\sqrt{\frac{c}{2\beta}}\xi\right) \tag{52}$$

and

$$u_3=B_1 g=\pm\sqrt{-\frac{3c}{\alpha}}\tan\left(\sqrt{\frac{c}{2\beta}}\xi\right). \tag{53}$$

**Type 3.** For $\delta=-1$, if $\alpha\beta<0$ and $c\beta<0$, then the solution to mKdV equation (33) is

$$u_4=B_1 g=\mp\sqrt{\frac{3c}{\alpha}}\frac{\sinh(\sqrt{-\frac{2c}{\beta}}\xi)}{\cosh(\sqrt{-\frac{2c}{\beta}}\xi)\pm 1}. \tag{54}$$

**Type 4.** For $\delta=-1$, if $\alpha\beta<0$ and $c\beta>0$, then the solution to mKdV equation (33) is

$$u_5=B_1 g=\mp\sqrt{-\frac{3c}{\alpha}}\frac{\cos(\sqrt{\frac{2c}{\beta}}\xi)}{\sin(\sqrt{\frac{2c}{\beta}}\xi)\pm 1} \tag{55}$$

and

$$u_6=B_1 g=\pm\sqrt{-\frac{3c}{\alpha}}\frac{\sin(\sqrt{\frac{2c}{\beta}}\xi)}{\cos(\sqrt{\frac{2c}{\beta}}\xi)\pm 1}. \tag{56}$$

**Type 5.** For $\delta=-1$, if $\alpha c>0$ and $c\beta>0$, then the solution to mKdV equation (33) is

$$u_7=A_1 f=\mp\sqrt{\frac{6c}{\alpha}}\operatorname{sech}\left(\sqrt{\frac{c}{\beta}}\xi\right). \tag{57}$$

**Type 6.** For $\delta=-1$, if $\alpha c>0$ and $c\beta<0$, then the solution to mKdV equation (33) is

$$u_8=A_1 f=\pm\sqrt{\frac{6c}{\alpha}}\csc\left(\sqrt{-\frac{c}{\beta}}\xi\right) \tag{58}$$

and

$$u_9=A_1 f=\pm\sqrt{\frac{6c}{\alpha}}\sec\left(\sqrt{-\frac{c}{\beta}}\xi\right). \tag{59}$$

**Type 7.** For $\delta = -1$, if $\alpha c > 0$ and $(r^2 - 1)c\beta < 0$, then the solution to mKdV equation (33) is

$$u_{10} = A_0 + A_1 f = \pm\sqrt{\frac{3cr^2}{\alpha(r^2+2)}} \pm \frac{2(r^2-1)c}{p\beta(r^2+2)}\sqrt{\frac{3\beta^2 p^2(r^2+2)}{\alpha c}}\frac{1}{\cosh\left(\sqrt{\frac{2(1-r^2)c}{\beta(r^2+2)}}\xi\right)+r} \tag{60}$$

with the constraint that $r \neq 0$ and $r^2 \neq 1$.

**Type 8.** For $\delta = -1$, if $\alpha c > 0$ and $(r^2 - 1)c\beta > 0$, then the solution to mKdV equation (33) is

$$u_{11} = A_0 + A_1 f = \pm\sqrt{\frac{3cr^2}{\alpha(r^2+2)}} \pm \frac{2(r^2-1)c}{p\beta(r^2+2)}\sqrt{\frac{3\beta^2 p^2(r^2+2)}{\alpha c}}\frac{1}{\sin\left(\sqrt{\frac{2(r^2-1)c}{\beta(r^2+2)}}\xi\right)+r} \tag{61}$$

and

$$u_{12} = A_0 + A_1 f = \pm\sqrt{\frac{3cr^2}{\alpha(r^2+2)}} \pm \frac{2(r^2-1)c}{p\beta(r^2+2)}\sqrt{\frac{3\beta^2 p^2(r^2+2)}{\alpha c}}\frac{1}{\cos\left(\sqrt{\frac{2(r^2-1)c}{\beta(r^2+2)}}\xi\right)+r} \tag{62}$$

with the constraint that $r \neq 0$ and $r^2 \neq 1$.

**Type 9.** For $\delta = -1$, if $\alpha\beta < 0$, $c\beta < 0$ and $r^2 > 1$, then the solution to mKdV equation (33) is

$$u_{13} = A_1 f + B_1 g = \pm\frac{2c}{\beta p}\sqrt{\frac{3\beta^2 p^2(r^2-1)}{4\alpha c}}\frac{1}{\cosh\left(\sqrt{-\frac{2c}{\beta}}\xi\right)+r} \mp \sqrt{\frac{3c}{\alpha}}\frac{\sinh\left(\sqrt{-\frac{2c}{\beta}}\xi\right)}{\cosh\left(\sqrt{-\frac{2c}{\beta}}\xi\right)+r} \tag{63}$$

with the constraint $r^2 \neq 1$.

**Type 10.** For $\delta = -1$, if $\alpha\beta < 0$, $c\beta > 0$ and $r^2 < 1$, then the solution to mKdV equation (33) is

$$u_{14} = A_1 f + B_1 g = \pm\frac{2c}{\beta p}\sqrt{\frac{3\beta^2 p^2(r^2-1)}{4\alpha c}}\frac{1}{\sin\left(\sqrt{\frac{2c}{\beta}}\xi\right)+r} \mp \sqrt{-\frac{3c}{\alpha}}\frac{\cos\left(\sqrt{\frac{2c}{\beta}}\xi\right)}{\sin\left(\sqrt{\frac{2c}{\beta}}\xi\right)+r} \tag{64}$$

and

$$u_{15} = A_1 f + B_1 g = \pm\frac{2c}{\beta p}\sqrt{\frac{3\beta^2 p^2(r^2-1)}{4\alpha c}}\frac{1}{\cos\left(\sqrt{\frac{2c}{\beta}}\xi\right)+r} \pm \sqrt{-\frac{3c}{\alpha}}\frac{\sin\left(\sqrt{\frac{2c}{\beta}}\xi\right)}{\cos\left(\sqrt{\frac{2c}{\beta}}\xi\right)+r} \tag{65}$$

with the constraint $r^2 \neq 1$.

**Type 11.** For $\delta = 1$, if $\alpha\beta < 0$ and $c\beta < 0$, then the solution to mKdV equation (33) is

$$u_{16} = B_1 g = \mp\sqrt{\frac{3c}{\alpha}}\coth\left(\sqrt{-\frac{c}{2\beta}}\xi\right). \tag{66}$$

**Type 12.** For $\delta = 1$, if $\alpha c < 0$ and $c\beta > 0$, then the solution to mKdV equation (33) is

$$u_{17} = A_1 f = \mp\sqrt{-\frac{6c}{\alpha}}\operatorname{csch}\left(\sqrt{\frac{c}{\beta}}\xi\right). \tag{67}$$

**Type 13.** For $\delta = 1$, if $\alpha c(r^2 - 2) > 0$ and $(r^2 - 2)c\beta < 0$, then the solution to mKdV equation (33) is

$$u_{18} = A_0 + A_1 f = \pm\sqrt{\frac{3cr^2}{\alpha(r^2-2)}} \pm \frac{2(r^2+1)c}{p\beta(r^2-2)}\sqrt{\frac{3\beta^2 p^2(r^2-2)}{\alpha c}} \frac{1}{\sinh\left(\sqrt{\frac{2(1+r^2)c}{\beta(2-r^2)}}\xi\right) + r} \tag{68}$$

with the constraint that $r \neq 0$ and $r^2 \neq 2$.

**Type 14.** For $\delta = 1$, if $\alpha\beta < 0$ and $c\beta < 0$, then the solution to mKdV equation (33) is

$$u_{19} = A_1 f + B_1 g = \pm\frac{2c}{\beta p}\sqrt{\frac{3\beta^2 p^2(r^2+1)}{4\alpha c}} \frac{1}{\sinh\left(\sqrt{-\frac{2c}{\beta}}\xi\right) + r} \mp \sqrt{\frac{3c}{\alpha}} \frac{\cosh\left(\sqrt{-\frac{2c}{\beta}}\xi\right)}{\sinh\left(\sqrt{-\frac{2c}{\beta}}\xi\right) + r}. \tag{69}$$

Obviously, the solutions $u_1$, $u_2$, $u_3$, $u_7$, $u_8$, $u_9$, $u_{16}$ and $u_{17}$ are general solitary wave solutions and periodic solutions expressed by sine–cosine functions which can be found in the usual expansion methods, such as the function transformation method [9,10], the hyperbolic function expansion method [11,12], the Jacobi elliptic function expansion method [13,14] and the sine–cosine method [15]. But the solutions $u_4$, $u_5$, $u_6$, $u_{10}$, $u_{11}$, $u_{12}$, $u_{13}$, $u_{14}$, $u_{15}$, $u_{18}$ and $u_{19}$ cannot be obtained in these expansion methods. These solutions are new type solitary wave solutions or new type periodic solutions expressed by sine–cosine functions, and some of them have not been found before.

## 4. Conclusion

In this Letter, we introduce a new transformation from the projective Riccati equations and apply it to solve mKdV equation. Many solutions are obtained for this mKdV equation, such as solitary wave solutions constructed in terms of hyperbolic functions, periodic solutions expressed in terms of sine and cosine functions, some solutions are not given in literature to our knowledge. Of course, this transformation can be also applied to other nonlinear wave equations. Furthermore, in this Letter, we correct some errors found in some literature.

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