粒子物理

4. 衰变宽度、散射截面和相空间

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本节内容基于Thomson教授的幻灯片

Special Relativity and 4-Vector Notation

- Will use 4-vector notation with p^0 as the time-like component, e.g.

$$p^\mu = \{E, \vec{p}\} = \{E, p_x, p_y, p_z\} \qquad \text{(contravariant)}$$

$$p_\mu = g_{\mu\nu} p^\mu = \{E, -\vec{p}\} = \{E, -p_x, -p_y, -p_z\} \qquad \text{(covariant)}$$

with

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- In particle physics, usually deal with relativistic particles. Require all calculations to be Lorentz Invariant. L.I. quantities formed from 4-vector scalar products, e.g.

$$p^\mu p_\mu = E^2 - p^2 = m^2 \qquad \text{Invariant mass}$$

$$x^\mu p_\mu = Et - \vec{p}.\vec{r} \qquad \text{Phase}$$

- A few words on NOTATION

  Four vectors written as either: p^μ or p

  Four vector scalar product: $p^\mu q_\mu$ or $p.q$

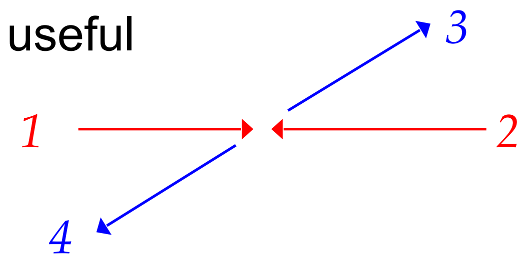
  Three vectors written as: $\vec{p}$

  Quantities evaluated in the centre of mass frame: $\vec{p}^*, p^*$ etc.

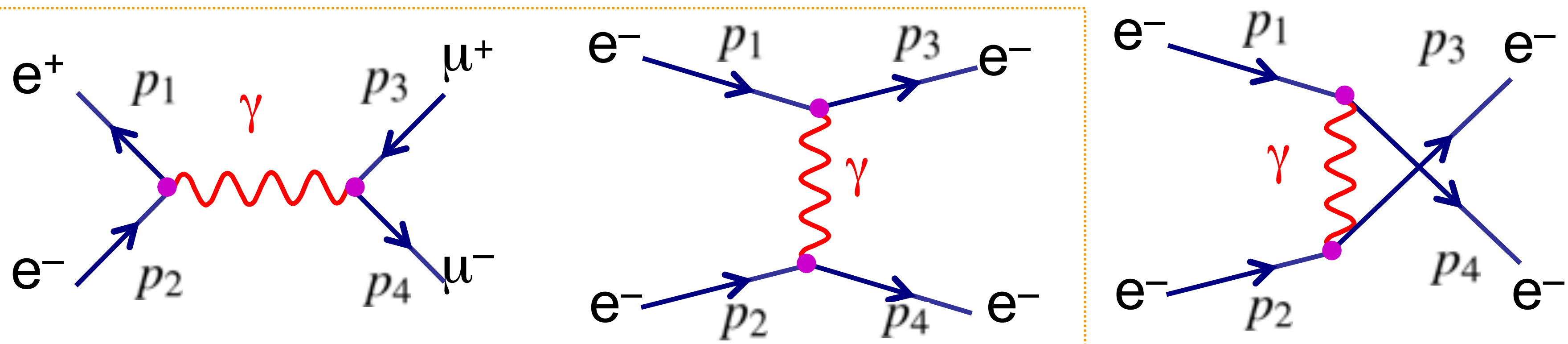
Mandelstam s, t and u

★ In particle scattering/annihilation there are three particularly useful Lorentz Invariant quantities: s, t and u

★ Consider the scattering process $1 + 2 \rightarrow 3 + 4$



★ (Simple) Feynman diagrams can be categorized according to the four-momentum of the exchanged particle



s-channel

t-channel

u-channel

• Can define three kinematic variables: s, t and u from the following four vector scalar products (squared four-momentum of exchanged particle)

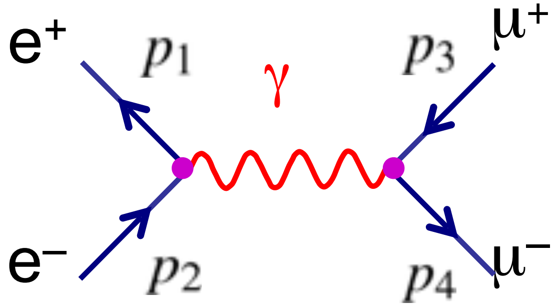
$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2$$

Example: Mandelstam s, t and u

$$\boxed{s=(p_1+p_2)^2, \quad t=(p_1-p_3)^2, \quad u=(p_1-p_4)^2}$$

Note: $s+t+u=m_1^2+m_2^2+m_3^2+m_4^2$

★ e.g. Centre-of-mass energy, s:



$$s=(p_1+p_2)^2=(E_1+E_2)^2-(\vec{p}_1+\vec{p}_2)^2$$

- This is a scalar product of two four-vectors ➡ Lorentz Invariant
- Since this is a L.I. quantity, can evaluate in any frame. Choose the most convenient, i.e. the centre-of-mass frame:

$$p_1^*=(E_1^*,\vec{p}^*) \quad p_2=(E_2^*,-\vec{p}^*)$$

$$\Rightarrow \quad \boxed{s=(E_1^*+E_2^*)^2}$$

★Hence $\sqrt{s}$ is the total energy of collision in the centre-of-mass frame

From Feynman diagrams to Physics

Particle Physics = Precision Physics

★ Particle physics is about building fundamental theories and testing their predictions against precise experimental data

- Dealing with fundamental particles and can make very precise theoretical predictions – not complicated by dealing with many-body systems
- Many beautiful experimental measurements
  → precise theoretical predictions challenged by precise measurements
- For all its flaws, the Standard Model describes all experimental data ! This is a (the?) remarkable achievement of late 20th century physics.

Requires understanding of theory and experimental data

★ Feynman diagrams mainly used to describe how particles interact

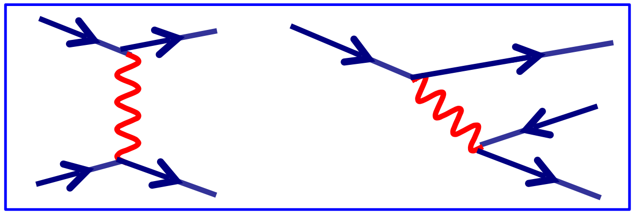
- ♦ will use Feynman diagrams and associated Feynman rules to perform calculations for many processes
- ♦ hopefully gain a fairly deep understanding of the Standard Model and how it explains all current data

Before we can start, need calculations for:

- Interaction cross sections
- Particle decay rates

Cross Sections and Decay Rates

- In particle physics we are mainly concerned with particle interactions and decays, i.e. transitions between states
  - these are the experimental observables of particle physics



- Calculate transition rates from Fermi's Golden Rule

$$\boxed{\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)}$$

Γ_{fi} is number of transitions per unit time from initial state $|i\rangle$ to final state $\langle f|$ – not Lorentz Invariant !

T_{fi} is Transition Matrix Element

$$T_{fi} = \langle f|\hat{H}|i\rangle + \sum_{j\neq i} \frac{\langle f|\hat{H}|j\rangle\langle j|\hat{H}|i\rangle}{E_i - E_j} + \ldots$$

$\hat{H}$ is the perturbing Hamiltonian

$\rho(E_f)$ is density of final states

★ Rates depend on MATRIX ELEMENT and DENSITY OF STATES

the ME contains the fundamental particle physics

just kinematics

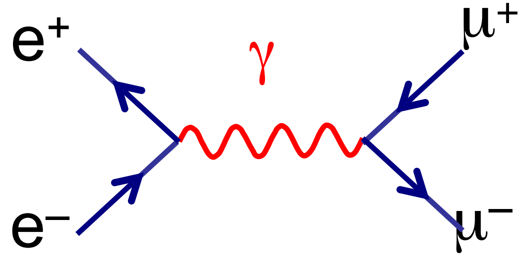
Next a few lectures

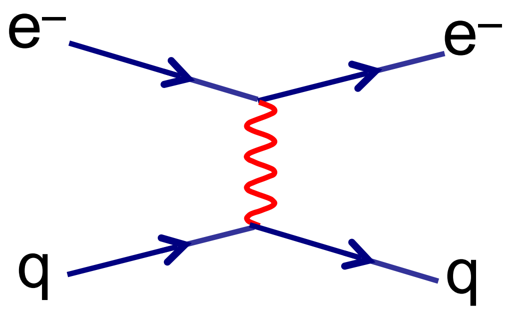
★ Aiming towards a proper calculation of decay and scattering processes

Will concentrate on:

- $e^+e^- \rightarrow \mu^+\mu^-$
- $e^-q \rightarrow e^-q$

(e⁻q→e⁻q to probe proton structure)





- Need <u>relativistic</u> calculations of particle decay rates and cross sections:

$$\sigma = \frac{|M_{fi}|^2}{\text{flux}} \times (\text{phase space})$$

- Need <u>relativistic</u> treatment of spin-half particles:

  Dirac Equation

- Need <u>relativistic</u> calculation of interaction Matrix Element:

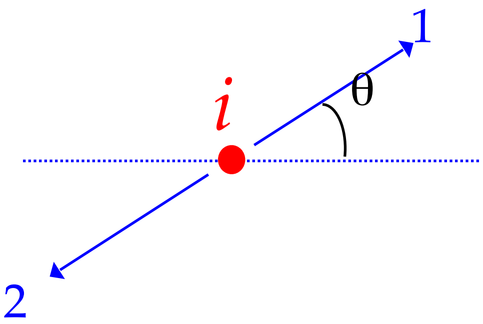
  Interaction by particle exchange and Feynman rules

\+ and a few mathematical tricks along, e.g. the Dirac Delta Function

Particle Decay Rates

- Consider the two-body decay

$$i \rightarrow 1 + 2$$



- Want to calculate the decay rate in first order perturbation theory using plane-wave descriptions of the particles (Born approximation):

$$\psi_1 = N e^{i(\vec{p}.\vec{r} - Et)} = N e^{-ip.x} \qquad (\vec{k}.\vec{r} = \vec{p}.\vec{r} \text{ as } \hbar = 1)$$

where N is the normalization and $p.x = p^\mu x_\mu$

<u>For decay rate calculation need to know:</u>

- Wave-function normalization
- Transition matrix element from perturbation theory
- Expression for the density of states

All in a Lorentz Invariant form

★ First consider wave-function normalization

- Non-relativistic: normalized to one particle in a cube of side a

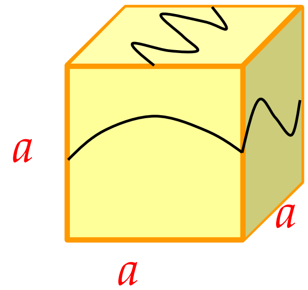
$$\int \psi \psi^* \mathrm{d}V = N^2 a^3 = 1 \quad \Rightarrow \quad \boxed{N^2 = 1/a^3}$$

Non-relativistic Phase Space

- Apply boundary conditions $(\vec{p} = \hbar\vec{k})$:
- Wave-function vanishing at box boundaries

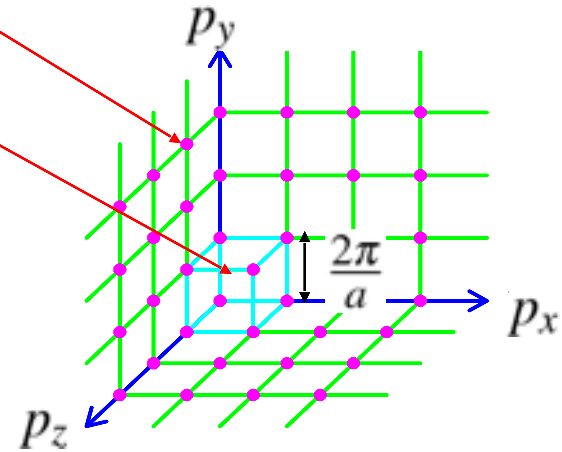
  ➡ quantized particle momenta:

$$p_x = \frac{2\pi n_x}{a};\; p_y = \frac{2\pi n_y}{a};\; p_z = \frac{2\pi n_z}{a}$$



- Volume of single state in momentum space:

$$\left(\frac{2\pi}{a}\right)^3 = \frac{(2\pi)^3}{V}$$



- Normalizing to one particle/unit volume gives number of states in element: $\mathrm{d}^3\vec{p} = \mathrm{d}p_x\mathrm{d}p_y\mathrm{d}p_z$

$$\mathrm{d}n = \frac{\mathrm{d}^3\vec{p}}{\frac{(2\pi)^3}{V}} \times \frac{1}{V} = \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}$$

- Therefore density of states in Golden rule:

$$\rho(E_f) = \left|\frac{\mathrm{d}n}{dE}\right|_{E_f} = \left|\frac{\mathrm{d}n}{\mathrm{d}|\vec{p}|}\frac{\mathrm{d}|\vec{p}|}{\mathrm{d}E}\right|_{E_f}$$

- Integrating over an elemental shell in momentum-space gives

  $(\mathrm{d}^3\vec{p} = 4\pi p^2 \mathrm{d}p)$

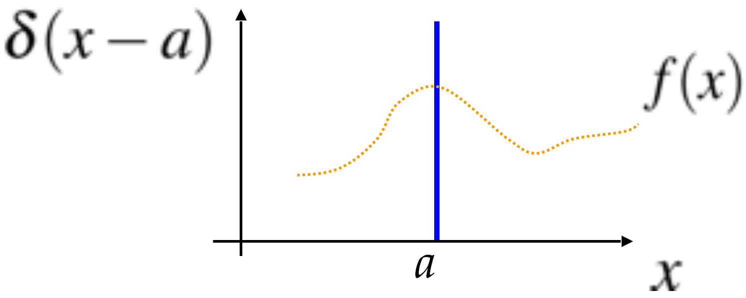
$$\rho(E_f) = \frac{4\pi p^2}{(2\pi)^3} \times \frac{1}{\beta}$$

with

$$E^2 = p^2 + m^2$$
$$2E\,\mathrm{d}E = 2p\,\mathrm{d}p$$
$$\frac{\mathrm{d}p}{\mathrm{d}E} = \frac{E}{p} = \frac{1}{\beta}$$

Dirac δ Function

- In the relativistic formulation of decay rates and cross sections we will make use of the Dirac δ function: *"infinitely narrow spike of unit area"*



$$\int_{-\infty}^{+\infty} \delta(x-a)\,\mathrm{d}x = 1$$

$$\int_{-\infty}^{+\infty} f(x)\,\delta(x-a)\,\mathrm{d}x = f(a)$$

- Any function with the above properties can represent $\delta(x)$

e.g.
$$\delta(x) = \lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x^2}{2\sigma^2}\right)}$$
(an infinitesimally narrow Gaussian)

- In relativistic quantum mechanics delta functions prove extremely useful for integrals over phase space, e.g. in the decay $a \to 1+2$

$$\int \ldots \delta(E_a - E_1 - E_2)\,\mathrm{d}E \qquad \text{and} \qquad \int \ldots \delta^3(\vec{p}_a - \vec{p}_1 - \vec{p}_2)\,\mathrm{d}^3\vec{p}$$

express energy and momentum conservation

Dirac δ Function

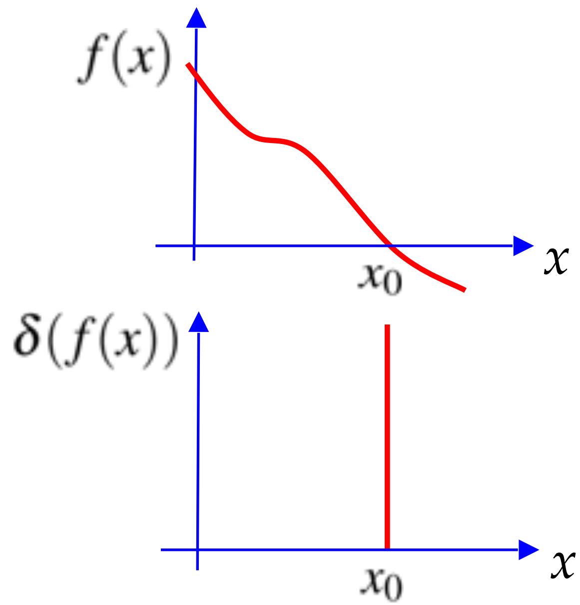
★ We will soon need an expression for the delta function of a function $\delta(f(x))$

- Start from the definition of a delta function

$$\int_{y_1}^{y_2} \delta(y)\mathrm{d}y = \begin{cases} 1 & \text{if } y_1 < 0 < y_2 \\ 0 & \text{otherwise} \end{cases}$$

- Now express in terms of $y = f(x)$ where $f(x_0) = 0$ and then change variables

$$\int_{x_1}^{x_2} \delta(f(x))\frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x = \begin{cases} 1 & \text{if } x_1 < x_0 < x_2 \\ 0 & \text{otherwise} \end{cases}$$



- From properties of the delta function (i.e. here only non-zero at x_0)

$$\left|\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_0} \int_{x_1}^{x_2} \delta(f(x))\mathrm{d}x = \begin{cases} 1 & \text{if } x_1 < x_0 < x_2 \\ 0 & \text{otherwise} \end{cases}$$

- Rearranging and expressing the RHS as a delta function

$$\int_{x_1}^{x_2} \delta(f(x))\mathrm{d}x = \frac{1}{|\mathrm{d}f/\mathrm{d}x|_{x_0}} \int_{x_1}^{x_2} \delta(x - x_0)\mathrm{d}x$$

$$\Rightarrow \quad \boxed{\delta(f(x)) = \left|\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_0}^{-1} \delta(x - x_0)} \quad (1)$$

The Golden Rule revisited

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)$$

- Rewrite the expression for density of states using a delta-function

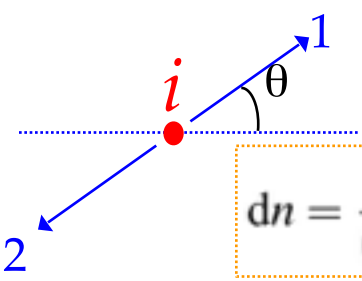
$$\rho(E_f) = \left|\frac{\mathrm{d}n}{\mathrm{d}E}\right|_{E_f} = \int \frac{\mathrm{d}n}{\mathrm{d}E}\delta(E - E_i)\mathrm{d}E \qquad \text{since} \quad E_f = E_i$$

Note : integrating over all final state energies but energy conservation now taken into account explicitly by delta function

- Hence the golden rule becomes: $\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E)\mathrm{d}n$

the integral is over all "allowed" final states of any energy

- For $\mathrm{d}n$ in a two-body decay, only need to consider one particle : mom. conservation fixes the other



$$\boxed{\mathrm{d}n = \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}}$$

$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E_1 - E_2)\frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3}$$

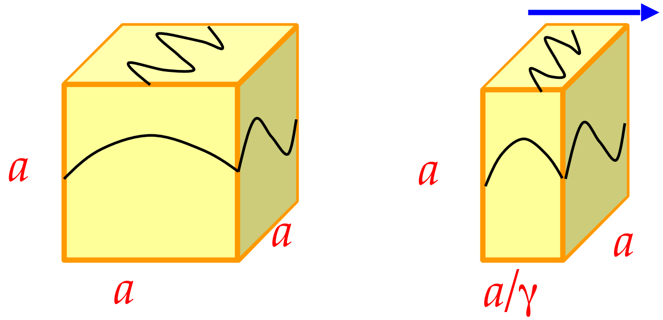
- However, can include momentum conservation explicitly by integrating over the momenta of both particles and using another δ-function

$$\Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2 \underbrace{\delta(E_i - E_1 - E_2)}_{\text{Energy cons.}}\underbrace{\delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2)}_{\text{Mom. cons.}}\underbrace{\frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3}\frac{\mathrm{d}^3\vec{p}_2}{(2\pi)^3}}_{\text{Density of states}}$$

Phase Space is Critical

Lorentz Invariant Phase Space

- In non-relativistic QM normalise to one particle/unit volume: $\int \psi^* \psi \mathrm{d}V = 1$
- When considering relativistic effects, volume contracts by $\gamma = E/m$



- Particle density therefore increases by $\gamma = E/m$
  - ★ Conclude that a relativistic invariant wave-function normalization needs to be proportional to E particles per unit volume
- Usual convention: Normalise to 2E particles/unit volume $\int \psi'^* \psi' \mathrm{d}V = 2E$
  - Previously used ψ normalized to 1 particle per unit volume $\int \psi^* \psi \mathrm{d}V = 1$
  - Hence $\psi' = (2E)^{1/2}\psi$ is normalized to $2E$ per unit volume
- Define Lorentz Invariant Matrix Element, M_{fi} in terms of the wave-functions normalized to $2E$ particles per unit volume

$$M_{fi} = \langle \psi_1'.\psi_2'...|\hat{H}|...\psi_{n-1}'\psi_n'\rangle = (2E_1.2E_2.2E_3....2E_n)^{1/2} T_{fi}$$

Two Body Decay

- For the two body decay

$$i \to 1 + 2$$

$$\begin{aligned} M_{fi} &= \langle \psi_1' \psi_2' | \hat{H}' | \psi_i' \rangle \\ &= (2E_i.2E_1.2E_2)^{1/2} \langle \psi_1 \psi_2 | \hat{H}' | \psi_i \rangle \\ &= (2E_i.2E_1.2E_2)^{1/2} T_{fi} \end{aligned}$$

★ Now expressing T_{fi} in terms of M_{fi} gives

$$\boxed{\Gamma_{fi} = \frac{(2\pi)^4}{2E_i} \int |M_{fi}|^2 \delta(E_i - E_1 - E_2) \delta^3(\vec{p}_a - \vec{p}_1 - \vec{p}_2) \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}}$$

<u>Note:</u>

- M_{fi} uses relativistically normalised wave-functions. It is Lorentz Invariant
- $\frac{d^3\vec{p}}{(2\pi)^3 2E}$ is the Lorentz Invariant Phase Space for each final state particle
  the factor of $2E$ arises from the wave-function normalization

  ⇨ This form of Γ_{fi} is simply a rearrangement of the original equation
  <u>but</u> the integral is now frame independent (i.e. L.I.)
- Γ_{fi} is inversely proportional to E_i, the energy of the decaying particle. This is exactly what one would expect from time dilation ($E_i = \gamma m$).
- Energy and momentum conservation in the delta functions

Decay Rate Calculations

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_i}\int |M_{fi}|^2 \delta(E_i - E_1 - E_2)\delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2)\frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3 2E_1}\frac{\mathrm{d}^3\vec{p}_2}{(2\pi)^3 2E_2}$$

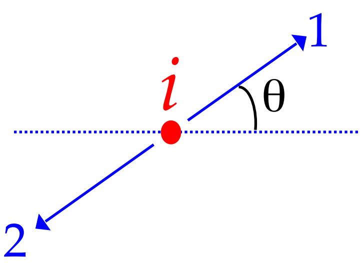
★ Because the integral is Lorentz invariant (i.e. frame independent) it can be evaluated in any frame we choose. The C.o.M. frame is most convenient

- In the C.o.M. frame $E_i = m_i$ and $\vec{p}_i = 0$

$$\Rightarrow \quad \Gamma_{fi} = \frac{1}{8\pi^2 E_i}\int |M_{fi}|^2 \delta(m_i - E_1 - E_2)\delta^3(\vec{p}_1 + \vec{p}_2)\frac{\mathrm{d}^3\vec{p}_1}{2E_1}\frac{\mathrm{d}^3\vec{p}_2}{2E_2}$$

- Integrating over $\vec{p}_2$ using the δ-function:

$$\Rightarrow \quad \Gamma_{fi} = \frac{1}{8\pi^2 E_i}\int |M_{fi}|^2 \delta(m_i - E_1 - E_2)\frac{\mathrm{d}^3\vec{p}_1}{4E_1E_2}$$



now $E_2^2 = (m_2^2 + |\vec{p}_1|^2)$ since the δ-function imposes $\vec{p}_2 = -\vec{p}_1$

- Writing $\mathrm{d}^3\vec{p}_1 = p_1^2 \mathrm{d}p_1 \sin\theta \mathrm{d}\theta \mathrm{d}\phi = p_1^2 \mathrm{d}p_1 \mathrm{d}\Omega$

For convenience, here $|\vec{p}_1|$ is written as p_1

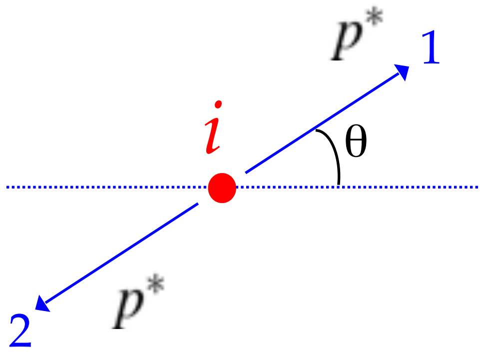
$$\Rightarrow \Gamma_{fi} = \frac{1}{32\pi^2 E_i}\int |M_{fi}|^2 \delta\left(m_i - \sqrt{m_1^2 + p_1^2} - \sqrt{m_2^2 + p_1^2}\right)\frac{p_1^2 \mathrm{d}p_1 \mathrm{d}\Omega}{E_1 E_2}$$

- Which can be written in the form

$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 g(p_1)\delta(f(p_1))\mathrm{d}p_1\mathrm{d}\Omega \qquad \textbf{(2)}$$

where $\quad g(p_1) = p_1^2/(E_1E_2) = p_1^2(m_1^2+p_1^2)^{-1/2}(m_2^2+p_1^2)^{-1/2}$

and $\quad f(p_1) = m_i - (m_1^2+p_1^2)^{1/2} - (m_2^2+p_1^2)^{1/2}$



<u>Note:</u>
- $\delta(f(p_1))$ imposes energy conservation.
- $f(p_1) = 0$ determines the C.o.M momenta of the two decay products
  i.e. $f(p_1) = 0$ for $p_1 = p^*$

★ Eq. (2) can be integrated using the property of δ–function derived earlier (eq. (1))

$$\int g(p_1)\delta(f(p_1))\mathrm{d}p_1 = \frac{1}{|\mathrm{d}f/\mathrm{d}p_1|_{p^*}} \int g(p_1)\delta(p_1 - p^*)\mathrm{d}p_1 = \frac{g(p^*)}{|\mathrm{d}f/\mathrm{d}p_1|_{p^*}}$$

where p^* is the value for which $f(p^*) = 0$

- All that remains is to evaluate $\mathrm{d}f/\mathrm{d}p_1$

$$\frac{\mathrm{d}f}{\mathrm{d}p_1} = -\frac{p_1}{(m_1^2+p_1^2)^{1/2}} - \frac{p_1}{(m_2^2+p_1^2)^{1/2}} = -\frac{p_1}{E_1} - \frac{p_1}{E_2} = -p_1\frac{E_1+E_2}{E_1E_2}$$

giving: $$\Gamma_{fi} = \frac{1}{32\pi^2 E_i}\int |M_{fi}|^2 \left|\frac{E_1E_2}{p_1(E_1+E_2)}\frac{p_1^2}{E_1E_2}\right|_{p_1=p^*} d\Omega$$
$$= \frac{1}{32\pi^2 E_i}\int |M_{fi}|^2 \left|\frac{p_1}{E_1+E_2}\right|_{p_1=p^*} d\Omega$$

- But from $f(p_1)=0$, i.e. energy conservation: $E_1+E_2=m_i$

$$\Gamma_{fi} = \frac{|\vec{p}^*|}{32\pi^2 E_i m_i}\int |M_{fi}|^2 d\Omega$$

In the particle's rest frame $E_i = m_i$

$$\boxed{\frac{1}{\tau} = \Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2}\int |M_{fi}|^2 d\Omega} \qquad (3)$$

VALID FOR ALL TWO-BODY DECAYS !

- p^* can be obtained from $f(p_1)=0$

$$(m_1^2+p^{*2})^{1/2}+(m_2^2+p^{*2})^{1/2}=m_i$$

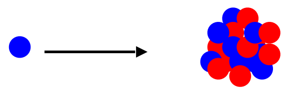
$$p^* = \frac{1}{2m_i}\sqrt{[(m_i^2-(m_1+m_2)^2][m_i^2-(m_1-m_2)^2]}$$

Cross section definition

$$\boxed{\sigma = \frac{\text{\# of interactions per unit time per target}}{\text{incident flux}}}$$

Flux = number of incident particles/ unit area/unit time

- The "cross section", σ , can be thought of as the effective cross-sectional area of the target particles for the interaction to occur.
- In general this has nothing to do with the physical size of the target although there are exceptions, e.g. neutron absorption

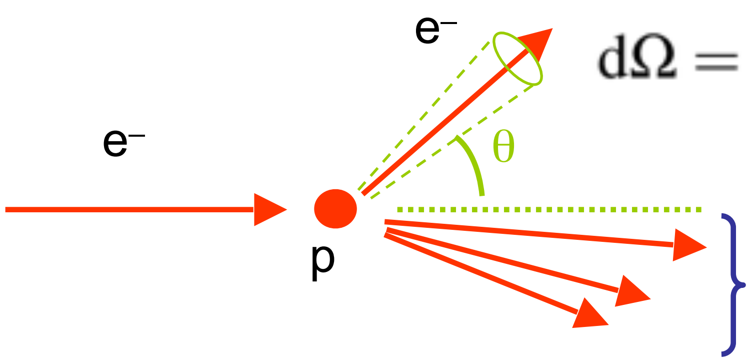


here σ is the projective area of nucleus

Differential Cross section

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\text{\# of particles per sec/per target into } d\Omega}{\text{incident flux}}}$$

or generally $\frac{d\sigma}{d\ldots}$



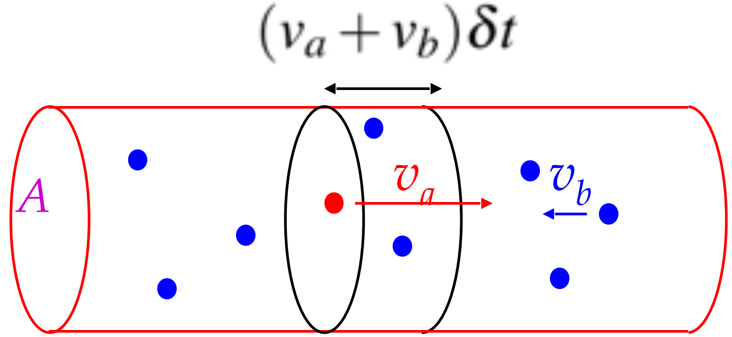
$$d\Omega = d(\cos\theta)d\phi$$

integrate over all other particles

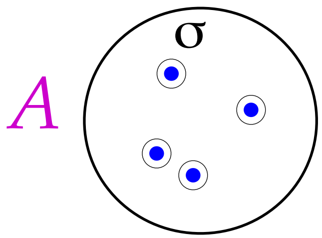
with $\boxed{\sigma = \int \frac{d\sigma}{d\Omega} d\Omega}$

Example

- Consider a single particle of type a with velocity, v_a, traversing a region of area A containing n_b particles of type b per unit volume



In time δt a particle of type a traverses region containing $n_b(v_a + v_b)A\delta t$ particles of type b



★Interaction probability obtained from effective cross-sectional area occupied by the $n_b(v_a + v_b)A\delta t$ particles of type b

- Interaction Probability = $\dfrac{n_b(v_a + v_b)A\delta t\sigma}{A} = n_b v \delta t \sigma$ $\quad [v = v_a + v_b]$

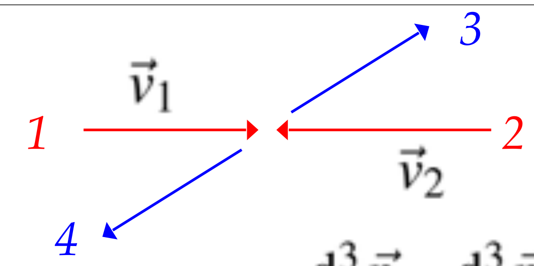
➡ Rate per particle of type a = $n_b v \sigma$

- Consider volume V, total reaction rate = $(n_b v \sigma).(n_a V) = (n_b V)(n_a v)\sigma$
  $= N_b \phi_a \sigma$

- As anticipated: Rate = Flux x Number of targets x cross section

Cross Section Calculations

- Consider scattering process



$$1 + 2 \rightarrow 3 + 4$$

- Start from Fermi's Golden Rule:

$$\Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2 \delta(E_1 + E_2 - E_3 - E_4)\delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)\frac{\mathrm{d}^3\vec{p}_3}{(2\pi)^3}\frac{\mathrm{d}^3\vec{p}_4}{(2\pi)^3}$$

where T_{fi} is the transition matrix for a normalization of 1/unit volume

- Now

$$\begin{aligned} \text{Rate/Volume} &= (\text{flux of 1}) \times (\text{number density of 2}) \times \sigma \\ &= n_1(v_1 + v_2) \times n_2 \times \sigma \end{aligned}$$

- For 1 target particle per unit volume $\text{Rate} = (v_1 + v_2)\sigma$

$$\sigma = \frac{\Gamma_{fi}}{(v_1 + v_2)}$$

$$\sigma = \frac{(2\pi)^4}{v_1 + v_2} \int |T_{fi}|^2 \delta(E_1 + E_2 - E_3 - E_4)\delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)\frac{\mathrm{d}^3\vec{p}_3}{(2\pi)^3}\frac{\mathrm{d}^3\vec{p}_4}{(2\pi)^3}$$

the parts are not Lorentz Invariant

- To obtain a Lorentz Invariant form use wave-functions normalized to $2E$ particles per unit volume

$$\psi' = (2E)^{1/2}\psi$$

- Again define L.I. Matrix element $M_{fi} = (2E_1\,2E_2\,2E_3\,2E_4)^{1/2}T_{fi}$

$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2(v_1+v_2)}\int |M_{fi}|^2\delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\frac{\mathrm{d}^3\vec{p}_3}{2E_3}\frac{\mathrm{d}^3\vec{p}_4}{2E_4}$$

- The integral is now written in a Lorentz invariant form
- The quantity $F = 2E_1 2E_2(v_1+v_1)$ can be written in terms of a four-vector scalar product and is therefore also Lorentz Invariant (the Lorentz Inv. Flux)

$$F = 4\left[(p_1^\mu p_{2\mu})^2 - m_1^2 m_2^2\right]^{1/2}$$

(see appendix I)

- Consequently cross section is a Lorentz Invariant quantity

Two special cases of Lorentz Invariant Flux:

- Centre-of-Mass Frame

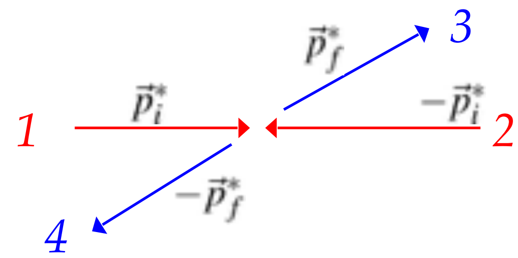
$$\begin{aligned} F &= 4E_1E_2(v_1+v_2) \\ &= 4E_1E_2(|\vec{p}^*|/E_1+|\vec{p}^*|/E_2) \\ &= 4|\vec{p}^*|(E_1+E_2) \\ &= 4|\vec{p}^*|\sqrt{s} \end{aligned}$$

- Target (particle 2) at rest

$$\begin{aligned} F &= 4E_1E_2(v_1+v_2) \\ &= 4E_1m_2v_1 \\ &= 4E_1m_2(|\vec{p}_1|/E_1) \\ &= 4m_2|\vec{p}_1| \end{aligned}$$

2→2 Body Scattering in C.o.M. Frame

- We will now apply above Lorentz Invariant formula for the interaction cross section to the most common cases used in the course. First consider 2→2 scattering in C.o.M. frame



- Start from

$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2(v_1+v_2)} \int |M_{fi}|^2 \delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\frac{d^3\vec{p}_3}{2E_3}\frac{d^3\vec{p}_4}{2E_4}$$

- Here $\vec{p}_1+\vec{p}_2=0$ and $E_1+E_2=\sqrt{s}$

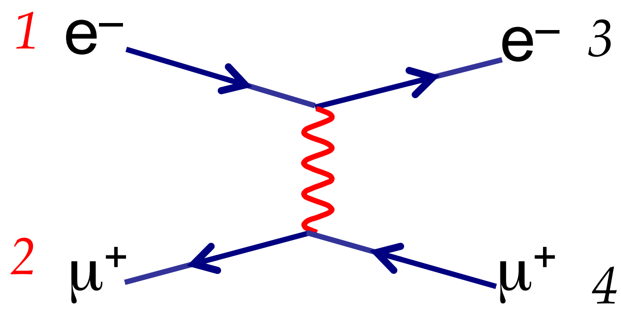
$$\Rightarrow \quad \sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i^*|\sqrt{s}} \int |M_{fi}|^2 \delta(\sqrt{s}-E_3-E_4)\delta^3(\vec{p}_3+\vec{p}_4)\frac{d^3\vec{p}_3}{2E_3}\frac{d^3\vec{p}_4}{2E_4}$$

★The integral is exactly the same integral that appeared in the particle decay calculation but with m_a replaced by $\sqrt{s}$

$$\Rightarrow \quad \sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i^*|\sqrt{s}}\frac{|\vec{p}_f^*|}{4\sqrt{s}} \int |M_{fi}|^2 d\Omega^*$$

$$\boxed{\sigma = \frac{1}{64\pi^2 s}\frac{|\vec{p}_f^*|}{|\vec{p}_i^*|}\int |M_{fi}|^2 d\Omega^*} \qquad (4)$$

- In the case of elastic scattering $|\vec{p}_i^*| = |\vec{p}_f^*|$



$$\sigma_{\text{elastic}} = \frac{1}{64\pi^2 s}\int |M_{fi}|^2 \mathrm{d}\Omega^*$$

- For calculating the total cross-section (which is Lorentz Invariant) the result on the previous page (eq. (4)) is sufficient. However, it is not so useful for calculating the differential cross section in a rest frame other than the C.o.M:

$$\mathrm{d}\sigma = \frac{1}{64\pi^2 s}\frac{|\vec{p}_f^*|}{|\vec{p}_i^*|}|M_{fi}|^2 \mathrm{d}\Omega^*$$

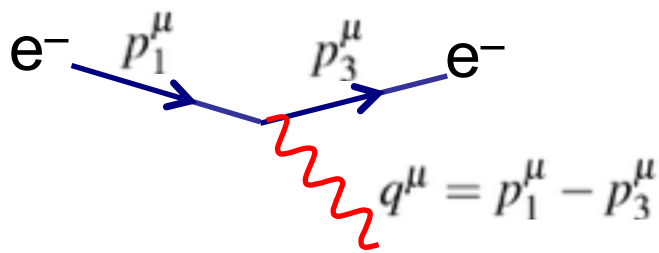
because the angles in $\mathrm{d}\Omega^* = \mathrm{d}(\cos\theta^*)\mathrm{d}\phi^*$ refer to the C.o.M frame

- For the last calculation in this section, we need to find a L.I. expression for $\mathrm{d}\sigma$

★ Start by expressing $\mathrm{d}\Omega^*$ in terms of Mandelstam t
i.e. the square of the four-momentum transfer

$$\boxed{t = q^2 = (p_1 - p_3)^2}$$

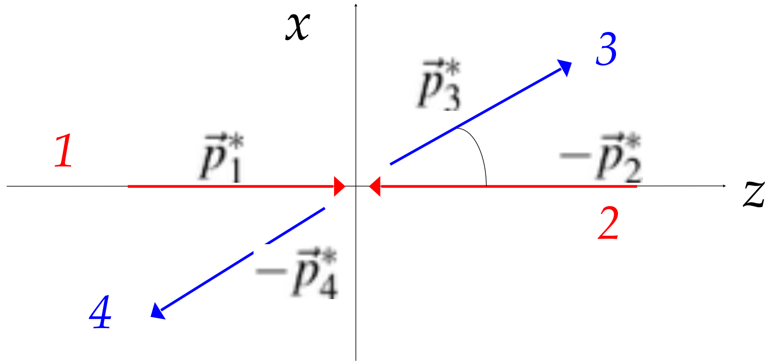
Product of four-vectors therefore L.I.



- Want to express $\mathrm{d}\Omega^*$ in terms of Lorentz Invariant $\mathrm{d}t$

where $t \equiv (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1.p_3 = m_1^2 + m_3^2 - 2p_1.p_3$

♦ In C.o.M. frame:



$$
\begin{aligned}
p_1^{*\mu} &= (E_1^*, 0, 0, |\vec{p}_1^*|) \\
p_3^{*\mu} &= (E_3^*, |\vec{p}_3^*|\sin\theta^*, 0, |\vec{p}_3^*|\cos\theta^*) \\
p_1^\mu p_{3\mu} &= E_1^* E_3^* - |\vec{p}_1^*||\vec{p}_3^*|\cos\theta^* \\
t &= m_1^2 + m_3^3 - E_1^* E_3^* + 2|\vec{p}_1^*||\vec{p}_3^*|\cos\theta^*
\end{aligned}
$$

giving $\mathrm{d}t = 2|\vec{p}_1^*||\vec{p}_3^*|\mathrm{d}(\cos\theta^*)$

therefore $\mathrm{d}\Omega^* = \mathrm{d}(\cos\theta^*)\mathrm{d}\phi^* = \dfrac{\mathrm{d}t\,\mathrm{d}\phi^*}{2|\vec{p}_1^*||\vec{p}_3^*|}$

hence $$\mathrm{d}\sigma = \frac{1}{64\pi^2 s}\frac{|\vec{p}_3^*|}{|\vec{p}_1^*|}|M_{fi}|^2\mathrm{d}\Omega^* = \frac{1}{2\cdot 64\pi^2 s|\vec{p}_1^*|^2}|M_{fi}|^2\mathrm{d}\phi^*\mathrm{d}t$$

- Finally, integrating over $\mathrm{d}\phi^*$ (assuming no ϕ^* dependence of $|M_{fi}|^2$) gives:

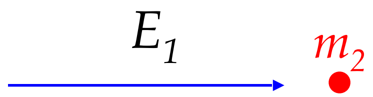
$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{64\pi s|\vec{p}_i^*|^2}|M_{fi}|^2}$$

Lorentz Invariant differential cross section

- All quantities in the expression for $\mathrm{d}\sigma/\mathrm{d}t$ are Lorentz Invariant and therefore, it applies to any rest frame. It should be noted that $|\vec{p}_i^*|^2$ is a constant, fixed by energy/momentum conservation

$$|\vec{p}_i^*|^2 = \frac{1}{4s}[s-(m_1+m_2)^2][s-(m_1-m_2)^2]$$

- As an example of how to use the invariant expression $\mathrm{d}\sigma/\mathrm{d}t$ we will consider elastic scattering in the laboratory frame in the limit where we can neglect the mass of the incoming particle $E_1 \gg m_1$



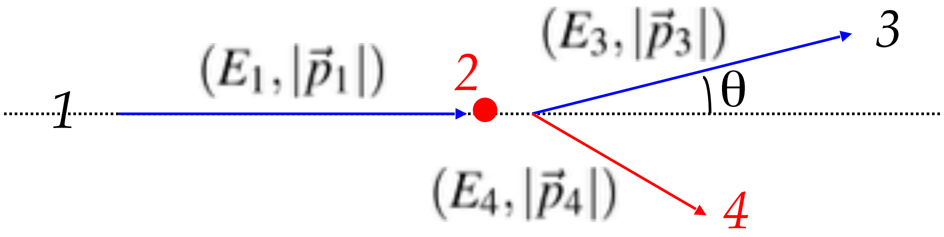
e.g. electron or neutrino scattering

In this limit $$|\vec{p}_i^*|^2 = \frac{(s-m_2^2)^2}{4s}$$

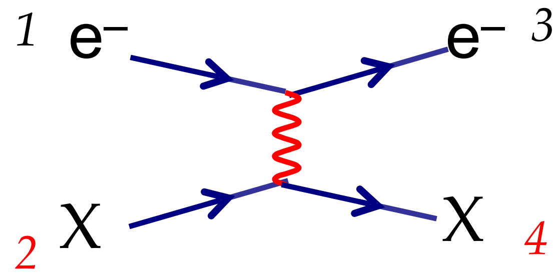
$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{16\pi(s-m_2^2)^2}|M_{fi}|^2} \qquad (m_1 = 0)$$

2→2 Body Scattering in Lab. Frame

- The other commonly occurring case is scattering from a fixed target in the Laboratory Frame (e.g. electron-proton scattering)
- First take the case of elastic scattering at high energy where the mass of the incoming particles can be neglected: $m_1 = m_3 = 0, \quad m_2 = m_4 = M$



e.g.



- Wish to express the cross section in terms of scattering angle of the e⁻

$$d\Omega = 2\pi d(\cos\theta)$$

therefore $$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{dt}\frac{dt}{d\Omega} = \frac{1}{2\pi}\frac{dt}{d(\cos\theta)}\frac{d\sigma}{dt}$$

Integrating over $d\phi$

- The rest is some rather tedious algebra…. start from four-momenta

$$p_1 = (E_1, 0, 0, E_1), \;\; p_2 = (M, 0, 0, 0), \;\; p_3 = (E_3, E_3\sin\theta, 0, E_3\cos\theta), \;\; p_4 = (E_4, \vec{p}_4)$$

so here $$t = (p_1 - p_3)^2 = -2p_1.p_3 = -2E_1E_3(1-\cos\theta)$$

But from (E,p) conservation $$p_1 + p_2 = p_3 + p_4$$

and, therefore, can also express t in terms of particles 2 and 4

$$t = (p_2 - p_4)^2 = 2M^2 - 2p_2.p_4 = 2M^2 - 2ME_4$$
$$= 2M^2 - 2M(E_1 + M - E_3) = -2M(E_1 - E_3)$$

Note E_1 is a constant (the energy of the incoming particle) so

$$\frac{dt}{d(\cos\theta)} = 2M\frac{dE_3}{d(\cos\theta)}$$

- Equating the two expressions for t gives $\boxed{E_3 = \frac{E_1 M}{M + E_1 - E_1\cos\theta}}$

so $$\frac{dE_3}{d(\cos\theta)} = \frac{E_1^2 M}{(M + E_1 - E_1\cos\theta)^2} = E_1^2 M\left(\frac{E_3}{E_1 M}\right)^2 = \frac{E_3^2}{M}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2\pi}\frac{dt}{d(\cos\theta)}\frac{d\sigma}{dt} = \frac{1}{2\pi}2M\frac{E_3^2}{M}\frac{d\sigma}{dt} = \frac{E_3^2}{\pi}\frac{d\sigma}{dt} = \frac{E_3^2}{\pi}\frac{1}{16\pi(s - M^2)^2}|M_{fi}|^2$$

Using $s = (p_1 + p_2)^2 = M^2 + 2p_1.p_2 = M^2 + 2ME_1$

Particle 1 massless $\rightarrow$ $(p_1^2 = 0)$

gives $(s - M^2) = 2ME_1$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2 |M_{fi}|^2}$$

In limit $m_1 \to 0$

In this equation, E_3 is a function of θ :

$$E_3 = \frac{E_1 M}{M + E_1 - E_1\cos\theta}$$

giving $$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M+E_1-E_1\cos\theta}\right)^2 |M_{fi}|^2} \quad (m_1 = 0)$$

General form for 2→2 Body Scattering in Lab. Frame

★The calculation of the differential cross section for the case where m_1 can not be neglected is longer and contains no more "physics" (see appendix II). It gives:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\cdot\frac{1}{p_1 m_1}\cdot\frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1+m_2) - E_3|\vec{p}_1|\cos\theta}\cdot|M_{fi}|^2}$$

Again there is only one independent variable, θ, which can be seen from conservation of energy

$$E_1 + m_2 = \sqrt{|\vec{p}_3|^2 + m_3^2} + \sqrt{\underbrace{|\vec{p}_1|^2 + |\vec{p}_3|^2 - 2|\vec{p}_1||\vec{p}_3|\cos\theta}_{\vec{p}_4 = \vec{p}_1 - \vec{p}_3} + m_4^2}$$

i.e. $|\vec{p}_3|$ is a function of θ

Summary

★ Used a Lorentz invariant formulation of Fermi′s Golden Rule to derive decay rates and cross-sections in terms of the Lorentz Invariant Matrix Element (wave-functions normalised to 2E/Volume)

Main Results:

★ Particle decay:

$$\boxed{\Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2}\int |M_{fi}|^2 \mathrm{d}\Omega}$$

Where p^* is a function of particle masses

$$p^* = \frac{1}{2m_i}\sqrt{\left[(m_i^2-(m_1+m_2)^2\right]\left[m_i^2-(m_1-m_2)^2\right]}$$

★ Scattering cross section in C.o.M. frame:

$$\boxed{\sigma \quad = \quad \frac{1}{64\pi^2 s}\frac{|\vec{p}_f^*|}{|\vec{p}_i^*|}\int |M_{fi}|^2 \mathrm{d}\Omega^*}$$

★ Invariant differential cross section (valid in all frames):

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{64\pi s|\vec{p}_i^*|^2}|M_{fi}|^2}$$

$$|\vec{p}_i^*|^2 = \frac{1}{4s}[s-(m_1+m_2)^2][s-(m_1-m_2)^2]$$

Summary cont.

★Differential cross section in the lab. frame ($m_1=0$)

$$\boxed{\frac{d\sigma}{d\Omega}=\frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2|M_{fi}|^2} \Leftrightarrow \boxed{\frac{d\sigma}{d\Omega}=\frac{1}{64\pi^2}\left(\frac{1}{M+E_1-E_1\cos\theta}\right)^2|M_{fi}|^2}$$

★Differential cross section in the lab. frame ($m_1\neq 0$)

$$\boxed{\frac{d\sigma}{d\Omega}=\frac{1}{64\pi^2}\cdot\frac{1}{|\vec{p}_1|m_1}\cdot\frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1+m_2)-E_3|\vec{p}_1|\cos\theta}\cdot|M_{fi}|^2}$$

with $E_1+m_2=\sqrt{|\vec{p}_3|^2+m_3^2}+\sqrt{|\vec{p}_1|^2+|\vec{p}_3|^2-2|\vec{p}_1||\vec{p}_3|\cos\theta+m_4^2}$

<u>Summary of the summary:</u>

★Have now dealt with kinematics of particle decays and cross sections

★The fundamental particle physics is in the matrix element

★The above equations are the basis for all calculations that follow

Appendix I : Lorentz Invariant Flux

- Collinear collision: $a \longrightarrow \qquad \longleftarrow b$

$$v_a, \vec{p}_a \qquad\qquad v_b, \vec{p}_b$$

$$\begin{aligned} F = 2E_a 2E_b (v_a + v_b) &= 4E_a E_b \left(\frac{|\vec{p}_a|}{E_a} + \frac{|\vec{p}_b|}{E_b} \right) \\ &= 4(|\vec{p}_a| E_b + |\vec{p}_b| E_a) \end{aligned}$$

To show this is Lorentz invariant, first consider

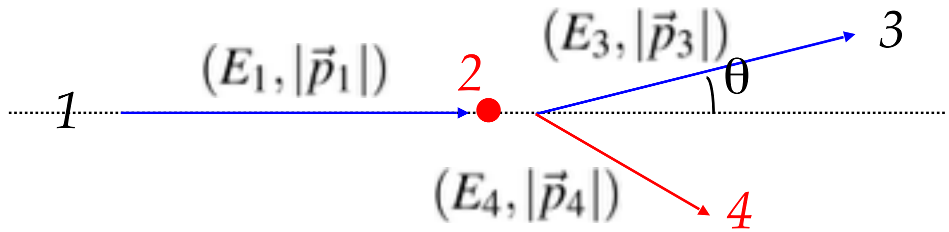
$$p_a . p_b = p_a^\mu p_{b\mu} = E_a E_b - \vec{p}_a . \vec{p}_b = E_a E_b + |\vec{p}_a||\vec{p}_b|$$

Giving

$$\begin{aligned} F^2/16 - (p_a^\mu p_{b\mu})^2 &= (|\vec{p}_a| E_b + |\vec{p}_b| E_a)^2 - (E_a E_b + |\vec{p}_a||\vec{p}_b|)^2 \\ &= |\vec{p}_a|^2 (E_b^2 - |\vec{p}_b|^2) + E_a^2 (|\vec{p}_b|^2 - E_b^2) \\ &= |\vec{p}_a|^2 m_b^2 - E_a^2 m_b^2 \\ &= -m_a^2 m_b^2 \end{aligned}$$

$$F = 4\left[(p_a^\mu p_{b\mu})^2 - m_a^2 m_b^2 \right]^{1/2}$$

Appendix II : general 2→2 Body Scattering in lab frame



$$p_1 = (E_1, 0, 0, |\vec{p}_1|), \; p_2 = (M, 0, 0, 0), \; p_3 = (E_3, E_3 \sin\theta, 0, E_3 \cos\theta), \; p_4 = (E_4, \vec{p}_4)$$

again $$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{dt}\frac{dt}{d\Omega} = \frac{1}{2\pi}\frac{dt}{d(\cos\theta)}\frac{d\sigma}{dt}$$

But now the invariant quantity t:

$$\begin{aligned} t &= (p_2 - p_4)^2 = m_2^2 + m_4^2 - 2p_2.p_4 = m_2^2 + m_4^2 - 2m_2 E_4 \\ &= m_2^2 + m_4^2 - 2m_2(E_1 + m_2 - E_3) \end{aligned}$$

$$\Rightarrow \quad \frac{dt}{d(\cos\theta)} = 2m_2 \frac{dE_3}{d(\cos\theta)}$$

Which gives $$\frac{d\sigma}{d\Omega} = \frac{m_2}{\pi}\frac{dE_3}{d(\cos\theta)}\frac{d\sigma}{dt}$$

To determine $dE_3/d(\cos\theta)$, first differentiate $E_3^2 - |\vec{p}_3|^2 = m_3^2$

$$2E_3\frac{dE_3}{d(\cos\theta)} = 2|\vec{p}_3|\frac{d|\vec{p}_3|}{d(\cos\theta)} \qquad \text{(AII.1)}$$

Then equate $t = (p_1 - p_3)^2 = (p_4 - p_2)^2$ to give

$$m_1^2 + m_3^2 - 2(E_1E_3 - |\vec{p}_1||\vec{p}_3|\cos\theta) = m_4^2 + m_2^2 - 2m_2(E_1 + m_2 - E_3)$$

Differentiate wrt. $\cos\theta$

$$(E_1 + m_2)\frac{dE_3}{d\cos\theta} - |\vec{p}_1|\cos\theta\frac{d|\vec{p}_3|}{d\cos\theta} = |\vec{p}_1||\vec{p}_3|$$

Using (1) $\Rightarrow$ $$\frac{dE_3}{d(\cos\theta)} = \frac{|\vec{p}_1||\vec{p}_3|^2}{|\vec{p}_3|(E_1 + m_2) - E_3|\vec{p}_1|\cos\theta} \qquad \text{(AII.2)}$$

$$\frac{d\sigma}{d\Omega} = \frac{m_2}{\pi}\frac{dE_3}{d(\cos\theta)}\frac{d\sigma}{dt} = \frac{m_2}{\pi}\frac{dE_3}{d(\cos\theta)}\frac{1}{64\pi s|\vec{p}_i^*|^2}|M_{fi}|^2$$

It is easy to show $|\vec{p}_i^*|\sqrt{s} = m_2|\vec{p}_1|$

$$\frac{d\sigma}{d\Omega} = \frac{dE_3}{d(\cos\theta)} \frac{m_2}{64\pi^2 m_2^2 |\vec{p}_1|^2} |M_{fi}|^2$$

and using (AII.2) obtain

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \cdot \frac{1}{p_1 m_1} \cdot \frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1 + m_2) - E_3|\vec{p}_1|\cos\theta} \cdot |M_{fi}|^2}$$