粒子物理

16. 规范对称性

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Quantum Mechanics

- Group operations represented by unitary operators (u) in a linear vector space of state vector $|\alpha\rangle$

state vector transformation: $|\alpha\rangle \to |\alpha'\rangle = u\,|\alpha\rangle$

operator transformation: $\theta \to \theta' = u\theta u^{-1}$

- If system is symmetric under group, $[H, u] = 0$

- Of particular interest are symmetry groups with representation like

$$u(\epsilon) = e^{-i\sum_j \epsilon^j Q^j}$$

infinitesimal parameters

Generators of the group & operators having quantum #'s as eigenvalues

- Connection through 'charge' & conserved 'current'

$$Q \equiv \int d^3x j^0(x) \qquad \partial_\mu j^\mu(x) = 0$$

Quantum Field Theory

$\phi(x)$ is an operator

$$\phi \to \phi' = u\phi u^{-1}$$

$$= (1 - i\sum_j \epsilon^j Q^j)\phi(1 + i\sum_j \epsilon^j Q^j)$$

$$= \cdots = \phi + i\sum_j \epsilon^j [Q^j, \phi(x)]$$

so $[Q^j, \phi(x)] = 0$ → symmetry → conservation law

Note: often $u\phi u^{-1} = \exp(i\sum_j \epsilon^j q^j)\phi(x)$

eigenvalues of Q^j

• 如果 $[H, Q] = 0 \Rightarrow \partial_0 Q = 0$

令 $H|\vec{p}_n\rangle = E_n|\vec{p}_n\rangle$

则 $QH|\vec{p}_n\rangle = E_n \underline{Q|\vec{p}_n\rangle}$
$\quad \|$
$H\underline{Q|\vec{p}_n\rangle}$

$|\vec{p}_n\rangle$ 和 $Q|\vec{p}_n\rangle$ 都是 $\hat{H}$ 的本征值 E_n 的本征态（简并）

→ 它们有可能表示具有不同量子数的正交态

• 真空 —— H 的特殊本征态

$$H|0\rangle = 0$$

我们常假设 $U|0\rangle = 0$ ，$U = e^{i\hat{Q}\alpha}$

$$\Rightarrow Q|0\rangle = 0$$

如果 $\hat{Q}|0\rangle \neq 0$ 而且 $[\hat{H}, Q] = 0$，那么一定存在简并的真空

Internal Symmetry

- **Symmetries whose transformation parameters do not affect the point of space and time x**
- **It is more natural in QM and QFT. For example, the phase of the wave function. Equation of Motion (Dirac or Schrodinger), normalization condition are invariant under the transformation:**

$$\Psi(x) \rightarrow e^{i\theta}\Psi(x)$$

- **It implies the conservation of the probability current.**

Heisenberg Isospin Theory

- **Assume the strong interaction are invariant under a group of SU(2) transformation in which the proton and neutron form a doublet N(x)**

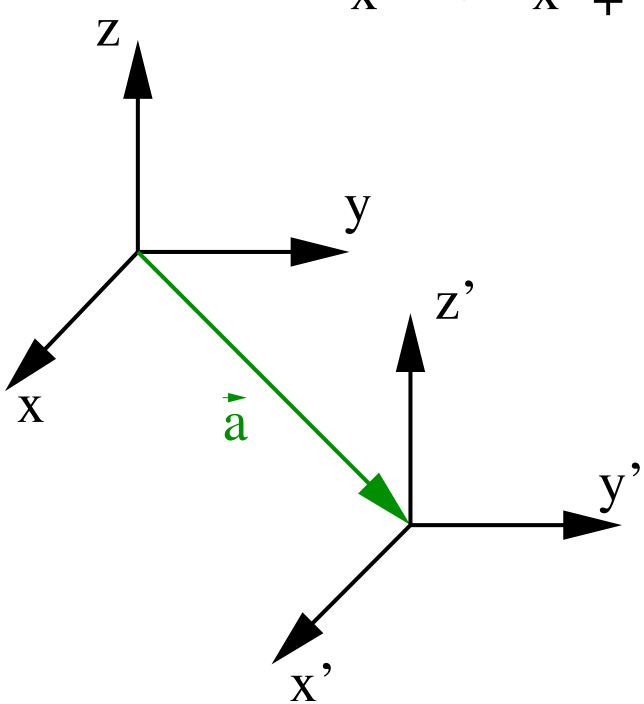
$$N(x) = \begin{pmatrix} p(x) \\ n(x) \end{pmatrix} \quad ; \quad N(x) \to e^{i\vec{\tau}\cdot\vec{\theta}} N(x)$$

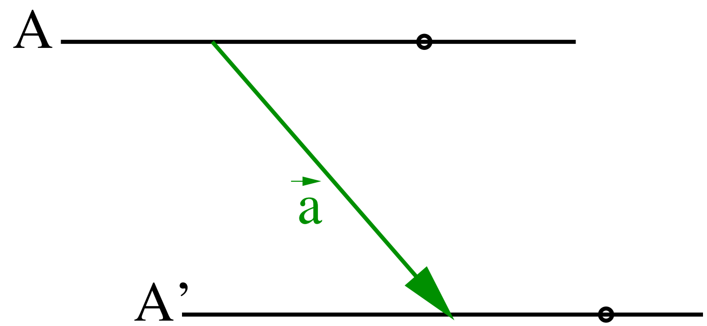
$\vec{\tau}$ are proportional to Pauli matrices

$\vec{\theta}$ are the three angles of a general rotation in a three dimensional Euclidean space

Global Symmetry

$$\vec{x} \longrightarrow \vec{x} + \vec{a}$$





A is trajectory of a free particle in the (x,y,z) system

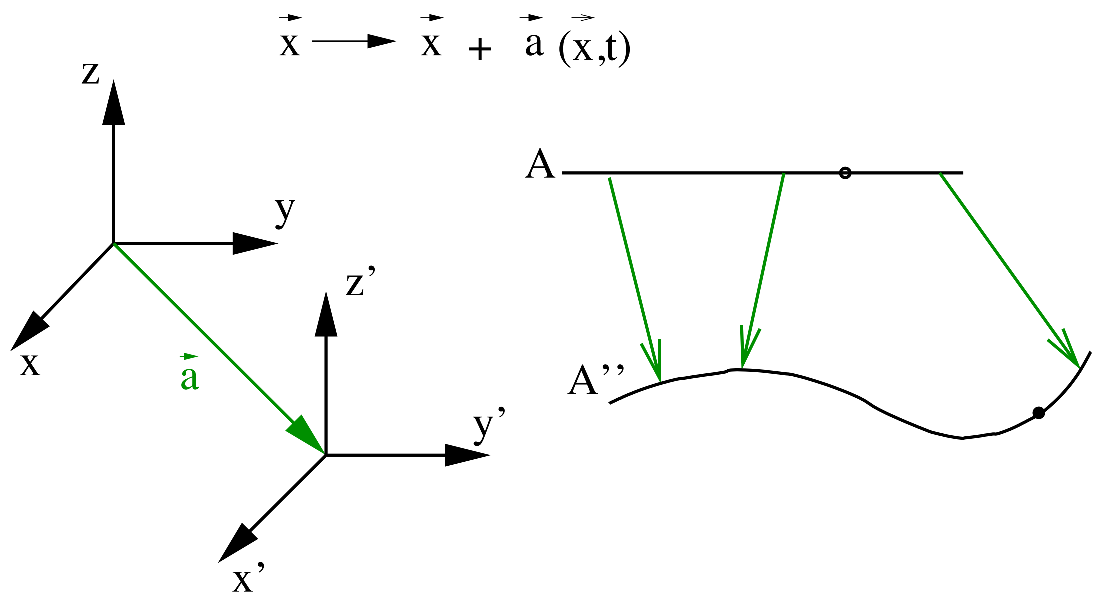
A' is also a possible trajectory of a free particle in the new system

The dynamics of free particles is invariant under space translations by a constant vector

Gauge Transformation

The transformation parameters are functions of the space-time point x

A free particle dynamics is not invariant under translations in which $\vec{a}$ is replaced by $\vec{a}(x)$.



For A" to be a trajectory, the particle must be subject to external forces

Symmetry= Force

Neither Dirac nor Schrodinger equation are invariant under a local change of phase $\theta(x)$

Free Dirac Lagrangian

$$\mathcal{L} = \bar{\Psi}(x)(i\partial\!\!\!/ - m)\Psi(x)$$

is not invariant under the transformation

$$\Psi(x) \rightarrow e^{i\theta(x)}\Psi(x) \quad \Longrightarrow \quad \partial_\mu \theta(x)$$

In order to restore invariance, we must modify free Dirac Lagrangian such that it is no longer describe a free Dirac Field.

$\Longrightarrow$ **Invariance under gauge symmetry leads to the introduction of interactions.**

Weyl's Gauge Transformation

Soon after GR was written by Einstein,
Weyl proposed a modification …

He added invariance with respect to

a) $g'_{\mu\nu} = \lambda(x) g_{\mu\nu}$

b) $A'_\mu = A_\mu - \dfrac{\partial \lambda(x)}{\partial x^\mu}$

same $\lambda(x)$ phase

b) is the regular ambiguity required of EM potentials

a) is weird $\Rightarrow$ $ds^2 = g_{\mu\nu} dx^\mu dx^\nu \to \lambda ds^2$

Lengths are
re-'gauged'

Weyl's Gauge Transformation

- **suggests an invariance even though space & time can change over all space and time**
- **the mediator which holds the space-time structure together would be the electromagnetic field**

An early attempt to unify gravitation with electromagnetism

The brilliant idea did not work but the name stuck.

In 1927 London revived the idea … but the symmetry isn't the scale of space-time, rather the phase of the wave function.

⊛ 全局对称性

阿贝尔对称性是因为 $e^{i\varepsilon} \equiv e^{i\mathbb{I}\varepsilon}$

其中 $\mathbb{I}$ 是 U(1) 群的生成元满足 $[\mathbb{I}, \mathbb{I}] = 0$

（U 是幺正矩阵，$UU^{\dagger}=1$，$U=U^{-1}$）

1) Abelian U(1) 对称性

以标量场为例，拉氏量为

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - \mu^2(\phi^*\phi) - \lambda(\phi^*\phi)^2$$

其中 ϕ^* 是复标量场 ϕ 的复共轭

在 U(1) 变换下，

$$\phi \longrightarrow \phi' = e^{i\varepsilon}\phi \equiv U\phi$$

$$\phi^* \longrightarrow \phi^{*\prime} = e^{-i\varepsilon}\phi^* = U^{\dagger}\phi$$

（ε 是实数且与时空坐标 x 无关）

所以

$$\phi^*\phi \longrightarrow \phi^*\phi$$

$$\partial_\mu\phi \longrightarrow \partial_\mu(e^{i\varepsilon}\phi) = e^{i\varepsilon}\partial_\mu\phi$$

$$\partial_\mu\phi^* \longrightarrow \partial_\mu(e^{-i\varepsilon}\phi^*) = e^{-i\varepsilon}\partial_\mu\phi^*$$

$\Rightarrow$ $\mathcal{L}$ 在 U(1) 变换下保持不变

2) 非阿贝尔 SU(2) 对称性（同位旋对称性）

令 ϕ 是一个同位旋二重态 (isodoublet) $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$

$$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - \mu^2 (\phi^\dagger \phi) - \frac{\lambda}{2} (\phi^\dagger \phi)^2$$

$$\phi^\dagger = (\phi^*)^T$$

此拉氏量在同位旋空间中无穷小转动下不变

$$\phi_j \to \phi_j' = \phi_j + i\varepsilon^a \frac{\tau^a_{jk}}{2} \phi_k \equiv V\phi$$

$j, k = 1, 2$

$a = 1, 2, 3$

ε^a 是实数且与 x 无关

τ^a 是泡利矩阵 $\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

证明：$\phi_j^{+} \to \phi_j^{+\prime} = \phi_j^{+} - i\varepsilon^b \phi_l^{+} \frac{\tau^b_{lj}}{2}$

$$\phi_j^{+}\phi_j \to \phi_j^{+}\phi_j + i\underbrace{\left(\varepsilon^a \frac{\tau^a_{jk}}{2}\phi_k \phi_j^{+} - \varepsilon^b \phi_l^{+}\frac{\tau^b_{lj}}{2}\phi_j\right)}_{\text{将 } l \text{ 和 } j \text{ 指标交换后为 } 0}$$

因为 ε^a 是与 x_μ 无关，所以 $(\partial_\mu \phi^{+})(\partial^\mu \phi)$ 保持不变

—— QED

注意：(1) 我们可以将上述变换记作为 $\phi \to \phi' = V\phi$, $V = e^{i\varepsilon^a \frac{\tau^a}{2}}$

则有 $\phi^+ \to \phi^{+\prime} = \phi^+ V^+$

$$\Rightarrow \phi^+\phi \to \phi^+ V^+ V \phi = \phi^+ \phi \qquad (\because V^+ V = 1)$$

(2) 此变换是非阿贝尔的，因为 $\left[\frac{\tau^a}{2}, \frac{\tau^b}{2}\right] = i\varepsilon_{abc}\frac{\tau^c}{2}$

(3) 任何 2×2 矩阵都可记作为

$$A = c_0 \mathbb{1} + c^a \tau^a \qquad \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(4) $\mathrm{Tr}\left(\frac{\tau^a}{2}\right) = 0 \qquad \mathrm{Tr}\left(\frac{\tau^a}{2}\frac{\tau^b}{2}\right) = \frac{\delta_{ab}}{2}$

$$\{\tau^a, \tau^b\} = \tau^a\tau^b + \tau^b\tau^a = 2\delta_{ab}\mathbb{1}$$

3) Non-abelian SU(3) 对称性

$$\phi \longrightarrow \phi' = U\phi$$

其中 $U = e^{iH}$, $tr(H) = 0$, $det(U) = 1$

对于 SU(3) 群的基础表示 $\underline{3}$，我们可以将 H 表示为 8 个 3×3 的无迹的矩阵 λ^a：

$$U = e^{i\varepsilon^a \frac{\lambda^a}{2}}, \quad a = 1, 2, \cdots, 8$$

其中 ε^a 是实的连续参数

λ^a 是 SU(3) 的生成元

1962年 Gell-mann 将 λ^a 取作如下8个

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

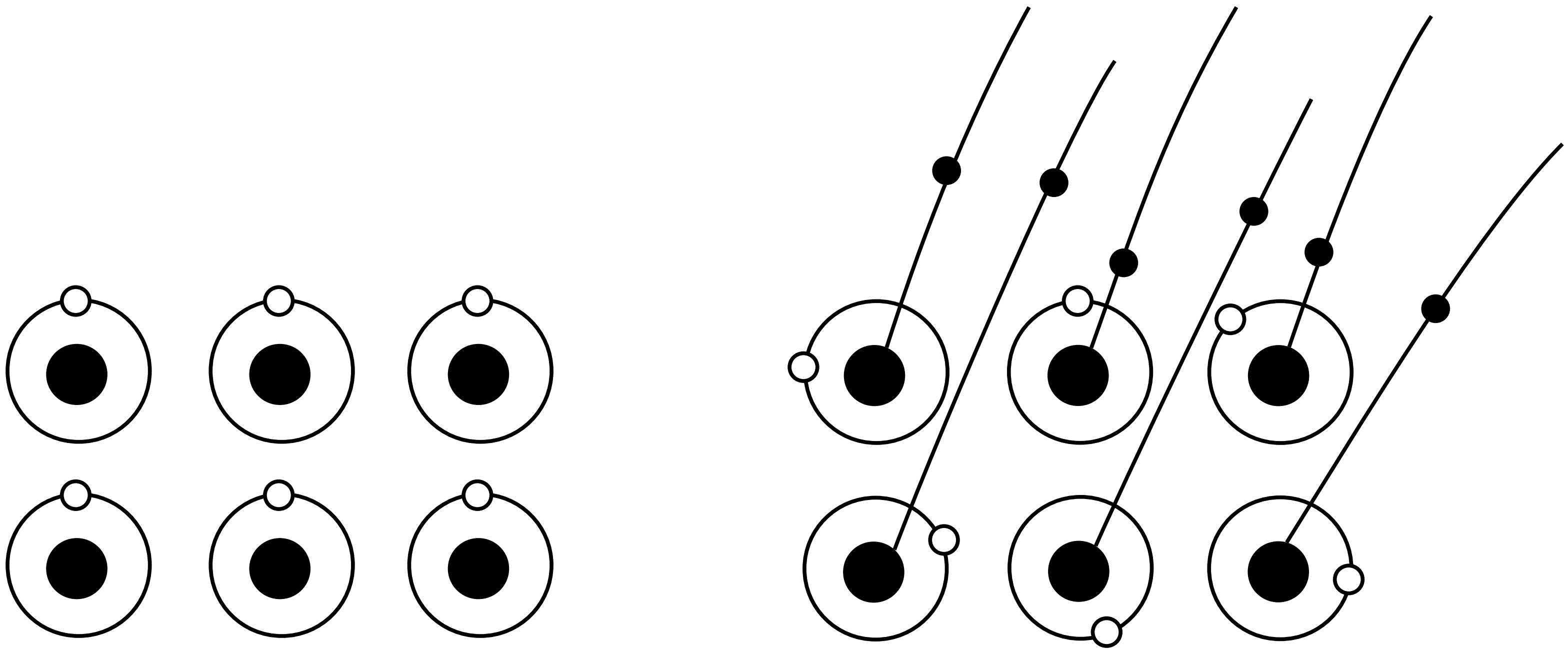
$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Note: (1) $\left[\frac{\lambda^a}{2}, \frac{\lambda^b}{2}\right] = i f^{abc} \lambda^c$, f^{abc} 是 SU(3) 的结构常数, 全反对称

$$\{\lambda^a, \lambda^b\} = \lambda^a\lambda^b + \lambda^b\lambda^a = 2 \underbrace{d_{abc}}_{\text{totally symmetric}} \lambda^c + \frac{4}{3}\delta_{ab}$$

(2) $\mathrm{Tr}\left(\frac{\lambda^a}{2}\right) = 0 \qquad \mathrm{Tr}\left(\frac{\lambda^a}{2}\frac{\lambda^b}{2}\right) = \frac{1}{2}\delta^{ab}$

Global versus Local



Global U(1) gauge transformation

Local U(1) gauge transformation

局域对称性 (Gauge Symmetries)

前面我们所讨论的相位变换的参数 ε^a 都与时空无关

$\Rightarrow$ 不同时空点上的场要同时变化相同的相位

下面我们考虑 $\varepsilon^a = \varepsilon^a(x)$，这样的局域对称性（与 x 相关）

可以给出动力学，即规范相互作用

(1) Abelian U(1) local symmetry （例如 Quantum Electrodynamics）

考虑电荷为 (eQ) 的自由费米子场的拉氏量密度

$$\mathcal{L}_0 = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x) \qquad \bar{\psi}(x) \equiv \psi^+\gamma_0$$

1.2) Euler-Lagrangian field equation

$$\partial_\mu \frac{\delta \mathcal{L}_0}{\delta(\partial_\mu \bar{\psi})} - \frac{\delta \mathcal{L}_0}{\delta \bar{\psi}} = 0$$

$$\partial(\bar{\psi}\psi) = \bar{\psi}\partial\psi + (\partial\bar{\psi})\psi$$

$$\mathcal{L}_0 = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

$$= \underbrace{\partial_\mu(\bar{\psi} i\gamma^\mu \psi)}_{\text{全导数项，并不影响作用量 } \int \mathcal{L}_0 d^4x} - [(\partial_\mu\bar{\psi}) i\gamma^\mu\psi] - \bar{\psi} m \psi$$

$$\frac{\delta \mathcal{L}_0}{\delta(\partial_\mu \bar{\psi})} = -i\gamma^\mu\psi, \quad \frac{\delta \mathcal{L}_0}{\partial \bar{\psi}} = -m\psi$$

$$\Longrightarrow \quad -i\gamma^\mu\partial_\mu\psi - (-m\psi) = 0 \quad \text{即} \quad (i\not{\partial} - m)\psi = 0$$

1.2) Euler-Lagrangian field equation

$$\partial_\mu \frac{\delta \mathcal{L}_0}{\delta(\partial_\mu \bar{\psi})} - \frac{\delta \mathcal{L}_0}{\delta \bar{\psi}} = 0$$

$$\partial(\bar{\psi}\psi) = \bar{\psi}\partial\psi + (\partial\bar{\psi})\psi$$

$$\mathcal{L}_0 = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

$$= \underbrace{\partial_\mu(\bar{\psi} i\gamma^\mu \psi)}_{\text{全导数项，并不影响作用量 } \int \mathcal{L}_0 d^4x} - [(\partial_\mu \bar{\psi}) i\gamma^\mu \psi] - \bar{\psi} m \psi$$

同理，由 $\partial_\mu \left(\frac{\delta \mathcal{L}_0}{\delta(\partial_\mu \psi)}\right) - \frac{\delta \mathcal{L}_0}{\delta \psi} = 0$ 可得

$$(\partial_\mu \bar{\psi}) i\gamma^\mu + \bar{\psi} m = 0 \quad \text{即} \quad \bar{\psi}(i\gamma^\mu \overleftarrow{\partial_\mu} + m) = 0$$

$$\bar{\psi}(i\overleftarrow{\not{\partial}} + m) = 0$$

箭头表明偏微分做用在左边场量上

1.3) $\mathcal{L}_0$ 具有全局 U(1) 对称性

$$\left.\begin{aligned} \psi(x) &\longrightarrow \psi'(x) = e^{-iQ\alpha}\psi(x) \\ \bar{\psi}(x) &\longrightarrow \bar{\psi}'(x) = e^{+iQ\alpha}\psi(x) \end{aligned}\right\}$$

$\bar{\psi}\psi$ 和 $\bar{\psi}\partial\psi$ 都是不变的

1.4) 下面我们将此全局对称性改为局域对称性

"to gauge the symmetry" $\Longrightarrow$ "$\alpha \longrightarrow \alpha(x)$"

即，

$$\left.\begin{aligned} \psi(x) &\longrightarrow \psi'(x) = e^{-iQ\alpha(x)}\psi(x) \\ \bar{\psi}(x) &\longrightarrow \bar{\psi}'(x) = \bar{\psi}(x)\, e^{+iQ\alpha(x)} \end{aligned}\right\} \Longrightarrow \bar{\psi}\psi \text{ 不变}$$

现在导数项将出现较为复杂的变换形式

$$\bar{\Psi}(x)\,\partial_\mu \psi(x) \longrightarrow \bar{\Psi}'(x)\,\partial_\mu \psi'(x)$$

$$= \bar{\Psi}(x)\, e^{+iQ\alpha(x)}\,\partial_\mu\left(e^{-iQ\alpha(x)}\psi(x)\right)$$

$\partial_\mu e^{iQ\alpha(x)} = iQ\, e^{iQ\alpha(x)}\,\partial_\mu \alpha(x) \Rightarrow$

$$= \bar{\Psi}(x)\,\partial_\mu \psi(x) \underbrace{- iQ\,\bar{\Psi}(x)\left(\partial_\mu \alpha(x)\right)\psi(x)}_{\text{此项破坏了不变性}}$$

为了使导数项在规范变换下保持不变，我们需要定义新的规范协变的导数

$$\partial_\mu \longrightarrow D_\mu$$

并且要求

$$D_\mu \psi(x) \longrightarrow \left[D_\mu \psi(x)\right]' = e^{-iQ\alpha(x)} D_\mu \psi(x) \qquad (A)$$

$\Rightarrow \bar{\Psi} D_\mu \psi(x)$ 是规范不变的

我们引入新的矢量场 $A_\mu(x)$ (规范场) 来定义协变导数

$$D_\mu\psi \equiv (\partial_\mu + ieQA_\mu)\psi \qquad (B)$$

$$[D_\mu\psi]' = (\partial_\mu + ieQA'_\mu)\psi', \quad \psi' = U\psi$$

$$U \equiv e^{-iQ\alpha(x)}$$

$$(A) \Rightarrow (\partial_\mu + ieQA'_\mu)(U\psi) = U(\partial_\mu + ieQA_\mu)\psi$$

$$\Rightarrow (\partial_\mu U)\psi + \underline{U(\partial_\mu\psi)} + ieQA'_\mu U\psi = \underline{U(\partial_\mu\psi)} + ieQUA_\mu\psi$$

$$\Rightarrow [\partial_\mu U + ieQA'_\mu U]\psi = [+ieQUA_\mu]\psi$$

$$\Rightarrow \underbrace{[\partial_\mu U + ieQA'_\mu U - ieQUA_\mu]}_{=0}\psi = 0$$

$$\Rightarrow \underbrace{\left[\partial_\mu U + ieQA_\mu' U - ieQUA_\mu\right]}_{=0}\psi = 0$$

从右方乘以 U^{-1}，可得

$$(\partial_\mu U)U^{-1} + ieQA_\mu' UU^{-1} - ieQUA_\mu U^{-1} = 0$$

因为 $UU^{-1} = 1$，$\partial_\mu(UU^{-1}) = 0 = (\partial_\mu U)U^{-1} + U(\partial_\mu U^{-1})$

$$\Rightarrow (\partial_\mu U)U^{-1} = -U(\partial_\mu U^{-1})$$

所以

$$ieQA_\mu' = ieQUA_\mu U^{-1} - (\partial_\mu U)U^{-1}$$

$$\Rightarrow A_\mu' = UA_\mu U^{-1} - \frac{1}{ieQ}(\partial_\mu U)U^{-1} = U\left(A_\mu + \frac{1}{ieQ}\partial_\mu\right)U^{+}$$

对于无穷小相位变换 $\alpha \ll 1$，

$$U \equiv e^{-iQ\alpha} \simeq 1 - iQ\alpha$$

$$U^{\dagger} = e^{+iQ\alpha} \approx 1 + iQ\alpha$$

$\alpha(x)$ 和 Q 都是实数

则有

$$A'_{\mu}(x) = (1 - iQ\alpha)\left(A_{\mu} - \frac{i}{eQ}\partial_{\mu}\right)(1 + iQ\alpha)$$

$$= (1 - iQ\alpha) A_{\mu} (1 + iQ\alpha) - \frac{i}{eQ}\partial_{\mu}(+iQ\alpha) + O(\alpha^2)$$

$$\Rightarrow A'_{\mu} = A_{\mu} + \frac{1}{e}\partial_{\mu}\alpha(x) + O(\alpha^2)$$

现在我们得到 U(1) 规范不变的拉氏量

$$\mathcal{L}_0' = \bar{\Psi} i\gamma^\mu D_\mu \psi - m\bar{\Psi}\psi$$

$$D_\mu = \partial_\mu + ieQA_\mu$$

So far so good, BUT 我们并不知道 A_μ 的动力学!!!

因为上面的拉氏量中并不保含 A_μ 的导数项，所以

Euler-Lagrange Equation $\Rightarrow \frac{\partial \mathcal{L}}{\partial A_\mu} = -g\bar{\Psi}\gamma_\mu\psi = -g j^\mu = 0$

光子场的运动方程告诉我们，电磁流严格为 0 ($j^\mu = 0$)

$\Rightarrow$ 自由电子的拉氏量 $\bar{\Psi}(i\gamma_\mu\partial^\mu - m)\psi$

为了使规范场 A_μ 成为真正的动力学变量，而不是一个辅助工具，
我们需要在拉氏量中加入 A_μ 的导数项（动能项，kinetic term）

（所有的物理场都具有其特有的动力学。如果某物理场出现在 $\mathcal{L}$ 中却没有动能项，那么该场仅提供一个常数的背景场。）

最简单的量纲为 4 的规范不变项为

$$\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

规范场的反对称 2 阶张量和协变导数之间关系为

$$(D_\mu D_\nu - D_\nu D_\mu)\psi = +ieQ(F_{\mu\nu})\psi$$

$$\Rightarrow (\partial_\mu + ieQA_\mu)(\partial_\nu + ieQA_\nu) - (\partial_\nu + ieQA_\nu)(\partial_\mu + ieQA_\mu) = +ieQF_{\mu\nu}$$

$$\Rightarrow F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

可验证 $$\left[(D_\mu D_\nu - D_\nu D_\mu)\psi\right]' = U(D_\mu D_\nu - D_\nu D_\mu)\psi$$

因为 $$[D_\mu \psi]' \equiv D_\mu' \psi' = U(D_\mu \psi)$$

所以，

$$\left.\begin{aligned} F_{\mu\nu}'\psi' &= U F_{\mu\nu}\psi \\ &= F_{\mu\nu}'(U\psi) \end{aligned}\right\} \Rightarrow \left(F_{\mu\nu}' U - U F_{\mu\nu}\right)\psi = 0$$

从右边乘以 U^{-1} 可得

$$F_{\mu\nu}' = U F_{\mu\nu} U^{-1} = U F_{\mu\nu} U^{+}$$

在QED中，$U = e^{-iQ\alpha(x)}$，故而

$$U F_{\mu\nu} U^{\dagger} = U U^{\dagger} F_{\mu\nu} \Longrightarrow F'_{\mu\nu} = F_{\mu\nu}$$

即，$F_{\mu\nu}$ 本身就是规范不变的

一般而言，$F_{\mu\nu} \equiv [D_\mu, D_\nu] \frac{1}{ieQ}$

$$\mathrm{Tr}(F_{\mu\nu} F^{\mu\nu}) \longrightarrow \mathrm{Tr}(F'_{\mu\nu} F'^{\mu\nu}) = \mathrm{Tr}\left[(U F_{\mu\nu} U^{\dagger})(U F^{\mu\nu} U^{\dagger})\right]$$

$$= \mathrm{Tr}(F_{\mu\nu} F^{\mu\nu})$$

小结：QED 拉氏量为

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

其中 $D_\mu = \partial_\mu + ieQA_\mu$

$$F_{\mu\nu} = \frac{1}{ieQ}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu$$

（e是电荷，Q是费米子 ψ 的电荷，$Q_{e^-} = -1$）

规范变换为

$$D_\mu\psi \to [D_\mu\psi]' = U(D_\mu\psi)$$

$$\psi \to \psi' = U\psi \qquad F_{\mu\nu} \to F'_{\mu\nu} = UF_{\mu\nu}U^+$$

$$\bar{\psi} \to \bar{\psi}' = \bar{\psi}U^+ \qquad A_\mu \to A'_\mu = U\left(A_\mu + \frac{1}{ieQ}\partial_\mu\right)U^+$$

$$U = e^{-iQ\alpha}$$

在 Abelian 理论中，$F'_{\mu\nu} = UF_{\mu\nu}U^+ = F_{\mu\nu}$

$$A'_\mu = A_\mu + \frac{1}{e}\partial_\mu\alpha + O(\alpha^2)$$

小结：QED 拉氏量为

$$\mathcal{L} = \bar{\psi}\,(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

其中 $D_\mu = \partial_\mu + ieQA_\mu$

$$F_{\mu\nu} = \frac{1}{ieQ}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu$$

（e是电荷，Q是费米子ψ的电荷，$Q_{e^-} = -1$）

注意：(1) 光子无质量，因为 $A_\mu A^\mu$ 破坏规范对称性

$$(A_\mu A^\mu)' \neq (A_\mu A^\mu)$$

(2) 光子和光子之间没有相互作用

2) 非阿贝耳规范场 — Yang-Mills fields

Non-abelian SU(3) gauge sym (QCD) Quantum Chromodynamics

1954年 Yang-Mills 将规范原理扩充到非阿贝耳对称性

考虑 SU(3) 色对称性

$$\psi = \begin{pmatrix} \psi_{red} \\ \psi_{green} \\ \psi_{blue} \end{pmatrix}$$

在 SU(3) 变换下，我们有

$$\psi(x) \longrightarrow \psi'(x) = U\psi(x), \quad U = e^{-i\frac{\lambda^a}{2}\theta^a(x)}$$

变换参数

标符

$a = 1, 2, \cdots, 8$

其中，λ^a 是 Gell-mann 矩阵

$$\left[\frac{\lambda^a}{2}, \frac{\lambda^b}{2}\right] = i f_{abc} \frac{\lambda^c}{2}$$

同理，$\bar{\psi}(x) \longrightarrow \bar{\psi}'(x) = \bar{\psi}(x) U^{+}$

$$D_\mu \psi = (\partial_\mu - i\underbrace{g_s}_{\text{强作用耦合常数}} \frac{\lambda^a}{2} \underbrace{G_\mu^a}_{\to\text{胶子场}})\psi$$

要求 $D_\mu \psi \longrightarrow (D_\mu \psi)' = U D_\mu \psi$

$$\Rightarrow \left(\frac{\lambda^a}{2} G_\mu^{a\prime}\right) = U\left(\frac{\lambda^a}{2} G_\mu^a - \frac{1}{(ig_s)}\partial_\mu\right) U^{+}$$

对于无穷小变换有

$$G_\mu^{a\prime} = G_\mu^a + f^{abc}\theta^b G_\mu^c - \frac{1}{g_s}\partial_\mu \theta^a + O(\theta^2)$$

胶子场的二阶反对称张量

$$(D_\mu D_\nu - D_\nu D_\mu)\psi \equiv i g_s \left(\frac{\lambda^a}{2} G^a_{\mu\nu}\right)\psi$$

$$\Rightarrow G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc} G^b_\mu G^c_\nu$$

因为 $(D_\mu \psi)$ 具有和 ψ 相同的规范变换行为，所以

$$[(D_\mu D_\nu - D_\nu D_\mu)\psi]' = U(D_\mu D_\nu - D_\nu D_\mu)\psi$$

$$\Rightarrow \left(\frac{\lambda^a}{2} G^a_{\mu\nu}\right)' = U\left(\frac{\lambda^a}{2} G^a_{\mu\nu}\right)U^+$$

对于无穷小变换，

$$G^{a\,\prime}_{\mu\nu} = G^a_{\mu\nu} + f^{abc}\theta^b G^c_{\mu\nu}$$

同时，

$$\mathrm{Tr}\left(\left(\frac{\lambda^a}{2}G^a_{\mu\nu}\right)\left(\frac{\lambda^b}{2}G^b_{\mu\nu}\right)\right) = G^a_{\mu\nu}G^{b\mu\nu}\underbrace{\mathrm{Tr}\left(\frac{\lambda^a}{2}\frac{\lambda^b}{2}\right)}_{=\frac{1}{2}\delta_{ab}}$$

$$= \frac{1}{2}G^a_{\mu\nu}G^{a\mu\nu}$$

在SU(3)变换下不变

$$\mathrm{Tr}\left(U\left(\frac{\lambda^a}{2}G^a_{\mu\nu}\right)\underbrace{U^\dagger U}_{\mathbb{1}}\left(\frac{\lambda^b}{2}G^b_{\mu\nu}\right)U^\dagger\right)$$

故而，QCD拉氏量为

$$\mathcal{L} = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \bar{\psi}(i\not{D} - m)\psi$$

$$= -\frac{1}{2}\mathrm{Tr}\left(\left(\frac{\lambda^a}{2}G^a_{\mu\nu}\right)\left(\frac{\lambda^b}{2}G^{b\mu\nu}\right)\right) + \bar{\psi}(i\not{D} - m)\psi$$

注意：

① 对于SU(2)规范理论，我们也可得到类似的结论：

生成元	$\lambda^a \ (a=1,2,\cdots,8)$	$\longrightarrow$	$\tau^a \ (a=1,2,3)$
规范玻色子	G_μ^a		W_μ^a
群结构常数	f_{abc}		ε_{abc}
耦合强度	g_s		g_2
	$G_{\mu\nu}^a$		$W_{\mu\nu}^a$

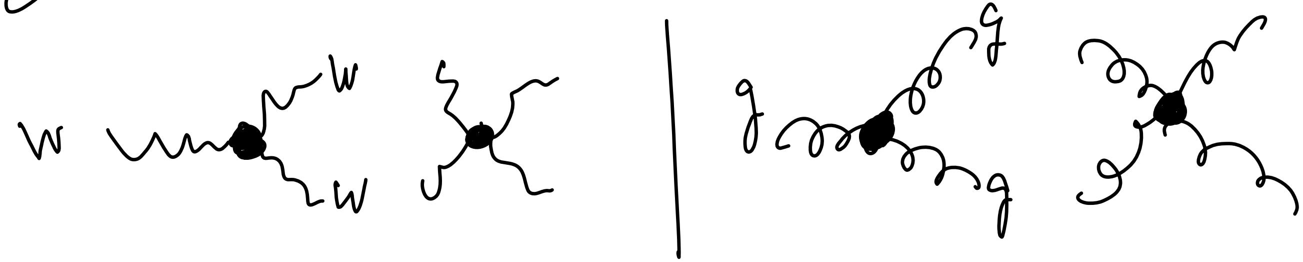
② Yang-Mills项 $-\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu}$ (in SU(3)) 或

$-\frac{1}{4} W^a_{\mu\nu} W^{a\mu\nu}$ (in SU(2)) 中包含规范场的三次和四次项

SU(3): $-g_s f^{abc}(\partial_\mu G^a_\nu) G^{b\mu} G^{c\nu} - \frac{g_s^2}{4} f^{abc} f^{ade} G^b_\mu G^c_\nu G^{d\mu} G^{e\nu}$

SU(2): $-g_2 \varepsilon^{abc}(\partial_\mu W^a_\nu) W^{b\mu} W^{c\nu} - \frac{g_2^2}{4} \varepsilon^{abc} \varepsilon^{ade} W^b_\mu W^c_\nu W^{d\mu} W^{e\nu}$

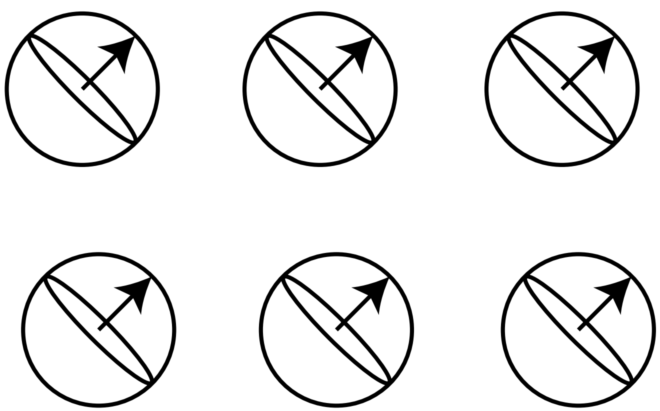
$\Rightarrow$ 非阿贝尔规范场的自相互作用



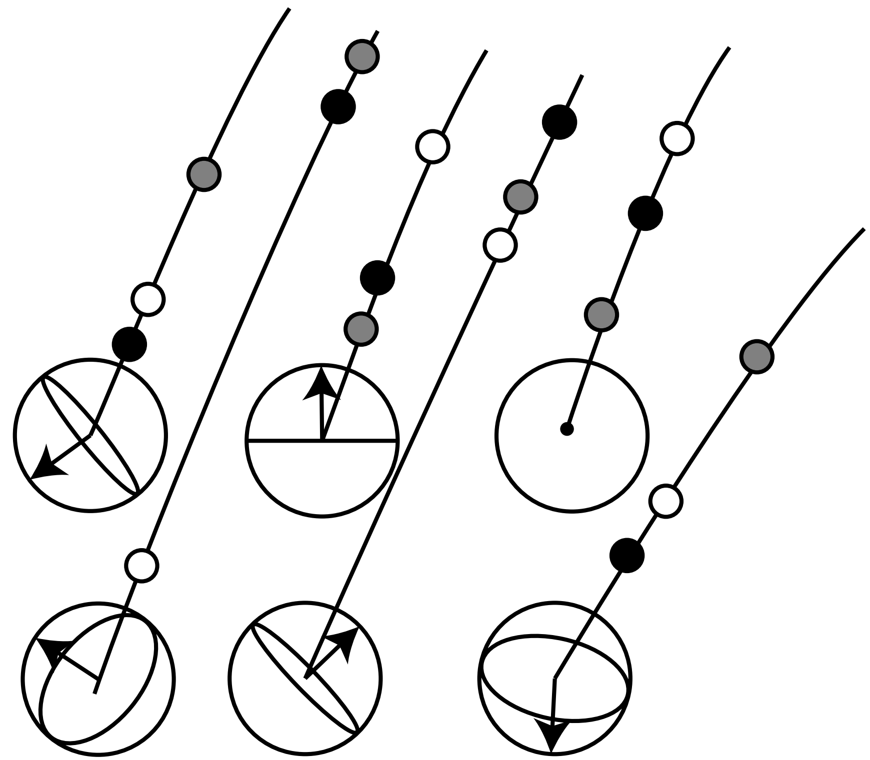
(和阿贝尔规范场不同之处)

$\Rightarrow$ 色禁闭和渐近自由

SU(2): Global versus Local



Global SU(2) gauge transformation



Local SU(2) gauge transformation