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Periodic Solutions for Two Coupled Nonlinear-Partial Differential Equations*

LIU Shi-Da,[1] FU Zun-Tao,[1,2,†] and LIU Shi-Kuo[1,2]

[1]School of Physics, Peking University, Beijing 100871, China

[2]State Key Laboratory for Turbulence and Complex Systems, Peking University, Beijing 100871, China

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Abstract *In this paper, by applying the Jacobi elliptic function expansion method, the periodic solutions for two coupled nonlinear partial differential equations are obtained.*

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1 Introduction

In 2002, Yao and Li[1] and Liu and Liu[2] presented a new method for finding exact travelling wave solutions of some coupled nonlinear differential equations. However, there only some soliton-like solutions were derived and some conditions are coarse. In this letter, by using the Jacobi elliptic function expansion method,[3–5] we obtain the periodic solutions for two coupled nonlinear partial differential equations, which play an important role in modern physics.

2 Periodic Solutions for DSW Equations

The classical Drinfel'd–Soklov–Wilson (DSW for short) equations[6] read

$$u_t + \alpha_1 v v_x = 0\,, \tag{1a}$$

$$u_t + \alpha_2 u v_x + \alpha_3 v u_x + \beta v_{xxx} = 0\,. \tag{1b}$$

We seek the travelling wave solutions of Eqs. (1) in the form

$$u = u(\xi)\,, \quad v = v(\xi)\,, \quad \xi = k(x - ct)\,, \tag{2}$$

where k and c are wave number and wave speed, respectively. Substituting Eqs. (2) into Eqs. (1), we have

$$-c\frac{\mathrm{d}u}{\mathrm{d}\xi} + \alpha_1 v\frac{\mathrm{d}v}{\mathrm{d}\xi} = 0\,, \tag{3a}$$

$$-c\frac{\mathrm{d}v}{\mathrm{d}\xi} + \alpha_2 u\frac{\mathrm{d}v}{\mathrm{d}\xi} + \alpha_3 v\frac{\mathrm{d}u}{\mathrm{d}\xi} + \beta\frac{\mathrm{d}^3 v}{\mathrm{d}\xi^3} = 0\,. \tag{3b}$$

By using the Jacobi elliptic function expansion method,[3–5] u and v can be expressed as

$$u = a_0 + a_1 \operatorname{sn}\xi + a_2 \operatorname{sn}^2\xi\,, \tag{4a}$$

$$v = b_0 + b_1 \operatorname{sn}\xi \tag{4b}$$

with $a_2^2 + b_1^2 \neq 0$, where $\operatorname{sn}\xi$ is the Jacobi elliptic sine function.[7–9]

Substituting Eqs. (4) into Eqs. (3) leads to

$$(-ca_1 + \alpha_1 b_0 b_1) + (-2ca_2 + \alpha_1 b_1^2)\operatorname{sn}\xi = 0\,, \tag{5a}$$

$$\begin{aligned}&[-cb_1 + \alpha_2 a_0 b_1 + \alpha_3 a_1 b_0 - \beta k^2(1+m^2)b_1] \\ &\quad + [\alpha_2 a_1 b_1 + \alpha_3(2a_2 b_0 + a_1 b_1)]\operatorname{sn}\xi \\ &\quad + (\alpha_2 a_2 b_1 + 2\alpha_3 a_2 b_1 + 6\beta k^2 m^2 b_1]\operatorname{sn}^2\xi = 0\,,\end{aligned} \tag{5b}$$

with m $(0 < m < 1)$ is the modulus.

From Eqs. (5), we have

$$a_1 = b_0 = 0\,, \quad a_0 = \frac{c + \beta k^2(1+m^2)}{\alpha_2}\,,$$

$$a_2 = -\frac{6\beta k^2 m^2}{\alpha_2 + 2\alpha_3}\,, \quad b_1 = \pm\sqrt{-\frac{12\beta k^2 m^2 c}{\alpha_1(\alpha_2 + 2\alpha_3)}}\,. \tag{6}$$

So, the periodic solutions to the classical DSW equations (1) are

$$u = \frac{c + \beta k^2(1+m^2)}{\alpha_2} - \frac{6\beta k^2 m^2}{\alpha_2 + 2\alpha_3}\operatorname{sn}^2\xi\,, \tag{7a}$$

$$v = \pm\sqrt{-\frac{12\beta k^2 m^2 c}{\alpha_1(\alpha_2 + 2\alpha_3)}}\operatorname{sn}\xi\,. \tag{7b}$$

When $m \to 1$, equations (7) reduce to the following solitary wave (shock wave) solutions:

$$u = \frac{c + 2\beta k^2}{\alpha_2} - \frac{6\beta k^2}{\alpha_2 + 2\alpha_3}\tanh^2\xi\,,$$

$$v = \pm\sqrt{-\frac{12\beta k^2 c}{\alpha_1(\alpha_2 + 2\alpha_3)}}\tanh\xi\,. \tag{8}$$

Similar to Eqs. (4), the ansatz solution can be taken as

$$u = c_0 + c_1 \operatorname{cn}\xi + c_2 \operatorname{cn}^2\xi\,, \tag{9a}$$

$$v = d_0 + d_1 \operatorname{cn}\xi \tag{9b}$$

with $c_2^2 + d_1^2 \neq 0$ and where $\operatorname{cn}\xi$ is the Jacobi elliptic cosine function.[7–9]

Substituting Eqs. (9) into Eqs. (3) yields

$$c_1 = d_0 = 0\,, \quad c_0 = \frac{c - \beta k^2(2m^2 - 1)}{\alpha_2}\,,$$

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†Correspondence author, E-mail: fuzt@pku.edu.cn

$$c_2 = \frac{6\beta k^2 m^2}{\alpha_2 + 2\alpha_3}\,, \quad d_1 = \pm\sqrt{\frac{12\beta k^2 m^2 c}{\alpha_1(\alpha_2 + 2\alpha_3)}}\,. \tag{10}$$

Then, the another periodic solutions to the classical DSW equations (1) are

$$u = \frac{c - \beta k^2(2m^2 - 1)}{\alpha_2} + \frac{6\beta k^2 m^2}{\alpha_2 + 2\alpha_3}\,\mathrm{cn}^2\xi\,, \tag{11a}$$

$$v = \pm\sqrt{\frac{12\beta k^2 m^2 c}{\alpha_1(\alpha_2 + 2\alpha_3)}}\,\mathrm{cn}\,\xi\,. \tag{11b}$$

When $m \to 1$, equations (11) reduce to the following solitary wave solutions:

$$u = \frac{c - \beta k^2}{\alpha_2} + \frac{6\beta k^2}{\alpha_2 + 2\alpha_3}\,\mathrm{sech}^2\xi\,,$$
$$v = \pm\sqrt{\frac{12\beta k^2 c}{\alpha_1(\alpha_2 + 2\alpha_3)}}\,\mathrm{sech}\,\xi\,. \tag{12}$$

The solutions (8) and (12) are the same as given in Ref. [1].

3 Periodic Solutions for Hirota–Satsuma Coupled KdV Equations

The Hirota–Satsuma coupled KdV equations[10,11] reads

$$u_t + \alpha(uu_x - vw_x - wv_x) + \beta u_{xxx} = 0\,, \tag{13a}$$
$$v_t - \alpha uv_x - 2\beta v_{xxx} = 0\,, \tag{13b}$$
$$w_t - \alpha uw_x - 2\beta w_{xxx} = 0\,. \tag{13c}$$

Similarly, the periodic solutions of Eqs. (13) in the travelling wave frame,

$$u = u(\xi)\,, \quad v = v(\xi)\,, \quad w = w(\xi)\,,$$
$$\xi = k(x - ct)\,, \tag{14}$$

can be written as

$$u = a_0 + a_1\,\mathrm{sn}\,\xi + a_2\,\mathrm{sn}^2\xi\,,$$
$$v = b_0 + b_1\,\mathrm{sn}\,\xi + b_2\,\mathrm{sn}^2\xi\,,$$
$$w = c_0 + c_1\,\mathrm{sn}\,\xi + c_2\,\mathrm{sn}^2\xi \tag{15}$$

with the constraint $a_2 \neq 0$.

Substituting Eq. (15) into Eqs. (13) leads to the following results

$$[-c + \alpha a_0 - \beta k^2(1 + m^2)]a_1 - \alpha(b_0c_1 + b_1c_0) = 0\,, \tag{16a}$$
$$\alpha a_1^2 - 2\alpha(b_0c_2 + b_1c_1 + b_2c_0) + 2[-c + \alpha a_0 - 4\beta k^2(1 + m^2)]a_2 = 0\,, \tag{16b}$$
$$3(\alpha a_2 + 2\beta k^2 m^2)a_1 - 3\alpha(b_1c_2 + b_2c_1) = 0\,, \tag{16c}$$
$$(\alpha a_2 + 12\beta k^2 m^2)a_2 - 2\alpha b_2c_2 = 0\,, \tag{16d}$$
$$[c + \alpha a_0 - 2\beta k^2(1 + m^2)]b_1 = 0\,, \tag{16e}$$
$$[c + \alpha a_0 - 2\beta k^2(1 + m^2)]c_1 = 0\,, \tag{16f}$$
$$2[c + 2\alpha a_0 - 8\beta k^2(1 + m^2)]b_2 + \alpha a_1b_1 = 0\,, \tag{16g}$$
$$2[c + 2\alpha a_0 - 8\beta k^2(1 + m^2)]c_2 + \alpha a_1c_1 = 0\,, \tag{16h}$$
$$(\alpha a_2 + 12\beta k^2 m^2)b_1 + 2\alpha a_1b_2 = 0\,, \tag{16i}$$
$$(\alpha a_2 + 12\beta k^2 m^2)c_1 + 2\alpha a_1c_2 = 0\,, \tag{16j}$$
$$(\alpha a_2 + 24\beta k^2 m^2)b_2 = 0\,, \tag{16k}$$
$$(\alpha a_2 + 24\beta k^2 m^2)c_2 = 0\,. \tag{16l}$$

For the system (16), two cases must be considered. The first one is $a_1 = b_1 = c_1 = 0$, then we have

$$a_0 = \frac{8\beta k^2(1 + m^2) - c}{2\alpha}\,, \quad a_2 = -\frac{24\beta k^2 m^2}{\alpha}\,,$$
$$b_2c_2 = \frac{144\beta^2k^4m^4}{\alpha^2}\,, \quad b_0c_2 + b_2c_0 = \frac{36\beta k^2 m^2 c}{\alpha^2}\,. \tag{17}$$

So the periodic solution to the coupled system (13) is

$$u = \frac{8\beta k^2(1 + m^2) - c}{2\alpha} - \frac{24\beta k^2 m^2}{\alpha}\,\mathrm{sn}^2\xi\,,$$
$$v = b_0 + b_2\,\mathrm{sn}^2\xi\,, \quad w = c_0 + c_2\,\mathrm{sn}^2\xi \tag{18}$$

with b_0, b_2, c_0, and c_2 satisfying the constraint (17).

When $m \to 1$, equation (18) reduces to

$$u = \frac{16\beta k^2 - c}{2\alpha} - \frac{24\beta k^2}{\alpha}\tanh^2\xi\,,$$
$$v = b_0 + b_2\tanh^2\xi\,, \quad w = c_0 + c_2\tanh^2\xi\,. \tag{19}$$

The second case is $a_1 = b_2 = c_2 = 0$, from Eq. (16), one has

$$a_0 = \frac{2\beta k^2(1 + m^2) - c}{\alpha}\,, \quad a_2 = -\frac{12\beta k^2 m^2}{\alpha}\,,$$
$$b_1c_1 = \frac{24\beta k^2 m^2[c + \beta k^2(1 + m^2)]}{\alpha^2}\,,$$
$$b_0c_1 + b_1c_0 = 0\,. \tag{20}$$

So another periodic solution to the coupled system (13) is

$$u = \frac{2\beta k^2(1 + m^2) - c}{\alpha} - \frac{12\beta k^2 m^2}{\alpha}\,\mathrm{sn}^2\xi\,,$$
$$v = b_0 + b_1\mathrm{sn}\xi\,, \quad w = c_0 + c_1\,\mathrm{sn}\,\xi \tag{21}$$

with b_0, b_1, c_0, and c_1 satisfying the constraint (20).

When $m \to 1$, equation (21) reduces to

$$u = \frac{4\beta k^2 - c}{\alpha} - \frac{12\beta k^2}{\alpha}\tanh^2\xi\,,$$
$$v = b_0 + b_1\tanh\xi\,, \quad w = c_0 + c_1\tanh\xi\,. \tag{22}$$

Similarly, if the ansatz solution to the coupled system (13) is taken as

$$u = d_0 + d_1\,\mathrm{cn}\,\xi + d_2\,\mathrm{cn}^2\xi\,,$$
$$v = e_0 + e_1\,\mathrm{cn}\,\xi + e_2\,\mathrm{cn}^2\xi\,,$$
$$w = f_0 + f_1\,\mathrm{cn}\,\xi + f_2\,\mathrm{cn}^2\xi \tag{23}$$

with the constraint $d_2 \neq 0$, there are another two similar periodic solutions.

The first one is

$$u = -\frac{8\beta k^2(2m^2 - 1) + c}{2\alpha} + \frac{24\beta k^2 m^2}{\alpha}\,\mathrm{cn}^2\xi\,,$$
$$v = e_0 + e_2\,\mathrm{cn}^2\xi\,,$$
$$w = f_0 + f_2\,\mathrm{cn}^2\xi \tag{24}$$

with e_0, e_2, f_0, and f_2 satisfying the constraint

$$e_2 f_2 = \frac{144\beta^2 k^4 m^4}{\alpha^2}\,,$$
$$e_0 f_2 + e_2 f_0 = -\frac{36\beta k^2 m^2 c}{\alpha^2}\,. \tag{25}$$

The second one is

$$u = -\frac{2\beta k^2(2m^2-1)+c}{\alpha} + \frac{12\beta k^2 m^2}{\alpha}\,\mathrm{cn}^2\xi\,,$$
$$v = e_0 + e_1\,\mathrm{cn}\,\xi\,, \quad w = f_0 + f_1\,\mathrm{cn}\,\xi \tag{26}$$

with e_0, e_1, f_0, and f_1 satisfying the constraint

$$e_1 f_1 = \frac{24\beta k^2 m^2[\beta k^2(2m^2-1)-c]}{\alpha^2}\,,$$
$$e_0 f_1 + e_1 f_0 = 0\,. \tag{27}$$

When $m \to 1$, equation (26) reduces to

$$u = -\frac{2\beta k^2 + c}{\alpha} + \frac{12\beta k^2}{\alpha}\,\mathrm{sech}^2\xi\,,$$
$$v = e_0 + e_1\,\mathrm{sech}\,\xi\,,$$
$$w = f_0 + f_1\,\mathrm{sech}\,\xi\,. \tag{28}$$

Taking $\alpha = 3$, $\beta = -1/2$, the solutions (18) and (28) are the same as given in Ref. [1].

4 Conclusion

In this letter, we apply the Jacobi elliptic function expansion to solve two coupled nonlinear systems, and many periodic wave solutions and shock wave or solitary wave solutions are derived. These solutions are helpful in understanding the problems in modern physics.

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