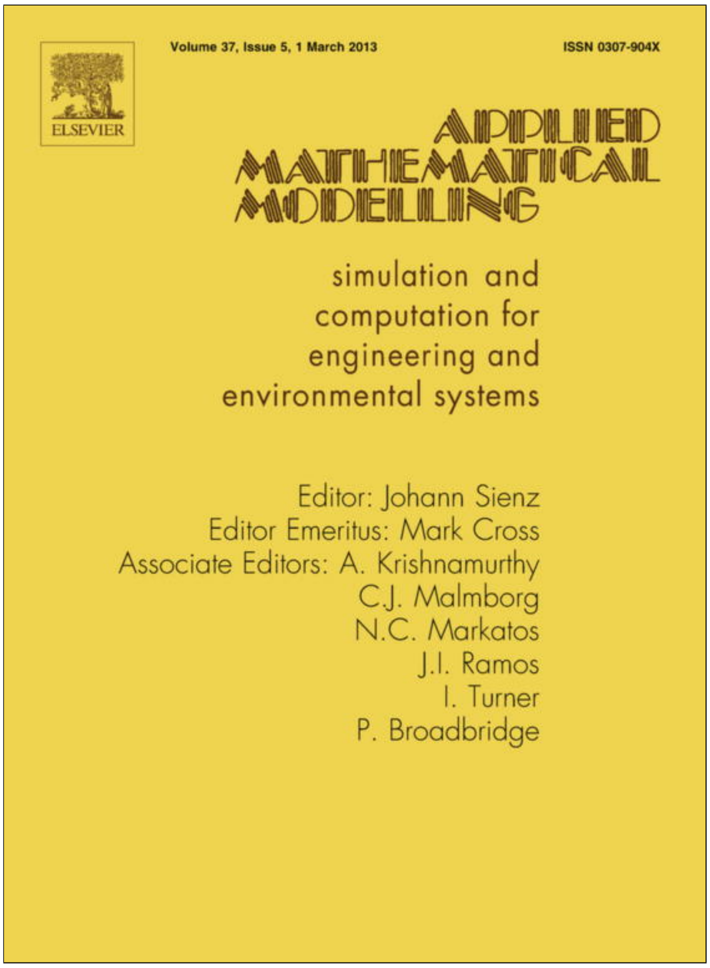
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Exact coherent structures in the (2 + 1)-dimensional KdV equations

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ABSTRACT

Different from the (1 + 1)-dimensional nonlinear systems, (2 + 1) or higher dimensional nonlinear systems admit more rich coherent structures. Taking (2 + 1)-dimensional Korteweg de Vries (KdV for short) equations as an example, the singular manifold method is applied to search these coherent structures in an analytical form. With the aid of symbolic computation and plot representation of Maple, some coherent structures expressed in terms of new forms, such as dromions and solitoffs, have been illustrated by means of arbitrary functions in the analytical forms. In the paper, we will show these results by changing some specific choices for three different special cases for singular variable in details.

1. Introduction

Different from the (1 + 1)-dimensional nonlinear systems, (2 + 1) or higher dimensional nonlinear systems admit more rich coherent structures, such as dromions [1,2] and solitoffs [3]. As we know, dromions are exact localized solutions of (2 + 1)-dimensional equations and decay exponentially in all directions [1,2], solitoffs constitute an intermediate state between dromions and plane solitons, since they decay exponentially in all directions except a preferred one [3].

Although the coherent structures in the (2 + 1) or higher dimensional nonlinear systems have been reported in both experimental or simulating and theoretical studies, there still exist a lot of open problems need to unreal. Especially the construction of analytical solutions for these coherent structures is still a challenging problem. Not like many methods proposed and widely applied to solve (1 + 1)-dimensional nonlinear wave equations extensively [4–13], there is still no systematic method for (2 + 1)- and higher dimensional equations. The singular manifold method [14] has been applied to construct the localized solutions in the (2 + 1) or higher dimensional nonlinear systems by Peng [15–18] for some specific choices and it is shown that the singular manifold method is powerful in this direction. In this paper, we will take the (2 + 1)-dimensional KdV equations [19] as an example to show there are more coherent structures by applying the singular manifold method [14] in details.

2. The (2 + 1)-dimensional KdV equations and coherent structures

The (2 + 1)-dimensional KdV equations

$$u_t + u_{xxx} - 3uv_x - 3u_x v = 0, \tag{1a}$$
$$u_x = v_y, \tag{1b}$$

were first derived by Botti et al. [19] using the idea of the weak Lax pair.

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According the singular manifold method [14], the solution to Eq. (1) can be truncated as

$$u = \phi^{-2}u_0 + \phi^{-1}u_1 + u_2, \tag{2a}$$

$$v = \phi^{-2}v_0 + \phi^{-1}v_1 + v_2, \tag{2b}$$

where $\phi = \phi(x, y, t)$ is the singular manifold variable, $u_i = u_i(x, y, t)$ and $v_i = v_i(x, y, t), i = 0, 1, 2.$

Although the (2 + 1)-dimensional KdV Eq. (1) have been studied by the singular manifold method extensively [15–18] there are still many problems unsolved. For example, the u_2 and/or v_2 were usually taken as zero, although this assumption will simplify the process in solving the equations, it will also let us lose some solutions.

Substituting Eq. (2) into Eq. (1) equating the coefficients with the same powers of ϕ, one gets

$$u_0 = 2\phi_x\phi_y, \quad v_0 = 2\phi_x^2, \tag{3a}$$

$$u_1 = -\phi_{xy}, \quad v_1 = -2\phi_{xx}. \tag{3b}$$

and u_2 and v_2 satisfy Eq. (1), where ϕ satisfies the following set of equations

$$[(\phi_t + \phi_{xxx})_y - 3(u_2\phi_{xx} + v_2\phi_{xy})]_x = 0, \tag{4a}$$

$$[\phi_y(\phi_t + \phi_{xxx}) + 3(\phi_x\phi_{xxy} - \phi_{xx}\phi_{xy}) - 3\phi_x(u_2\phi_x + v_2\phi_y)]_x + \phi_x[(\phi_t + \phi_{xxx})_y - 3(u_2\phi_{xx} + v_2\phi_{xy})] = 0, \tag{4b}$$

$$\phi_y(\phi_t + \phi_{xxx}) + 3(\phi_x\phi_{xxy} - \phi_{xx}\phi_{xy}) - 3\phi_x(u_2\phi_x + v_2\phi_y) = 0. \tag{4c}$$

If ϕ satisfies

$$(\phi_t + \phi_{xxx})_y - 3(u_2\phi_{xx} + v_2\phi_{xy}) = 0, \tag{5a}$$

$$\phi_y(\phi_t + \phi_{xxx}) + 3(\phi_x\phi_{xxy} - \phi_{xx}\phi_{xy}) - 3\phi_x(u_2\phi_x + v_2\phi_y) = 0, \tag{5b}$$

then the ϕ also satisfies (4).

Remark 1. It is obvious that Eq. (4b) is a combination of (4a) and (4c). But Eq. (4b) is not a simple combination of (4a) and (4c), if (4a) and (4c) hold, we know the Eq. (4b) is correct, too. However, if the Eq.(4b) hold, we cannot derive (4a) and (4c), directly.

Remark 2. It is obvious that (5b) is just the same as (4c). (5a) is special case of (4a), where if (5a) holds, then (4a) is correct, but we can not derive (5a) from (4a) directly.

Contrary to making any assumption on u_2 and v_2, we can derive u_2 and v_2 from Eqs. (5), i.e.

$$u_2 = \frac{1}{3(\phi_{xx}\phi_y - \phi_x\phi_{xy})}\left[\phi_y(\phi_t + \phi_{xxx})_y - \frac{\phi_{xy}\phi_y(\phi_t + \phi_{xxx})}{\phi_x} + \frac{3\phi_{xy}(\phi_x\phi_{xxy} - \phi_{xx}\phi_{xy})}{\phi_x}\right], \tag{6a}$$

$$v_2 = -\frac{1}{3(\phi_{xx}\phi_y - \phi_x\phi_{xy})}\left[\phi_x(\phi_t + \phi_{xxx})_y - \frac{\phi_{xx}\phi_y(\phi_t + \phi_{xxx})}{\phi_x} + \frac{3\phi_{xx}(\phi_x\phi_{xxy} - \phi_{xx}\phi_{xy})}{\phi_x}\right]. \tag{6b}$$

If ϕ satisfies

$$\phi_x\phi_{xxy} - \phi_{xx}\phi_{xy} = 0, \tag{7}$$

then (6) becomes

$$u_2 = \frac{1}{3(\phi_{xx}\phi_y - \phi_x\phi_{xy})}\left[\phi_y(\phi_t + \phi_{xxx})_y - \frac{\phi_{xy}\phi_y(\phi_t + \phi_{xxx})}{\phi_x}\right], \tag{8a}$$

$$v_2 = -\frac{1}{3(\phi_{xx}\phi_y - \phi_x\phi_{xy})}\left[\phi_x(\phi_t + \phi_{xxx})_y - \frac{\phi_{xx}\phi_y(\phi_t + \phi_{xxx})}{\phi_x}\right]. \tag{8b}$$

As mentioned in Ref. [14], if the arbitrary function ϕ takes a separable form, then from (4) we can derive u_2 and v_2. For any given ϕ, we can substitute u_2 and v_2 derived from (6) or (7) and (8) back into Eqs. (1), if u_2 and v_2 are solutions of Eq. (1), then we get a set of solutions for Eq. (1) expressed by Eq. (2).

In the following part, we will show that three different choices of separable ϕ will result in more coherent structures.

2.1. $\phi = e^{x+t}g(y)$ and coherent structures

If ϕ takes the following separable form

$$\phi = e^{x+t}g(y), \tag{9}$$

where $g(y)$ is an arbitrary function, then we have

$$u_2 = u_2(y) = \frac{2g_y(y)}{3g(y)}, \quad v_2 = 0, \tag{10}$$

obviously, this is a solution to Eq. (1).

Substitute $\phi = e^{x+t}g(y)$ back into (2), the solution to Eq. (1) can be written as

$$u = \frac{2g_y(y)}{3g(y)}, \quad v = 0. \tag{11}$$

Thanks to the arbitrariness of function $g(y)$, we may obtain a diversity of exact solutions to Eq. (1) by choosing this function.

Case 1: If $g(x) = \mathrm{dn}(y, m)$, then the solution to Eq. (1) can be expressed as

$$u = -\frac{2m^2 \mathrm{sn}(y, m)\mathrm{cn}(y, m)}{3\mathrm{dn}(y, m)}, \tag{12}$$

where $\mathrm{sn}(y, m), \mathrm{cn}(y, m)$ and $\mathrm{dn}(y, m)$ are the Jacobi elliptic sine function, the Jacobi elliptic cosine function and the Jacobi elliptic function of the third kind with its modulus m $(0 < m < 1)$ [20,21], respectively.

In this case, field u is a periodic wave along y-direction, which is also one kind of breather lattice solutions [22,23], Fig. 1a illustrates shock wave spatial structure for u when $m = 1$.

Case 2: If $g(x) = 1 + \mathrm{dn}(y, m)$, then the solution to Eq. (1) can be expressed as

$$u = -\frac{2m^2 \mathrm{sn}(y, m)\mathrm{cn}(y, m)}{3(1 + \mathrm{dn}(y, m))}. \tag{13}$$

In this case, field u is also a periodic wave along y-direction, which is also one kind of breather lattice solutions [22,23], Fig. 1b illustrates its spatial structure when $m = 1$, which is not a single shock wave.

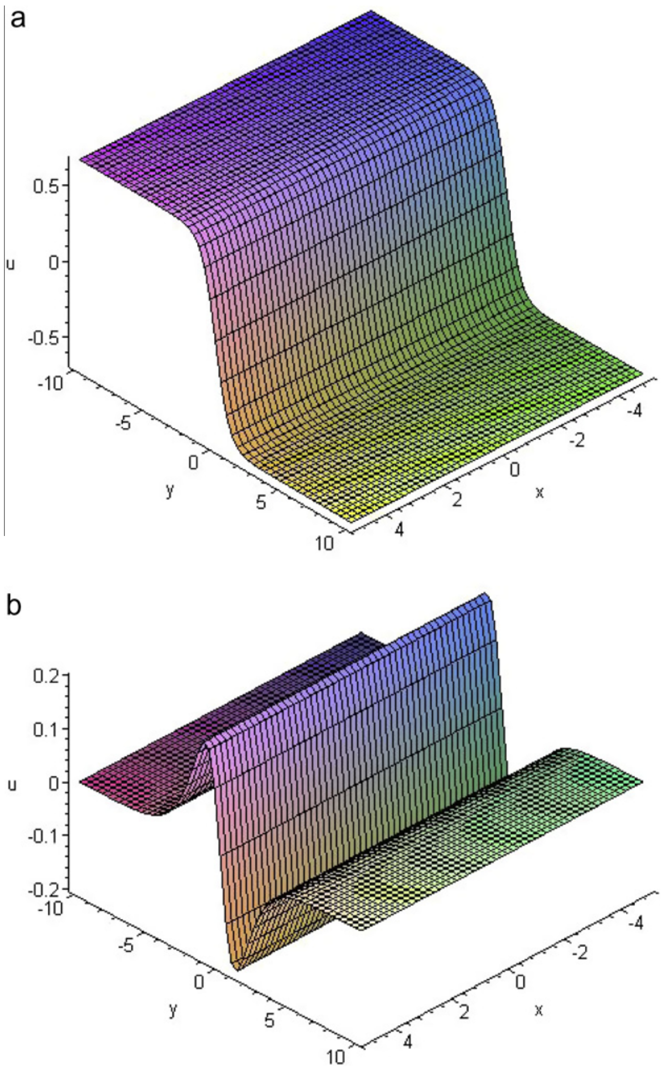

Fig. 1. A typical spatial structure of (a) Eq. (12) for u and (b) Eq. (13) for u.

Case 3: If $g(x) = 2 + \mathrm{sn}(y, m)$, then the solution to Eq. (1) can be expressed as

$$u = \frac{2\mathrm{dn}(y, m)\mathrm{cn}(y, m)}{3(2 + \mathrm{sn}(y, m))}. \tag{14}$$

In this case, field u is also a periodic wave along y-direction, which is also one kind of breather lattice solutions [22,23], Fig. 2 illustrates its spatial structure when $m = 1$, which is bell-shaped solitary wave.

2.2. $\phi = f(x)e^{y+t} + h(y)$ and coherent structures

If ϕ takes the following separable form

$$\phi = f(x)e^{y+t} + h(y), \tag{15}$$

where $f(x)$ and $h(y)$ are two arbitrary functions, then we have

$$u_2 = 0, \quad v_2 = v_2(x) = \frac{f + f_{xxx}}{3f_x}, \tag{16}$$

obviously, this is another solution to Eq. (1).

Substitute (15) back into (2), the solution to Eq. (1) can be written as

$$u = \frac{2f_x(h_y - h)e^{y+t}}{[fe^{y+t} + h]^2}, \tag{17a}$$

$$v = \frac{2f_x^2 e^{2(y+t)}}{[fe^{y+t} + h]^2} - \frac{2f_{xx}e^{y+t}}{fe^{y+t} + h} + \frac{f + f_{xxx}}{3f_x}. \tag{17b}$$

Thanks to the arbitrariness of functions $f(x)$ and $h(y)$, we may obtain a diversity of exact solutions to Eq. (1) by choosing these functions.

Case 1: If $f(x) = e^x$ and $h(y) = \mathrm{sech}\, y$, then the solution to Eq. (1) can be expressed as

$$u = \frac{2\,\mathrm{sech}\, y(\tanh y + 1)e^{x+y+t}}{(e^{x+y+t} + \ \mathrm{sech}\, y)^2}, \tag{18a}$$

$$v = \frac{2e^{2(x+y+t)}}{(e^{x+y+t} + \mathrm{sech}\, y)^2} - \frac{2e^{x+y+t}}{e^{x+y+t} + \mathrm{sech}\, y} + \frac{2}{3}. \tag{18b}$$

In this case, field u is a anti-solitoff while the field v is a two-interacting-soliton structure, Fig. 3 illustrates intuitively the typical spatial structure u and v at $t = 1$.

Case 2: If $f(x) = e^{-x^2}$ and $h(y) = \mathrm{sech}\, y$, then the solution to Eq. (1) can be expressed as

$$u = \frac{4x\,\mathrm{sech}\, y(\tanh y + 1)e^{-x^2+y+t}}{(e^{-x^2+y+t} + \mathrm{sech}\, y)^2}. \tag{19}$$

In this case, field u is a symmetric solitoff and anti-solitoff with hamonic motions while the field v is a line-soliton structure (with expression and figure not shown), Fig. 4 illustrates intuitively the typical spatial structure u at $t = 1$, this is a new coherent structure, which has not been reported by Peng [15–18].

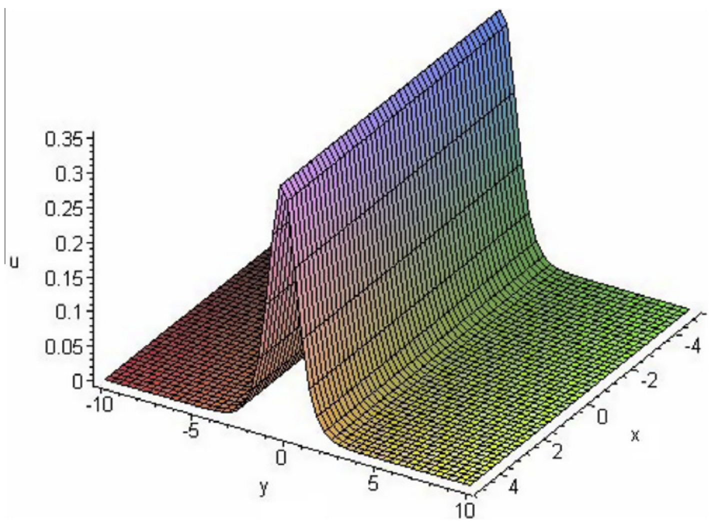

Fig. 2. A typical spatial structure of Eq. (14).

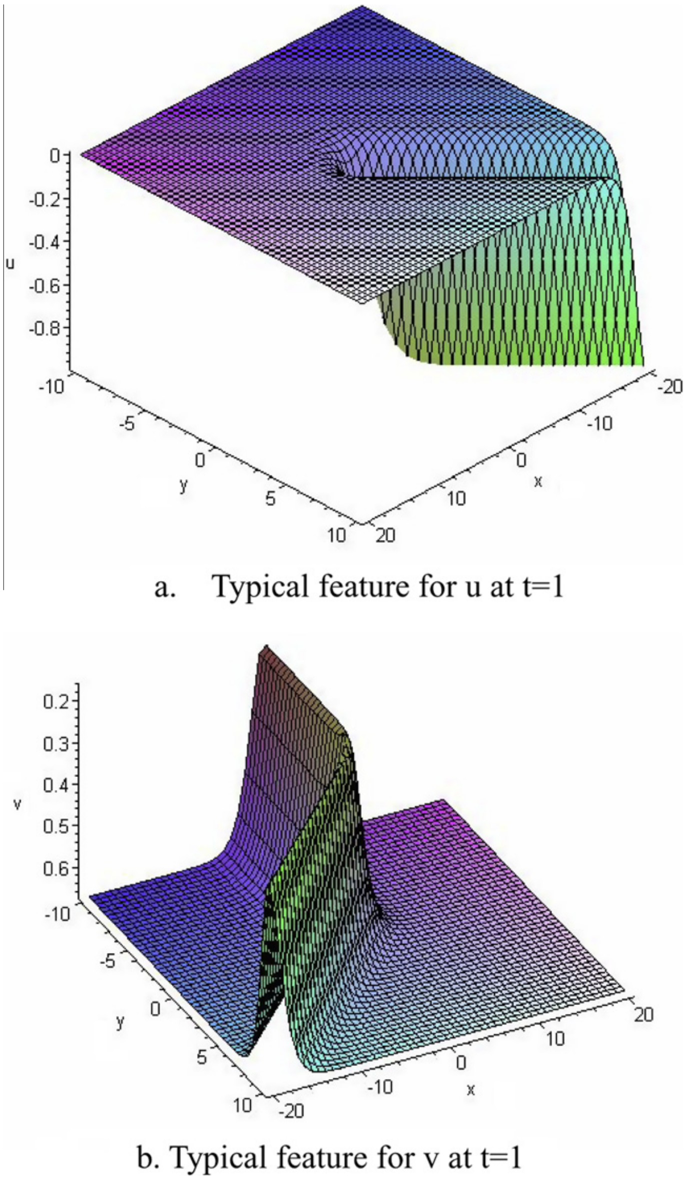

Fig. 3. A typical spatial structure of Eq. (18): (a) for u and (b) for v.

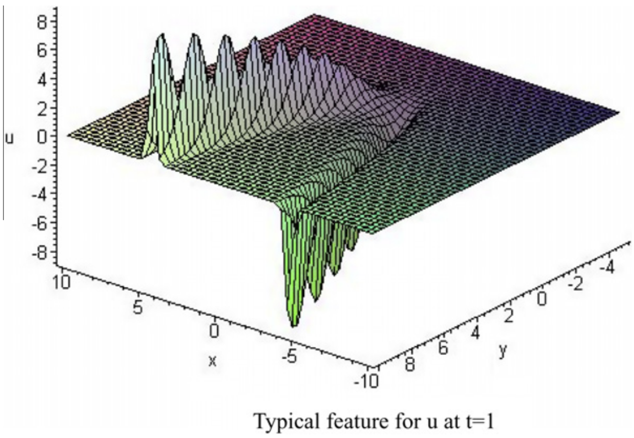

Fig. 4. A typical spatial structure of Eq. (19).

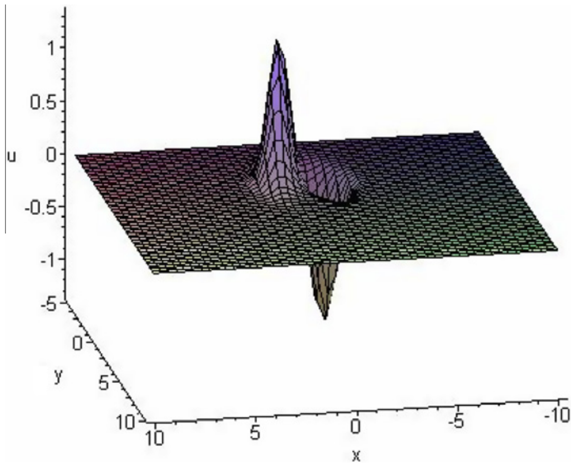

Fig. 5. A typical spatial structure of Eq. (20).

Case 3: If $f(x) = e^{-x^2}$ and $h(y) = \cosh y$, then the solution to Eq. (1) can be expressed as

$$u = -\frac{4x(\sinh y - \cosh y)e^{-x^2+y+t}}{(e^{-x^2+y+t} + \cosh y)^2}. \tag{20}$$

In this case, field u is a symmetric dromion and anti-dromion structure while the field v is a line-soliton structure (with expression and figure not shown), Fig. 5 illustrates intuitively the typical spatial structure u at $t = 1$.

Case 4: If $f(x) = \cosh x$ and $h(y) = \cosh y$, then the solution to Eq. (1) can be expressed as

$$u = \frac{2\sinh x(\sinh y - \cosh y)e^{y+t}}{(\cosh x e^{y+t} + \cosh y)^2}. \tag{21}$$

In this case, field u is a symmetric solitoff and anti-solitoff while the field v is a line-soliton structure (with expression and figure not shown), Fig. 6 illustrates intuitively the typical spatial structure u at $t = 1$.

2.3. $\phi = f(x)g(y,t) + h(y)$ and coherent structures

In fact, ϕ can also be extended to

$$\phi = f(x)g(y,t) + h(y), \tag{22}$$

where $f(x), g(y,t)$ and $h(y)$ are three arbitrary functions, then we have

$$u_2 = 0, \quad v_2 = v_2(x,t) = \frac{fg_t + gf_{xxx}}{3gf_x}, \tag{23}$$

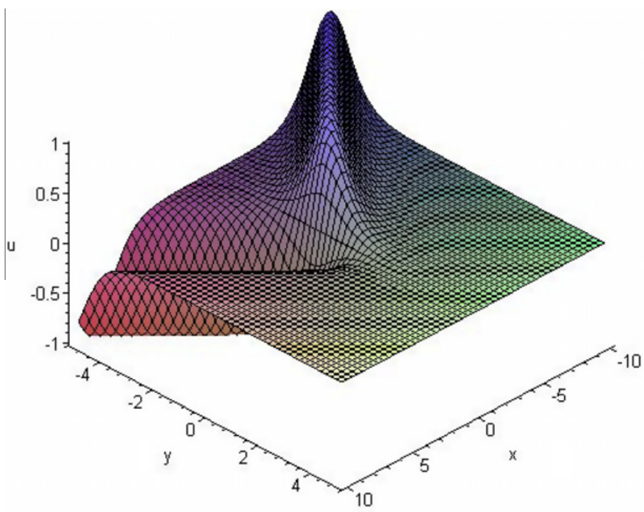

Fig. 6. A typical spatial structure of Eq. (21).

obviously, this is another solution to Eq. (1).

Substitute (22) back into (2), the solution to Eq. (1) can be written as

$$u = \frac{2f_x(gh_y - g_y h)}{(fg+h)^2}, \tag{24a}$$

$$v = \frac{2f_x^2 g^2}{(fg+h)^2} - \frac{2f_{xx}g}{fg+h} + \frac{fg_t + f_{xxx}g}{3f_x g}. \tag{24b}$$

Thanks to the arbitrariness of functions $f(x), g(y,t)$ and $h(y)$, we may obtain a diversity of exact solutions to Eq. (1) by choosing these functions.

Case 1: If $f(x) = \operatorname{sech} x, g(y,t) = \operatorname{sech}(y+t)$ and $h(y) = \operatorname{sech} y$, then the solution to Eq. (1) can be expressed as

$$u = \frac{2\operatorname{sech} x \tanh x \operatorname{sech}(y+t)\operatorname{sech} y[\tanh y - \tanh(y+t)]}{[\operatorname{sech} x \operatorname{sech}(y+t) + \operatorname{sech} y]^2}, \tag{25a}$$

$$v = \frac{2\operatorname{sech}^2 x \tanh^2 x \operatorname{sech}^2(y+t)}{[\operatorname{sech} x \operatorname{sech}(y+t) + \operatorname{sech} y]^2} - \frac{2\operatorname{sech} x[2\tanh^2 x - 1]}{\operatorname{sech} x \operatorname{sech}(y+t) + \operatorname{sech} y} + \frac{\tanh(y+t) + 6\tanh^3 x - 5\tanh x}{3\tanh x}. \tag{25b}$$

In this case, field u is a dromion and anti-dromion, which takes different analytical form from (20) while the field v is a new kind of line-soliton structure, Fig. 7a and b illustrates intuitively the typical spatial structure u and v at $t = 1$.

Case 2: If $f(x) = \operatorname{sech} x, g(y,t) = e^{-(y+t)^2}$ and $h(y) = \operatorname{sech} y$, then the solution to Eq. (1) can be expressed as

$$u = \frac{2\operatorname{sech} x \tanh x \operatorname{sech} y e^{-(y+t)^2}[\tanh y - 2(y+t)]}{[\operatorname{sech} x e^{-(y+t)^2} + \operatorname{sech} y]^2}. \tag{26}$$

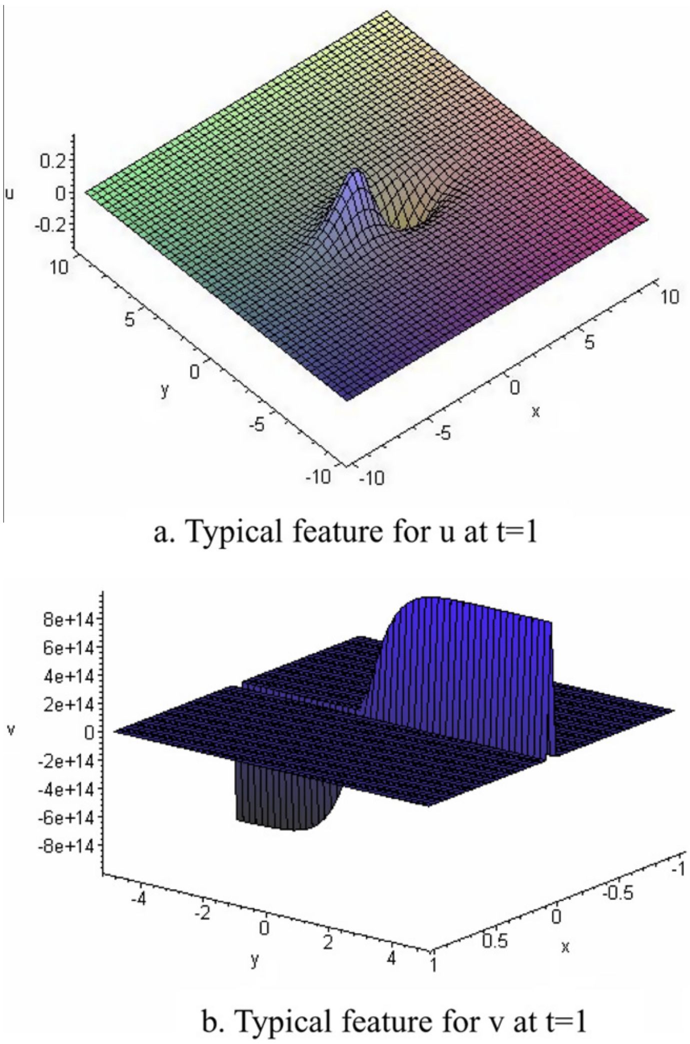

Fig. 7. A typical spatial structure of Eq. (25): (a) for u and (b) for v.

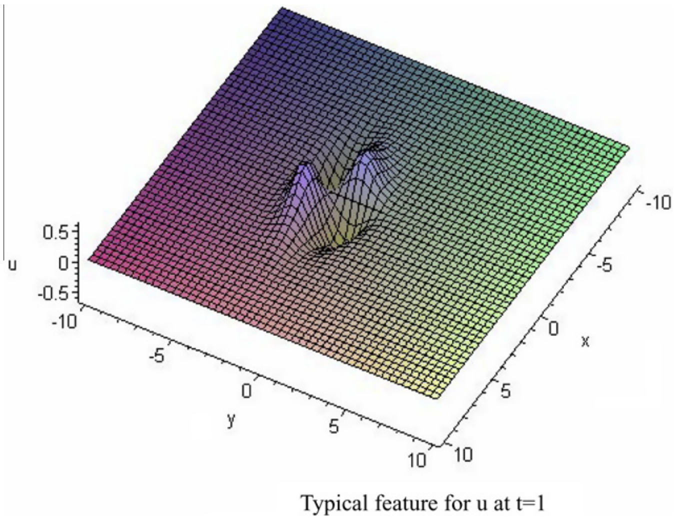

Fig. 8. A typical spatial structure of Eq. (26).

In this case, field u is a symmetric two-dromion and two-anti-dromion structure while the field v is a line-soliton structure (with expression and figure not shown), Fig. 8 illustrates intuitively the typical spatial structure u at $t = 1$.

Case 3: If $f(x) = \operatorname{sech} x, g(y,t) = e^{-(y+t)^2}$ and $h(y) = e^{-y^2}$, then the solution to Eq. (1) can be expressed as

$$u = -\frac{4t\operatorname{sech} x \tanh x e^{-(y+t)^2-y^2}}{[\operatorname{sech} x e^{-(y+t)^2} + e^{-y^2}]^2}. \tag{27}$$

In this case, field u is a solitoff and anti-solitoff structure, which takes different analytical form from (21) while the field v is a line-soliton structure (with expression and figure not shown), Fig. 9 illustrates intuitively the typical spatial structure u at $t = 1$.

Case 4: If $f(x) = e^{-x^2}, g(y,t) = e^{-(y+t)^2}$ and $h(y) = e^{-y^2}$, then the solution to Eq. (1) can be expressed as

$$u = -\frac{8xte^{-[x^2+(y+t)^2+y^2]}}{[e^{-x^2-(y+t)^2} + e^{-y^2}]^2}. \tag{28}$$

In this case, field u is another solitoff and anti-solitoff with hamonic motions, which takes different analytical form from (19) while the field v is a line-soliton structure (with expression and figure not shown), Fig. 10 illustrates intuitively the typical spatial structure u at $t = 1$, which has not been reported by Peng [15–18].

Case 5: If $f(x) = \cosh x, g(y,t) = \cosh 2(y+t) + \cosh(y-t)$ and $h(y) = \cosh y$, then the solution to Eq. (1) can be expressed as

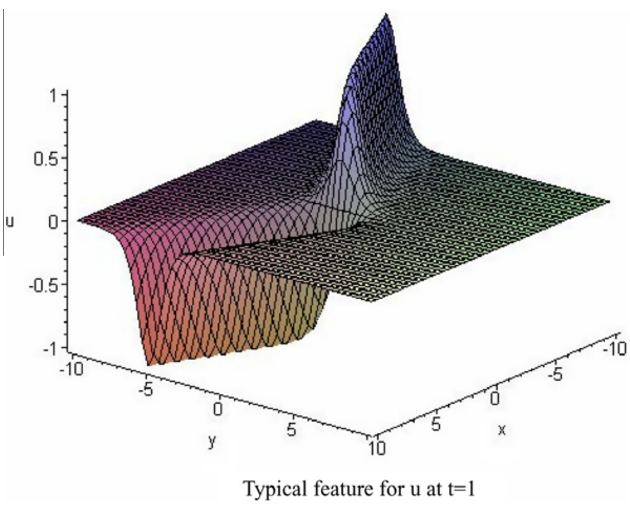

Fig. 9. A typical spatial structure of Eq. (27).

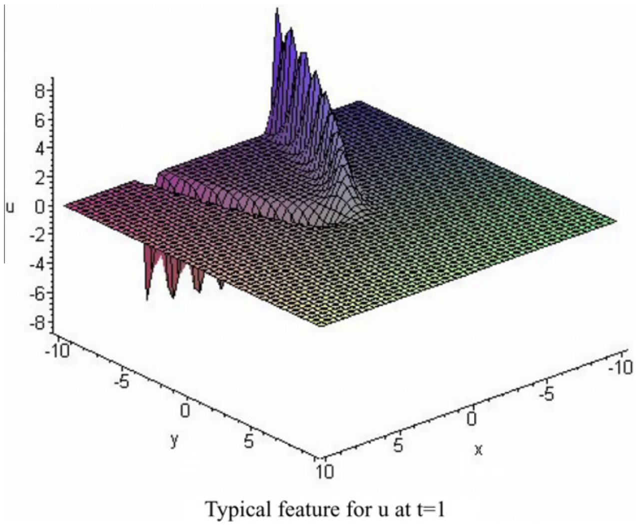

Fig. 10. A typical spatial structure of Eq. (28).

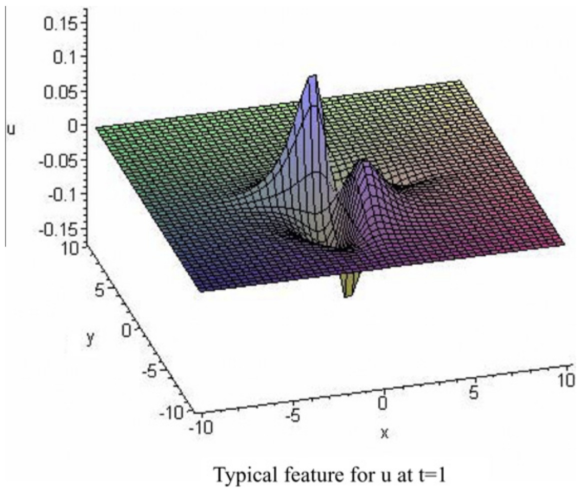

Fig. 11. A typical spatial structure of Eq. (29a).

$$u = \frac{2\,\mathrm{sech}x\,\tanh x\,\mathrm{sech}(y+t)\mathrm{sech}y[\tanh y - \tanh(y+t)]}{[\mathrm{sech}x\,\mathrm{sech}(y+t) + \mathrm{sech}y]^2}, \tag{29a}$$

$$v = \frac{2\sinh x\{[\cosh 2(y+t) + \cosh(y-t)]\sinh y - \cosh y[2\sinh(y+t) + \sinh(y-t)]\}}{\{\cosh x[\cosh 2(y+t) + \cosh(y-t)] + \cosh y\}^2}. \tag{29b}$$

In this case, field u is an asymmetric two-dromion and two-anti-dromion while the field v is a new kind of line-soliton structure, Fig. 11 illustrates intuitively the typical spatial structure u at $t = 1$.

3. Conclusion and discussion

In this paper, the singular manifold method is applied to the (2 + 1)-dimensional KdV equations, certain special coherent structures have been obtained because of the existence of arbitrary functions in singular variable ϕ. Three cases for the singular variable ϕ have been considered, it has only one arbitrary function, two arbitrary functions and three arbitrary functions, respectively. Here we can find that even the analytical expression is different, the similar coherent structures can be presented, and more new coherent structures have been obtained, too. So more applications of this method to other (2 + 1)-dimensional or higher dimensional nonlinear equations to derive more new structures deserves to be studied further.

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