Project 1.

1 Floquet Theory

Consider the $n \times n$ linear equation

$$\mathbf{y}'(t) = \mathbf{A}(t)\mathbf{y}(t), t \in \mathbb{R}, \tag{1}$$

where $\mathbf{A}(t)$ is a T-periodic function. Let $\Phi(t)$ be a fundamental solution of (1), i.e. $\Phi'(t) = \mathbf{A}(t)\Phi(t)$ and Φ is nonsingular. Then from the periodicity of $\mathbf{A}(\cdot)$, $\Phi(t+T)$ is again satisfies a fundamental solution. Since there are only n independent solutions of (1), thus there exists a constant nonsingular matrix C such that

$$\Phi(t+T) = \Phi(t)C.$$

Since any solution $\mathbf{y}$ can be expressed as $\Phi(t)c$ for some constant vector c, we get

$$\mathbf{y}(t+T) = \Phi(t)Cc.$$

Now if we choose c to be an eigenvector of C with eigenvalue ρ. Then the corresponding $\mathbf{y}(t) := \Phi(t)c$ satisfies

$$\mathbf{y}(t+T) = \rho\mathbf{y}(t).$$

Let C has eigenvalues $\rho_1, ..., \rho_n$. with eigenvectors $c_1, ..., c_n$. Then the solution

$$\mathbf{y}_i(t) := \Phi(t)c_i$$

satisfies

$$\mathbf{y}_i(t+T) = \rho_i\mathbf{y}_i(t),$$

The above definition of ρ_i are independent of the choice of a particular choice of the fundamental solution. If $\Psi(t)$ is another fundamental solution, then there exists a constant D such that $\Psi(t) = \Phi(t)D$. It is easy to check from this that

$$C := \Phi(T)\Phi(0)^{-1} = \Psi(T)\Psi(0)^{-1}.$$

t=0

The eigenvalues ρ_i are called the characteristic values of (1). Notice that the Wronskian $W(t) := det\,\Phi(t)$ satisfies

$$W'(t) = (tr(A(t))W(t).$$

Hence, $W(t) = \exp\left(\int_0^t (tr(A(s))\,ds\right) \neq 0$. Thus, $\rho_i \neq 0$ for all i. It is convinient to express $\rho_i = e^{T\mu_i}$. The constants μ_i, $i = 1, ..., n$ are called the Floquet exponents of (1). If we define $\phi_i(t) = \mathbf{y}(t)e^{-\mu_i t}$, then ϕ_i is a T-periodic function.

$$\begin{aligned}
\phi_i(t+T) &= \mathbf{y}(t+T)e^{-\mu_i(t+T)} \\
&= e^{\mu_i T}\mathbf{y}_i(t)e^{-\mu_i(t+T)} \\
&= \mathbf{y}_i(t)e^{-\mu_i t} \\
&= \phi_i(t).
\end{aligned}$$

Thus, we may express

$$\mathbf{y}_i(t) = e^{\mu_i t}\phi_i(t).$$

If the real part of $\mu_i \leq 0$, then the corresponding $\mathbf{y}_i(t)$ is bounded for $t \geq 0$.

2 Sturm-Liouville equations

We consider one dimensional Sturm-Liouville equation over periodic structure:

$$(p(x)u')' + q(x)u = 0 \tag{2}$$

where $p(\cdot) \neq 0$ and $q(\cdot)$ are periodics function with period d. By considering

$$\mathbf{y}(x) = \begin{pmatrix} u \\ p(x)u' \end{pmatrix},$$

this second order equation can be reduced to a first-order 2×2 system:

$$\mathbf{y}' = \mathbf{A}(x)\mathbf{y},\ \mathbf{A} = \begin{pmatrix} 0 & \frac{1}{p} \\ q & 0 \end{pmatrix},$$

According to the Floquet theory, there exist solutions $\mathbf{y}_i$, $i = 1, 2$ and Floquet exponents μ_i such that

$$\mathbf{y}_i(x + d) = e^{\mu_i d}\mathbf{y}_i(x).$$

The characteristic values e^{μ_i}, $i = 1, 2$ are the eigenvalues of the fundamental matrix $\Phi(d)$, where $\Phi(0) = id$. Notice that the Wronskian $W(x) := det\,\Phi(x)$ satisfies $W'(x) = tr(\mathbf{A}(x))W(x) = 0$. Hence the Wranskian $W(x) \equiv 1$. In other words, $e^{\mu_1 d}e^{\mu_2 d} = 1$. We get

$$\mu_1 + \mu_2 = 0.$$

We may express

$$\mu_1 = ik,\ \mu_2 = -ik$$

for some complex number k.

In terms of u, we get

$$u(x + d) = e^{\pm ikd}u(x),$$

If we express $u(x) = e^{\pm ikx}\phi(x, k)$, then $\phi(\cdot, k)$ is a d-periodic function.

We are interested in those bounded solutions. These correspond to those k which are real. Notice that we can just consider the class

$$u(x) = e^{ikx}\phi(x, k)$$

with $k \in [0, 2\pi/d]$. This is because

$$e^{i(k+2n\pi/d)x} = e^{ikx}e^{2n\pi x/d}$$

The last term is a d-periodic function which can be embeded into ϕ. Thus, for those k not in $[0, 2\pi/d]$ (including negative k's), we can always shift it into $[0, 2\pi/d]$.

To conclude, all bounded solutions of (2) can be expressed as

$$u(x) = e^{ikx}\phi(x, k)$$

for some $k \in [0, 2\pi/d]$ and $\phi(\cdot, k)$ is a d-periodic function.

Notice that the function $\phi(\cdot, k)$ satisfies

$$(\partial_x + ik)[p(x)(\partial_x + ik)\phi] + q(x)\phi = 0.$$

We denote this operator by A_k, called the shift cell operator. This operator with periodic boundary condition is self-adjoint. Thus, its eigenvalues are real and its engenvectors $\{\phi_m(\cdot, k)\}_{m=0}^{\infty}$ constitute an orthonormal basis in $L^2_p[0, d]$, the space of all square summable periodic functions.

3 Project 1: Wave motion on a periodic structure

Consider a string which is made of two materials periodically. The period is d. The governing equation for the string is

$$\rho(x)u_{tt} = (T(x)u_x)_x, x \in \mathbb{R}$$

where the density $\rho(x)$ and the tension $T(x)$ are d-periodic functions and piecewise constants in a period:

$$\rho(x) = \begin{cases} \rho_1 & 0 < x < a \\ \rho_2 & a < x < d \end{cases} \quad T(x) = \begin{cases} T_1 & 0 < x < a \\ T_2 & a < x < d \end{cases}$$

The natural interface condition at $x = a$ is

$$[u]_a = 0, \quad [Tu_x]_a = 0. \tag{3}$$

Here $[u]_a := u(a+) - u(a-)$.

We look for solution of the form: $u(x,t) = e^{i\omega t}u(x)$. This reduces the problem to the follwing eigenvalue problem:

$$(Tu(x)')' = -\omega^2\rho(x)u(x).$$

At the interfaces, the above interface conditions (3) should be satisfied. Since the physical domain is periodic, we look for bounded solutions. According to Floquet theory, those bounded solutions have the form:

$$u(x+d) = e^{ikd}u(x). \tag{4}$$

We thus impose this Bloch boundary condition and the problem is reduced to an eigenvalue problem on $[0,d]$.

Project goal: This project is to find all eigenvalues $\omega_m(k)$ and eigenvectors $u_m(x,k)$ *explicitly*, or equivalently, $\phi_m(x,k)$, which is a periodic function and is defined as $u_m(x,k)e^{-ikx}$. If possible, you may plot the functions $\omega_m(k)$ and $u_m(x,k)$ for $m = 0,1,2,3$.

Remarks. The remarks below help you to study this problem further. It is optional to write anything about these remarks.

1. The shift cell operator A_k is self-adjoint. Its eigenfunctions $\phi_m(\cdot,k)$ constitutes an orthonormal basis for all periodic functions on $[0,d]$.

2. Any nice function on $\mathbb{R}$ can be expanded as

$$f(x) = \sum_m \frac{d}{2\pi}\int_0^{2\pi/d} \hat{f}_m(k)e^{ikx}\phi_m(x,k)\,dk$$

where

$$\hat{f}_m(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\phi_m(x,k)\,dx.$$

Further, we have the following Parseval equality:

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \frac{d}{2\pi}\int_0^{2\pi/d} |\hat{f}_m(k)|^2\,dk$$

3. The general solutions can be expressed as

$$u(x,t) = \sum_m \frac{d}{2\pi}\int_0^{2\pi/d} (a_m(k)e^{-i\omega_m(k)t} + b_m(k)e^{i\omega_m(k)t})e^{ikx}\phi_m(x,k)\,dk \tag{5}$$

where the coefficients can be determined from the initial data

$$u(x,0) = f(x), \ u_t(x,0) = g(x), x \in \mathbb{R}.$$