Appendix C

Scalar functions

C.1 Definition

We define the one-point, two-point, three-point and four-point functions as [69, 105]

$$\mu^{2\epsilon}\int\frac{d^n\ell}{(2\pi)^n}\frac{1}{\ell^2-m^2}=\frac{i}{16\pi^2}A_0(m), \tag{C.1}$$

$$\mu^{2\epsilon}\int\frac{d^n\ell}{(2\pi)^n}\frac{1,\ell_\mu,\ell_\mu\ell_\nu}{(\ell^2-m_1^2)\left[(\ell+k)^2-m_2^2\right]}=\frac{i}{16\pi^2}B_0,B_\mu,B_{\mu\nu}(k,m_1,m_2) \tag{C.2}$$

$$\mu^{2\epsilon}\int\frac{d^n\ell}{(2\pi)^n}\frac{\ell^2,\ell^2\ell_\mu}{(\ell^2-m_1^2)\left[(\ell+k)^2-m_2^2\right]}=\frac{i}{16\pi^2}\tilde{B}_0,\tilde{B}_\mu(k,m_1,m_2) \tag{C.3}$$

$$\mu^{2\epsilon}\int\frac{d^n\ell}{(2\pi)^n}\frac{1,\ell_\mu,\ell_\mu\ell_\nu}{(\ell^2-m_1^2)\left[(\ell+k)^2-m_2^2\right]\left[(\ell+k+s)^2-m_3^2\right]}$$

$$=\frac{i}{16\pi^2}C_0,C_\mu,C_{\mu\nu}(k,s,m_1,m_2,m_3) \tag{C.4}$$

$$\mu^{2\epsilon}\int\frac{d^n\ell}{(2\pi)^n}\frac{1,\ell_\mu,\ell_\mu\ell_\nu,\ell_\mu\ell_\nu\ell_\alpha,\ell_\mu\ell_\nu\ell_\alpha\ell_\beta}{(\ell^2-m_1^2)\left[(\ell+k)^2-m_2^2\right]\left[(\ell+k+s)^2-m_3^2\right]\left[(\ell+k+s+p)^2-m_4^2\right]}$$

$$=\frac{i}{16\pi^2}D_0,D_\mu,D_{\mu\nu},D_{\mu\nu\alpha},D_{\mu\nu\alpha\beta}(k,s,p,m_1,m_2,m_3,m_4) \tag{C.5}$$

The integration formula of the scalar functions A_0, B_0, B_1, C_0 and D_0 are given as follows:

$$A_0(m) = m^2 \left[\Delta - \ln \frac{m^2}{\mu^2} + 1\right], \tag{C.6}$$

$$B_n(p_1, m_1, m_2) = (-1)^n \left[\frac{\Delta}{n+1} - \int_0^1 dx\, x^n \ln \frac{x^2 p_1^2 - x(p_1^2 + m_1^2 - m_2^2) + m_1^2}{\mu^2}\right], \tag{C.7}$$

$$C_0(p_1, p_2, m_1, m_2, m_3) = \int_0^1 dx \int_0^x dy \frac{1}{ax^2 + by^2 + cxy + dx + ey + f}, \tag{C.8}$$

where

$$\begin{aligned} &\Delta = \frac{1}{\epsilon} - \gamma_E + \ln 4\pi, \\ &a = -p_2^2,\ b = -p_1^2,\ c = -2p_1 \cdot p_2,\ d = -m_2^2 + m_3^2 + p_2^2, \\ &e = -m_1^2 + m_2^2 + p_1^2 + 2p_1 \cdot p_2,\ f = -m_3^2, \end{aligned} \tag{C.9}$$

and the four-point scalar function is given by

$$\begin{aligned} &D_0(p_1, p_2, p_3, m_1, m_2, m_3, m_4) \\ = &\int_0^1 dx \int_0^x dy \int_0^y dz \frac{1}{\left[ax^2 + by^2 + gz^2 + cxy + hxz + jyz + dx + ey + kz + f\right]^2}, \end{aligned}$$

where

$$\begin{aligned} &a = p_3^2,\ b = p_2^2,\ g = p_1^2, \\ &c = 2p_2 \cdot p_3,\ h = 2p_1 \cdot p_3,\ j = 2p_1 \cdot p_2, \\ &d = m_3^2 - m_4^2 - p_3^2,\ e = m_2^2 - m_3^2 - 2p_2 \cdot p_3 - p_2^2, \\ &k = m_1^2 - m_2^2 - 2p_1 \cdot p_2 - 2p_1 \cdot p_3 - p_1^2,\ f = m_4^2. \end{aligned}$$

C.2 Tensor reduction

Lorentz covariance of the integrals allows to decompose the tensor integrals into tensors constructed from the external momenta and the metric tensor $g_{\mu\nu}$ with the scalar coefficients. The explicit Lorentz decompositions for the vector and tensor integrals B_μ, C_μ and $C_{\mu\nu}$ are given below. We are not going to show the four-point function because our calculation does not involve the box diagrams, but they can be found in Ref. [106].

For the two-point functions we have the relations

$$
\begin{aligned}
B_\mu(\ell, m_1, m_2) &= \ell_\mu B_1(\ell, , m_1, m_2), && \text{(C.10)}\\
B_1(\ell, m_1, m_2) &= \frac{1}{2\ell^2}\left[A_0(m_1) - A_0(m_2) - (\ell^2 + m_1^2 - m_2^2)B_0(\ell, m_1, m_2)\right] && \text{(C.11)}\\
B_{\mu\nu}(\ell, m_1, m_2) &= \ell_\mu \ell_\nu B_{21} + g_{\mu\nu} B_{22}, && \text{(C.12)}\\
B_{21}(\ell, m_1, m_2) &= \frac{1}{3\ell^2}\left[A_0(m_2) - m_1^2 B_0 - 2(\ell^2 + m_1^2 - m_2^2)B_1 - \frac{m_1^2 + m_2^2}{2} + \frac{\ell^2}{6}\right], && \text{(C.13)}\\
B_{22}(\ell, m_1, m_2) &= \frac{1}{6}\left[A_0(m_2) + 2m_1^2 B_0 + (\ell^2 + m_1^2 - m_2^2)B_1 + m_1^2 + m_2^2 - \frac{\ell^2}{3}\right], && \text{(C.14)}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{B}_\mu(\ell, m_1, m_2) &= \ell_\mu \tilde{B}_1(\ell, m_1, m_2), && \text{(C.15)}\\
\tilde{B}_1(\ell, m_1, m_2) &= -A_0(m_2) + m_1^2 B_1(\ell, m_1, m_2), && \text{(C.16)}\\
\tilde{B}_0(\ell, m_1 m_2) &= A_0(m_2) + m_1^2 B_0(\ell, m_1, m_2). && \text{(C.17)}
\end{aligned}
$$

For the three-point functions we have the relations

$$
\begin{aligned}
C_\mu(\ell, s, m_1, m_2, m_3) &= \ell_\mu C_{11} + s_\mu C_{12}, && \text{(C.18)}\\
C_{\mu\nu}(\ell, s, m_1, m_2, m_3) &= \ell_\mu \ell_\nu C_{21} + s_\mu s_\nu C_{22} + (\ell_\mu s_\nu + \ell_\nu s_\mu)C_{23} + g_{\mu\nu} C_{24},
\end{aligned}
$$

where

$$
\begin{aligned}
C_{11} &= \frac{1}{\kappa}\left[s^2 R_1 - \ell \cdot s R_2\right], && \text{(C.19)}\\
C_{12} &= \frac{1}{\kappa}\left[-\ell \cdot s R_1 + \ell^2 R_2\right], && \text{(C.20)}\\
C_{24} &= \frac{1}{4}\left[B_0(s, m_2, m_3) + r_1 C_{11} + r_2 C_{12} + 2m_1^2 C_0 + 1\right], && \text{(C.21)}
\end{aligned}
$$

$$C_{21} = \frac{1}{\kappa}\left[s^2 R_3 - \ell \cdot s R_5\right], \tag{C.22}$$
$$C_{22} = \frac{1}{\kappa}\left[-\ell \cdot s R_4 + \ell^2 R_6\right], \tag{C.23}$$
$$C_{23} = \frac{1}{\kappa}\left[-\ell \cdot s R_3 + \ell^2 R_5\right] = \frac{1}{\kappa}\left[s^2 R_4 - \ell \cdot s R_6\right]. \tag{C.24}$$

with

$$\kappa = \ell^2 s^2 - (\ell \cdot s)^2. \tag{C.25}$$
$$r_1 = \ell^2 + m_1^2 - m_2^2. \tag{C.26}$$
$$r_2 = (\ell + s)^2 - \ell^2 + m_2^2 - m_3^2. \tag{C.27}$$
$$R_1 = \frac{1}{2}\left[B_0(\ell + s, m_1, m_3) - B_0(s, m_2, m_3) - (\ell^2 + m_1^2 - m_2^2)C_0\right]. \tag{C.28}$$
$$R_2 = \frac{1}{2}\left[B_0(\ell, m_1, m_2) - B_0(\ell + s, m_1, m_3) + (\ell^2 - (\ell + s)^2 - m_2^2 + m_3^2)C_0\right], \tag{C.29}$$
$$R_3 = -C_{24} - \frac{1}{2}\left[r_1 C_{11} - B_1(\ell + s, m_1, m_3) - B_0(s, m_2, m_3)\right], \tag{C.30}$$
$$R_4 = -\frac{1}{2}\left[r_1 C_{12} - B_1(\ell + s, m_1, m_3) + B_1(s, m_2, m_3)\right], \tag{C.31}$$
$$R_5 = -\frac{1}{2}\left[r_2 C_{11} - B_1(\ell, m_1, m_2) + B_1(\ell + s, m_1, m_3)\right], \tag{C.32}$$
$$R_6 = -C_{24} - \frac{1}{2}\left[r_2 C_{12} + B_1(\ell + s, m_1, m_3)\right]. \tag{C.33}$$

C.3 UV-divergent parts of tensor integrals

For practical calculations it is useful to know the UV-divergent parts of the tensor integrals explicitly. We give directly the pole structures of the divergent one-loop tensor coefficient integrals up to terms of the order $\mathcal{O}(\epsilon)$

$$A_0(m) = \frac{m^2}{\epsilon}, \tag{C.34}$$
$$B_0(\ell, m_1, m_2) = \frac{1}{\epsilon}, \tag{C.35}$$
$$B_1(\ell, m_1, m_2) = -\frac{1}{2\epsilon}, \tag{C.36}$$

$$B_{21}(\ell, m_1, m_2) = \frac{1}{3\epsilon}, \tag{C.37}$$

$$B_{22}(\ell, m_1, m_2) = -\frac{1}{12}\left(\ell^2 - 3m_1^2 - 3m_2^2\right), \tag{C.38}$$

$$C_{24}(\ell, s, m_1, m_2, m_3) = \frac{1}{4\epsilon}. \tag{C.39}$$

C.4 Scalar two-point function

For the two-point function B_0, we notice that the Feynman parameter integration itself cannot result in a pole, cf. Eq. C.7, therefore we can use a series expansion in ϵ to simplify the integration. The relevant B_0 functions used in this calculation are listed below.

$$B_0(0,0,0) = 0,$$

$$B_0(\hat{s}, 0, m_t^2) = \frac{iC_\epsilon}{16\pi^2}\left\{\frac{1}{\epsilon} + 2 - \frac{\hat{s}_1}{\hat{s}}\ln\frac{\hat{s}_1}{m_t^2}\right\},$$

$$B_0(m_t^2, 0, 0) = \frac{iC_\epsilon}{16\pi^2}\left\{\frac{1}{\epsilon} + 2 - i\pi\right\},$$

$$B_0(\hat{s}, 0, 0) = \frac{iC_\epsilon}{16\pi^2}\left\{\frac{1}{\epsilon} + 2 - \ln\frac{\hat{s}}{m_t^2} + i\pi\right\},$$

$$B_0(0, m_t^2, 0) = \frac{iC_\epsilon}{16\pi^2}\left\{\frac{1}{\epsilon} + 1\right\},$$

$$B_0(m_t^2, m_t^2, 0) = \frac{iC_\epsilon}{16\pi^2}\left\{\frac{1}{\epsilon} + 2\right\},$$

$$B_0(\hat{s}, m_t^2, m_t^2) = \frac{iC_\epsilon}{16\pi^2}\left\{\frac{1}{\epsilon} + 2 + \beta\ln\xi\right\}, \tag{C.40}$$

$$B_0(0, m_t^2, m_t^2) = \frac{iC_\epsilon}{16\pi^2}\frac{1}{\epsilon}, \tag{C.41}$$

where

$$
\begin{aligned}
C_\epsilon &= \left(\frac{4\pi\mu^2}{m_t^2}\right)^\epsilon \Gamma(1+\epsilon), \\
\hat{s}_1 &= \hat{s} - m_t^2, \\
\beta &= \sqrt{1 - \frac{4m^2}{\hat{s}}}, \\
\xi &= \frac{1-\beta}{1+\beta}.
\end{aligned}
$$

The B_1 function can be easily derived from B_0 function through the following relation,

$$
B_1(x, m_1, m_2) = \frac{(m_2 - m_1)\,(B_0(x, m_1, m_2) - B_0(0, m_1, m_2))}{2x} - \frac{1}{2} B_0(x, m_1, m_2). \tag{C.42}
$$

Then, we obtain

$$
B_1(0,0,0) = 0, \tag{C.43}
$$

$$
B_1(m_t^2, m_t^2, 0) = \frac{iC_\epsilon}{16\pi^2}\left\{-\frac{1}{2\epsilon} - \frac{3}{2}\right\}, \tag{C.44}
$$

$$
B_1(\hat{s}, 0, 0) = \frac{iC_\epsilon}{16\pi^2}\left\{-\frac{1}{2\epsilon} - 1 + \frac{1}{2}\ln\frac{\hat{s}}{m_t^2}\right\}, \tag{C.45}
$$

$$
B_1(\hat{s}, 0, m_t^2) = \frac{iC_\epsilon}{16\pi^2}\left\{-\frac{1}{2\epsilon} - \frac{\hat{s}_1}{\hat{s}} - \frac{m_t^2}{2\hat{s}} + \frac{\hat{s}_1^2}{2\hat{s}^2}\ln\frac{\hat{s}_1}{m_t^2}\right\}. \tag{C.46}
$$

C.5 Scalar three-point function

C.5.1 Analytical result of $C_0(p_1^2, p_2^2, 0, 0, 0)$

When the loop momentum in the scalar function goes to infinity, the ultra-violet divergence will appear and can be factorized from the rest finite components. Beside of the UV divergence, the infrared divergence may also arise in the case of the massless internal particles when the loop momentum goes to zero.

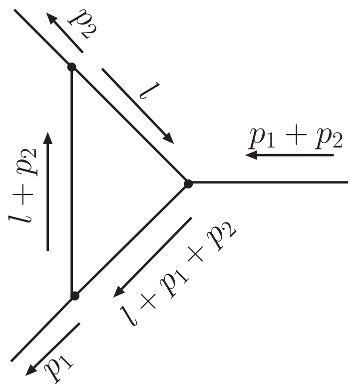

Figure C.1: The three-point green function with massless internal particles.

The exchange of gluon between the initial state massless quarks leads to the following divergent three-point integral:

$$C_0(p_1^2, p_2^2, 0, 0, 0) \equiv C_0^{\mathcal{A}} = \mu^{2\epsilon} \int \frac{d^n \ell}{(2\pi)^n} \frac{1}{\ell^2 (\ell + p_1)^2 (\ell + p_1 + p_2)^2}. \tag{C.47}$$

The integral looks impossible, and in fact it will not be easy. The evaluation of such integrals requires another piece of computational technology, known as the method of Feynman parameters *After the Feynman parameterization, we get

$$C_0^{\mathcal{A}} = \mu^{2\epsilon} \int_0^1 dx \int_0^1 dy \int \frac{d^n \ell}{(2\pi)^n} \frac{1}{\left\{ [\ell + (p_1 + x p_2) y]^2 + \hat{s} x y (1-y) \right\}^3}, \tag{C.48}$$

*Here is the MATHEMATICA code for Feynman parameterization:

$$\frac{1}{A_1^{m_1} A_2^{m_2} \cdots A_n^{m_n}} = \int_0^1 dx_1 \cdots dx_n \delta \left(\sum x_i - 1 \right) \frac{\prod x_i^{m_i - 1}}{[\sum x_i A_i]^{\sum m_i}} \frac{\Gamma(m_1 + \cdots + m_n)}{\Gamma(m_1) \cdots \Gamma(m_n)}.$$

The function FeynmanParameterization is defined as following:

```
FeynmanParameterization[p : {{_, _}..}, x_List]/; Length[x] === Length[p] :=
Module[
  {d, a}, {d, a} = Transpose[p];
```

$$\frac{\text{Gamma[Plus @@ a]}}{\text{Times @@ Gamma[a]}} (\text{DiracDelta[}1 - \text{Plus @@ x] Times @@ x}^{\text{a}-1})/(\text{Plus @@ (d x)})^{\text{Plus @@ a}}]$$

Usage:

FeynmanParameterization[$\{\{A_1, m_1\}, \cdots, \{A_n, m_n\}\}$, $\{x_1, \cdots, x_n\}$].

where $\hat{s} = (p_1 + p_2)^2$. Making the substitution,

$$\ell \to k - (p_1 + x p_4) y,$$

the scalar function becomes

$$\begin{aligned}
C_0^{\mathcal{A}} &= \mu^{2\epsilon} \int_0^1 dx \int_0^1 dy \, (2y) \int \frac{d^n k}{(2\pi)^n} \frac{1}{\left\{k^2 + \hat{s}xy(1-y)\right\}^3} \\
&= \mu^{2\epsilon} \int_0^1 dx \int_0^1 dy \, (2y) \frac{-i}{(4\pi)^{\frac{n}{2}}} \frac{\Gamma(3-\frac{n}{2})}{\Gamma(3)} \left[-\hat{s}xy(1-y)\right]^{\frac{n}{2}-3} \\
&= \frac{i}{16\pi^2} \left(\frac{4\pi\mu^2}{\hat{s}}\right)^{\epsilon} \frac{\Gamma(1+\epsilon)}{\Gamma(3)} \frac{2}{\hat{s}} (-1)^{\epsilon} \int_0^1 \frac{dx}{x^{1+\epsilon}} \int_0^1 dy \frac{y}{[y(1-y)]^{1+\epsilon}} \\
&= \frac{i}{16\pi^2} \left(\frac{4\pi\mu^2}{\hat{s}}\right)^{\epsilon} \frac{\Gamma(1+\epsilon)}{\Gamma(3)} \frac{2}{\hat{s}} (-1)^{\epsilon} \frac{\Gamma^2(-\epsilon)}{\Gamma(1-2\epsilon)}, \qquad \text{(C.49)}
\end{aligned}$$

where we have chosen $n = 4 - 2\epsilon$.

Using the properties of Γ-functions in F and the following relations

$$\Re(-1)^{\epsilon} = 1 - \frac{\pi^2}{2}\epsilon^2, \qquad \text{(C.50)}$$

$$\left(\frac{\hat{s}}{m_t^2}\right)^{-\epsilon} = 1 - \epsilon \ln \frac{\hat{s}}{m_t^2} + \frac{\epsilon^2}{2} \ln^2 \left(\frac{\hat{s}}{m_t^2}\right), \qquad \text{(C.51)}$$

we get the three-point scalar function with massless internal particles as

$$C_0^{\mathcal{A}} = \frac{i}{16\pi^2} \left(\frac{4\pi\mu^2}{m_t^2}\right)^{\epsilon} \frac{\Gamma(1+\epsilon)}{\hat{s}} \left\{\frac{1}{\epsilon^2} - \frac{1}{\epsilon} \ln \frac{\hat{s}}{m_t^2} + \frac{1}{2} \ln^2 \left(\frac{\hat{s}}{m_t^2}\right) - \frac{2\pi^2}{3}\right\}. \qquad \text{(C.52)}$$

Here the double poles shows up as one expects due to the soft and collinear singularities.

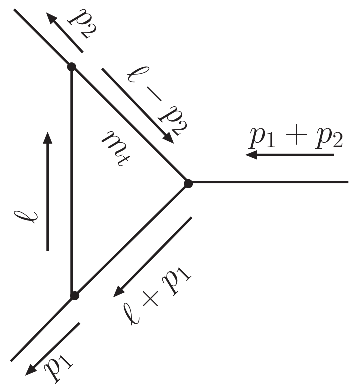

Figure C.2: The three-point green function with one massive internal particle.

C.5.2 Analytical result of $C_0\left((p_1+p_2)^2, (-p_2)^2, 0, m_t^2, 0\right)$

The exchange of gluon between the final state top quark and bottom quarks leads to the following divergent three-point integral (cf. Fig. C.2):

$$C_0\left((-p_2)^2, (p_1+p_2)^2, 0, m_t^2, 0\right) \equiv C_0^{\mathcal{B}} = \mu^{2\epsilon} \int \frac{d^n\ell}{(2\pi)^n} \frac{1}{\ell^2(\ell+p_1)^2\left((\ell-p_2)^2 - m_t^2\right)}. \tag{C.53}$$

After the Feynman parameterization we obtain

$$C_0^{\mathcal{B}} = \mu^{2\epsilon} \int_0^1 dx \int_0^1 dy\,(2y) \int \frac{d^n\ell}{(2\pi)^2} \frac{1}{\left\{\ell^2 + 2\ell\cdot p\right\}^3}, \tag{C.54}$$

where $p = [xp_1 - (1-x)p_2]\,y$. After substitution $\ell \to k - p$, we have

$$\begin{aligned} C_0^{\mathcal{B}} &= \mu^{2\epsilon} \int_0^1 dx \int_0^1 dy\,(2y) \int \frac{d^nk}{(2\pi)^n} \frac{1}{\left\{k^2 - p^2\right\}^3} \\ &= -\frac{i}{16\pi^2}(4\pi\mu^2)^\epsilon \Gamma(1+\epsilon) \int_0^1 dx\,(1-x)^{-1-\epsilon}(m_t^2 - \hat{s}x)^{-1-\epsilon} \int_0^1 dy\,y^{-1-2\epsilon} \\ &= \frac{i}{16\pi^2}\left(\frac{4\pi\mu^2}{m_t^2}\right)^\epsilon \frac{\Gamma(1+\epsilon)}{m_t^2} \frac{1}{2\epsilon} \frac{\Gamma(-\epsilon)}{\Gamma(1-\epsilon)} {}_2F_1\left(1+\epsilon, 1, 1-\epsilon, \frac{\hat{s}}{m_t^2}\right). \end{aligned} \tag{C.55}$$

In the limit $\epsilon \to 0$,

$$
\begin{aligned}
& {}_2F_1\left(1+\epsilon, 1, 1-\epsilon, \frac{\hat{s}}{m_t^2}\right) \\
= & \left(1-\frac{\hat{s}}{m_t^2}\right)^{-1-2\epsilon} {}_2F_1\left(-2\epsilon, -\epsilon, 1-\epsilon, \frac{\hat{s}}{m_t^2}\right) \\
= & \left(-\frac{\hat{s}_1}{m_t^2}\right)^{-1-2\epsilon} \times \left[1+2\epsilon^2 \mathrm{Li}\left(\frac{\hat{s}}{m_t^2}\right)\right],
\end{aligned}
\tag{C.56}
$$

therefore,

$$
C_0^{\mathcal{B}} = \frac{i}{16\pi^2}\left(\frac{4\pi\mu^2}{m_t^2}\right)^{\epsilon} \frac{\Gamma(1+\epsilon)}{\hat{s}_1}\left\{\frac{1}{2\epsilon^2} - \frac{1}{\epsilon}\ln\left(-\frac{\hat{s}_1}{m_t^2}\right) - \mathrm{Li}_2\left(\frac{\hat{s}}{\hat{s}_1}\right) + \frac{1}{2}\ln^2\left(-\frac{\hat{s}_1}{m_t^2}\right)\right\}. \tag{C.57}
$$

The logarithm is commonly defined with a branch cut along the negative real axis, therefore

$$
\ln\left(-\frac{\hat{s}}{m^2}\right) \to \ln\left(-\frac{\hat{s}+i\epsilon m^2}{m^2}\right) = \ln\left(\frac{\hat{s}}{m^2}\right) - i\pi. \tag{C.58}
$$

Applying Eq. C.58 into $C_0^{\mathcal{B}}$, we obtain

$$
C_0^{\mathcal{B}} = \frac{i}{16\pi^2} C_\epsilon \frac{1}{\hat{s}_1}\left\{\frac{1}{2\epsilon^2} - \frac{1}{\epsilon}\ln\left(\frac{\hat{s}_1}{m_t^2}\right) - \mathrm{Li}_2\left(\frac{\hat{s}}{\hat{s}_1}\right) + \frac{1}{2}\ln^2\left(\frac{\hat{s}_1}{m_t^2}\right) - \frac{\pi^2}{2}\right\}, \tag{C.59}
$$

where

$$
C_\epsilon = \left(\frac{4\pi\mu^2}{m_t^2}\right)^{\epsilon}\Gamma(1+\epsilon).
$$

Note that we only keep the real parts in the calculation and discard the imaginary parts since it will contribute at the next-leading order.