Periodic Solutions for Two Coupled Nonlinear-Partial Differential Equations

LIU Shi-Da,†FU Zun-Tao,1.2 and LIU Shi-Kuo1.2

1School of Physics, Peking University, Beijing 100871, China
2State Key Laboratory for Turbulence and Complex Systems, Peking University, Beijing 100871, China

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Abstract In this paper, by applying the Jacobi elliptic function expansion method, the periodic solutions for two coupled nonlinear partial differential equations are obtained.

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1 Introduction
In 2002, Yao and Li[1] and Liu and Liu[2] presented a new method for finding exact travelling wave solutions of some coupled nonlinear differential equations. However, there only some soliton-like solutions were derived and some conditions are coarse. In this letter, by using the Jacobi elliptic function expansion method,[3−5] we obtain the periodic solutions for two coupled nonlinear partial differential equations, which play an important role in modern physics.

2 Periodic Solutions for DSW Equations

\[ u_t + \alpha_1 uv_x = 0, \quad (1a) \]
\[ u_t + \alpha_2 uv_x + \alpha_3 v u_x + \beta v_{xxx} = 0. \quad (1b) \]

We seek the travelling wave solutions of Eqs. (1) in the form

\[ u = u(\xi), \quad v = v(\xi), \quad \xi = k(x - ct), \quad (2) \]

where \( k \) and \( c \) are wave number and wave speed, respectively. Substituting Eqs. (2) into Eqs. (1), we have

\[ -c \frac{du}{d\xi} + \alpha_1 v \frac{dv}{d\xi} = 0, \quad (3a) \]
\[ -c \frac{dv}{d\xi} + \alpha_2 u \frac{dv}{d\xi} + \alpha_3 v \frac{du}{d\xi} + \beta v_{\xi\xi\xi} = 0. \quad (3b) \]

By using the Jacobi elliptic function expansion method,[3−5] \( u \) and \( v \) can be expressed as

\[ u = a_0 + a_1 \sin \xi + a_2 \sin^2 \xi, \quad (4a) \]
\[ v = b_0 + b_1 \sin \xi \quad (4b) \]

with \( a_2^2 + b_1^2 \neq 0 \), where \( \sin \xi \) is the Jacobi elliptic sine function.[7−9]

Substituting Eqs. (4) into Eqs. (3) leads to

\[ (-c a_1 + \alpha_1 b_0 b_1) + (-2 c a_2 + \alpha_1 b_1^2) \sin \xi = 0, \quad (5a) \]
\[ \frac{d}{d\xi} \left[ \frac{\beta k^2 (1 + m^2) b_1}{\alpha_2^2} \right] + \frac{1}{\alpha_2^2} \left[ \frac{6 \beta k^2 m^2 c}{\alpha_1 (\alpha_2 + 2 \alpha_3)} \right] \sin \xi = 0, \quad (5b) \]

with \( m (0 < m < 1) \) is the modulus.

From Eqs. (5), we have

\[ a_1 = b_0 = 0, \quad a_0 = \frac{c + \beta k^2 (1 + m^2)}{\alpha_2}, \quad a_2 = \frac{6 \beta k^2 m^2 c}{\alpha_2 + 2 \alpha_3}, \quad b_1 = \pm \sqrt{-\frac{12 \beta k^2 m^2 c}{\alpha_1 (\alpha_2 + 2 \alpha_3)}}. \quad (6) \]

So, the periodic solutions to the classical DSW equations (1) are

\[ u = \frac{c + \beta k^2 (1 + m^2)}{\alpha_2} - \frac{6 \beta k^2 m^2 c}{\alpha_2 + 2 \alpha_3} \sin^2 \xi, \quad (7a) \]
\[ v = \pm \sqrt{-\frac{12 \beta k^2 m^2 c}{\alpha_1 (\alpha_2 + 2 \alpha_3)}} \sin \xi. \quad (7b) \]

When \( m \to 1 \), equations (7) reduce to the following solitary wave (shock wave) solutions:

\[ u = \frac{c + 2 \beta k^2}{\alpha_2} - \frac{6 \beta k^2}{\alpha_2 + 2 \alpha_3} \tanh^2 \xi, \quad (8) \]

Similar to Eqs. (4), the ansatz solution can be taken as

\[ u = c_0 + c_1 \cosh \xi + c_2 \sinh^2 \xi, \quad (9a) \]
\[ v = d_0 + d_1 \cosh \xi \quad (9b) \]

with \( c_2^2 + d_1^2 \neq 0 \) and where \( \cosh \xi \) is the Jacobi elliptic cosine function.[7−9]

Substituting Eqs. (9) into Eqs. (3) yields

\[ c_1 = d_0 = 0, \quad c_0 = \frac{c - \beta k^2 (2m^2 - 1)}{\alpha_2}, \quad (9c) \]
\[ c_2 = \frac{6\beta k^2 m^2}{\alpha_2 + 2\alpha_3}, \quad d_1 = \pm \sqrt{\frac{12\beta k^2 m^2 c}{\alpha_1 (\alpha_2 + 2\alpha_3)}}. \quad (10) \]

Then, the other periodic solutions to the classical DSW equations (1) are
\[ u = \frac{c - \beta k^2 (2m^2 - 1) + 6\beta k^2 m^2}{\alpha_2 + 2\alpha_3} \, \text{cn}^2 \xi , \quad (11a) \]
\[ v = \pm \sqrt{\frac{12\beta k^2 m^2 c}{\alpha_1 (\alpha_2 + 2\alpha_3)}} \, \text{cn} \xi . \quad (11b) \]

When \( m \to 1 \), equations (11) reduce to the following solitary wave solutions:
\[ u = \frac{c - \beta k^2 + 6\beta k^2 m^2}{\alpha_2 + 2\alpha_3} \, \text{sech}^2 \xi , \]
\[ v = \pm \sqrt{\frac{12\beta k^2 c}{\alpha_1 (\alpha_2 + 2\alpha_3)}} \, \text{sech} \xi . \quad (12) \]

The solutions (8) and (12) are the same as given in Ref. [1].

3 Periodic Solutions for Hirota–Satsuma Coupled KdV Equations

The Hirota–Satsuma coupled KdV equations\textsuperscript{[10,11]} reads
\[ u_t + \alpha(uu_x - vv_x - w v_x) + \beta u_{xxx} = 0, \quad (13a) \]
\[ v_t - \alpha u v_x - 2\beta v_{xxx} = 0, \quad (13b) \]
\[ w_t - \alpha u w_x - 2\beta w_{xxx} = 0. \quad (13c) \]

Similarly, the periodic solutions of Eqs. (13) in the travelling wave frame,
\[ u = u(\xi), \quad v = v(\xi), \quad w = w(\xi), \quad \xi = k(x - ct), \quad (14) \]

can be written as
\[ u = a_0 + a_1 \text{sn} \xi + a_2 \text{sn}^2 \xi, \]
\[ v = b_0 + b_1 \text{sn} \xi + b_2 \text{sn}^2 \xi, \]
\[ w = c_0 + c_1 \text{sn} \xi + c_2 \text{sn}^2 \xi \quad (15) \]

with the constraint \( a_2 \neq 0 \).

Substituting Eq. (15) into Eqs. (13) leads to the following results
\[ [-c + a a_0 - \beta k^2 (1 + m^2)] a_1 - \alpha(b_0 c_1 + b_1 c_0) = 0, \quad (16a) \]
\[ a a_1^2 - 2a(b_0 c_2 + b_1 c_1) + b_2 c_0 = 0, \quad (16b) \]
\[ 3(a a_0 + 2\beta k^2 m^2) a_1 - 3a_0 (b_1 c_2 + 2b_2 c_1) = 0, \quad (16c) \]
\[ (a a_2 + 12\beta k^2 m^2) a_2 - 2a b_2 c_0 = 0, \quad (16d) \]
\[ (a a_2 + 2\beta k^2 m^2) a_2 - 2a b_2 c_0 = 0, \quad (16e) \]
\[ [c + a a_0 - 2\beta k^2 (1 + m^2)] b_1 = 0, \quad (16f) \]
\[ [c + a a_0 - 2\beta k^2 (1 + m^2)] c_1 = 0, \quad (16g) \]
\[ 2(c + 2a a_0 - 8\beta k^2 (1 + m^2)) b_2 + a a_1 b_1 = 0, \quad (16h) \]
\[ 2(c + 2a a_0 - 8\beta k^2 (1 + m^2)) c_2 + a a_1 c_1 = 0, \quad (16i) \]
\[ (a a_2 + 12\beta k^2 m^2) b_1 + 2a a_1 b_2 = 0, \quad (16j) \]
\[ (a a_2 + 24\beta k^2 m^2) c_1 + 2a a_1 c_2 = 0, \quad (16k) \]
\[ (a a_2 + 24\beta k^2 m^2) b_2 = 0, \quad (16l) \]
\[ (a a_2 + 24\beta k^2 m^2) c_2 = 0. \quad (16m) \]

For the system (16), two cases must be considered.

The first one is \( a_1 = b_1 = c_1 = 0 \), then we have
\[ a_0 = \frac{8\beta k^2 (1 + m^2) - c}{2\alpha}, \quad a_2 = -\frac{24\beta k^2 m^2}{\alpha}, \quad b_2 c_0 = \frac{144\beta k^4 m^4}{\alpha^2}, \quad b_0 c_1 + b_1 c_0 = \frac{36\beta k^2 m^2 c}{\alpha^2}. \quad (17) \]

So the periodic solution to the coupled system (13) is
\[ u = \frac{8\beta k^2 (1 + m^2) - c}{2\alpha} - \frac{24\beta k^2 m^2}{\alpha} \, \text{sn}^2 \xi, \]
\[ v = b_0 + b_2 \text{sn}^2 \xi, \quad w = c_0 + c_2 \text{sn}^2 \xi \quad (18) \]
with \( b_0, \ b_2, \ c_0, \) and \( c_2 \) satisfying the constraint (17).

When \( m \to 1 \), equation (18) reduces to
\[ u = \frac{16\beta k^2 - c - 24\beta k^2}{2\alpha} \, \text{tanh}^2 \xi, \]
\[ v = b_0 + b_2 \text{tanh}^2 \xi, \quad w = c_0 + c_2 \text{tanh}^2 \xi. \quad (19) \]

The second case is \( a_1 = b_2 = c_2 = 0 \), from Eq. (16), one has
\[ a_0 = \frac{2\beta k^2 (1 + m^2) - c}{\alpha}, \quad a_2 = -\frac{12\beta k^2 m^2}{\alpha}, \quad b_1 c_0 = \frac{24\beta k^2 m^2 [c + \beta k^2 (1 + m^2)]}{\alpha^2}, \quad b_0 c_1 + b_1 c_0 = 0. \quad (20) \]

So another periodic solution to the coupled system (13) is
\[ u = \frac{2\beta k^2 (1 + m^2) - c - 12\beta k^2 m^2}{\alpha} \, \text{sn}^2 \xi, \]
\[ v = b_0 + b_1 \text{sn} \xi, \quad w = c_0 + c_1 \text{sn} \xi \quad (21) \]
with \( b_0, \ b_1, \ c_0, \) and \( c_1 \) satisfying the constraint (20).

When \( m \to 1 \), equation (21) reduces to
\[ u = \frac{4\beta k^2 - c - 12\beta k^2}{\alpha} \, \text{tanh}^2 \xi, \]
\[ v = b_0 + b_1 \text{tanh} \xi, \quad w = c_0 + c_1 \text{tanh} \xi. \quad (22) \]

Similarly, if the ansatz solution to the coupled system (13) is taken as
\[ u = d_0 + d_1 \text{cn} \xi + d_2 \text{cn}^2 \xi, \]
\[ v = c_0 + c_1 \text{cn} \xi + c_2 \text{cn}^2 \xi, \]
\[ w = f_0 + f_1 \text{cn} \xi + f_2 \text{cn}^2 \xi \quad (23) \]
with the constraint \( d_2 \neq 0 \), there are another two similar periodic solutions.

The first one is
\[ u = -\frac{8\beta k^2 (2m^2 - 1) + c + 24\beta k^2 m^2}{2\alpha} \, \text{sn}^2 \xi, \]
\[ v = c_0 + c_2 \text{cn}^2 \xi, \]
\[ w = f_0 + f_2 \text{cn}^2 \xi \quad (24) \]
with $e_0$, $e_2$, $f_0$, and $f_2$ satisfying the constraint

$$e_2 f_2 = \frac{144 \beta^2 k^4 m^4}{\alpha^2},$$
$$e_0 f_2 + e_2 f_0 = -\frac{36 \beta k^2 m^2 c}{\alpha^2}.$$  \hspace{1cm} (25)

The second one is

$$u = -\frac{2 \beta k^2 (2 m^2 - 1) + c}{\alpha} + \frac{12 \beta k^2 m^2}{\alpha} \text{cn}^2 \xi,$$
$$v = e_0 + e_1 \text{cn} \xi, \quad w = f_0 + f_1 \text{cn} \xi.$$  \hspace{1cm} (26)

with $e_0$, $e_1$, $f_0$, and $f_1$ satisfying the constraint

$$e_1 f_1 = \frac{24 \beta k^2 m^2 [\beta k^2 (2 m^2 - 1) - c]}{\alpha^2},$$
$$e_0 f_1 + e_1 f_0 = 0.$$  \hspace{1cm} (27)

When $m \to 1$, equation (26) reduces to

$$u = -\frac{2 \beta k^2 + c}{\alpha} + \frac{12 \beta k^2}{\alpha} \text{sech}^2 \xi,$$
$$v = e_0 + e_1 \text{sech} \xi, \quad w = f_0 + f_1 \text{sech} \xi.$$  \hspace{1cm} (28)

Taking $\alpha = 3$, $\beta = -1/2$, the solutions (18) and (28) are the same as given in Ref. [1].

4 Conclusion

In this letter, we apply the Jacobi elliptic function expansion to solve two coupled nonlinear systems, and many periodic wave solutions and shock wave or solitary wave solutions are derived. These solutions are helpful in understanding the problems in modern physics.

References