粒子物理

10. 电子-质子的弹性散射过程

曹庆宏

北京大学物理学院

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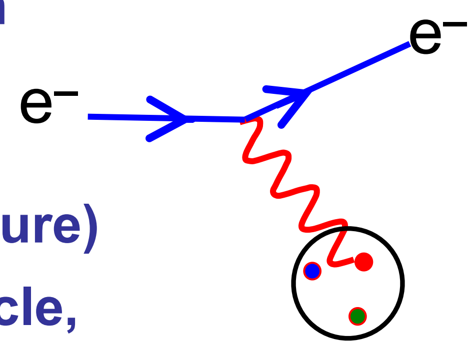
本节内容基于Thomson教授的幻灯片

Electron-Proton Scattering

- In this handout aiming towards a study of electron-proton scattering as a probe of the structure of the proton

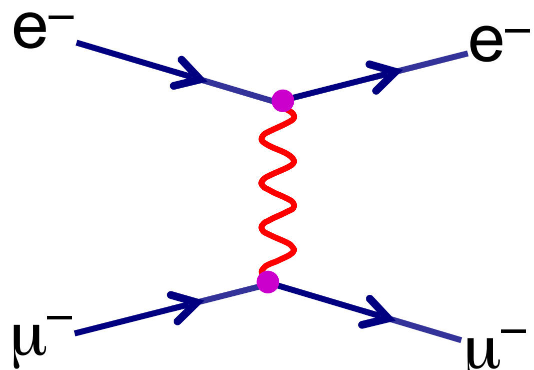


- Two main topics:
  - $e^-p \to e^-p$ elastic scattering
  - $e^-p \to e^-X$ deep inelastic scattering (next lecture)
- But first consider scattering from a point-like particle, e.g.

  $e^-\mu^- \to e^-\mu^-$

  i.e. the QED part of

  $(e^-q \to e^-q)$



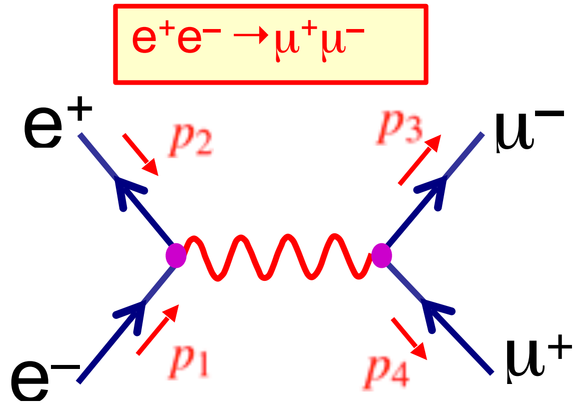
- Two ways to proceed:
  - perform QED calculation from scratch

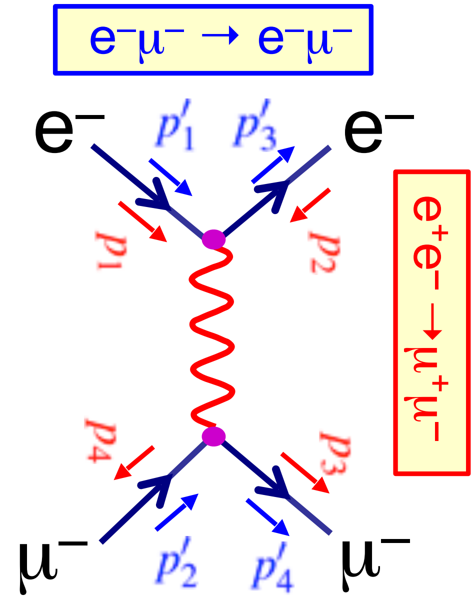
$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} \left[(p_1 . p_2)(p_3 . p_4) + (p_1 . p_4)(p_2 . p_3) \right] \qquad \textbf{(1)}$$

  - take results from $e^+e^- \to \mu^+\mu^-$ and use **"Crossing Symmetry"** to obtain the matrix element for $e^-\mu^- \to e^-\mu^-$

Crossing Symmetry

★ Having derived the Lorentz invariant matrix element for $e^+e^- \rightarrow \mu^+\mu^-$ "rotate" the diagram to correspond to $e^-\mu^- \rightarrow e^-\mu^-$ and apply the principle of crossing symmetry to write down the matrix element !





★ The transformation: $p_1 \rightarrow p_1'; \; p_2 \rightarrow -p_3'; \; p_3 \rightarrow p_4'; \; p_4 \rightarrow -p_2'$

Changes the spin averaged matrix element for

$e^-e^+ \rightarrow \mu^-\mu^+$ → $e^-\mu^- \rightarrow e^-\mu^-$

$p_1 \; p_2 \qquad p_3 \; p_4 \qquad\qquad p_1' \; p_2' \qquad p_3' \; p_4'$

- Take ME for $e^+e^- \rightarrow \mu^+\mu^-$ (page 143) and apply crossing symmetry:

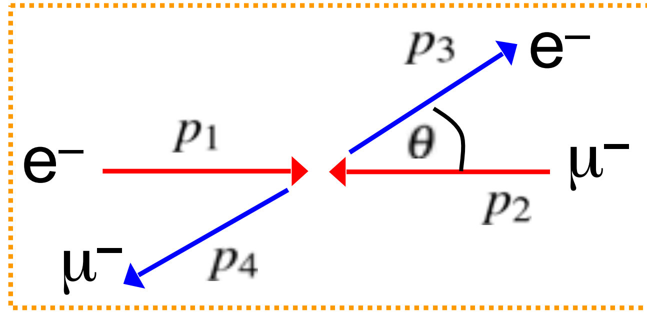
$$\langle|M_{fi}|^2\rangle = 2e^4 \frac{(p_1.p_3)^2 + (p_1.p_4)^2}{(p_1.p_2)^2} \rightarrow \langle|M_{fi}|^2\rangle = 2e^4 \frac{(p_1'.p_4')^2 + (p_1'.p_2')^2}{(p_1'.p_3')^2} \quad (1)$$

$$\langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p_1.p_4)^2 + (p_1.p_2)^2}{(p_1.p_3)^2} \quad \textbf{(2)}$$

$$\equiv 2e^4 \left(\frac{s^2 + u^2}{t^2} \right)$$

- **Work in the C.o.M:**

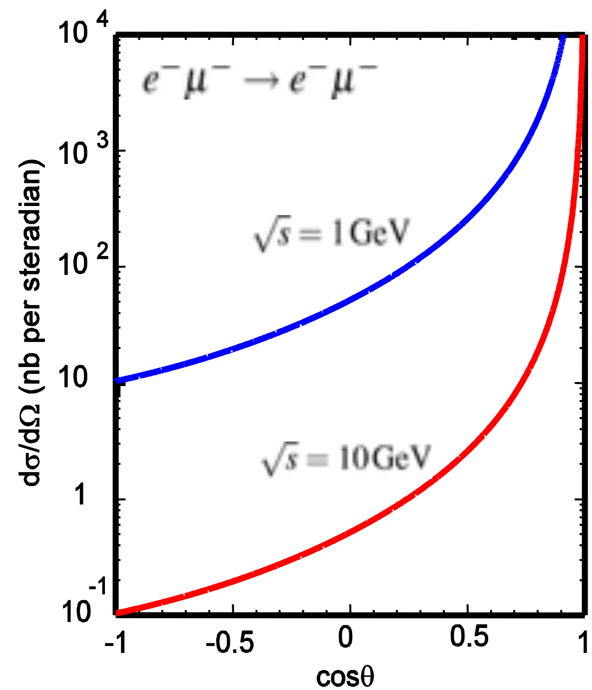
$$p_1 = (E, 0, 0, E) \qquad p_2 = (E, 0, 0, -E)$$
$$p_3 = (E, E\sin\theta, 0, E\cos\theta)$$
$$p_4 = (E, -E\sin\theta, 0, -E\cos\theta)$$



giving $p_1.p_2 = 2E^2; \quad p_1.p_3 = E^2(1-\cos\theta); \quad p_1.p_4 = E^2(1+\cos\theta)$

$$\rightarrow \quad \langle |M_{fi}|^2 \rangle = 2e^4 \frac{E^4(1+\cos\theta)^2 + 4E^4}{E^4(1-\cos\theta)^2}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \langle |M_{fi}|^2 \rangle = \frac{e^4}{8\pi^2 s} \frac{\left[1 + \frac{1}{4}(1+\cos\theta)^2\right]}{(1-\cos\theta)^2} \rightarrow$$



- **The <u>denominator</u> arises from the propagator** $-ig_{\mu\nu}/q^2$

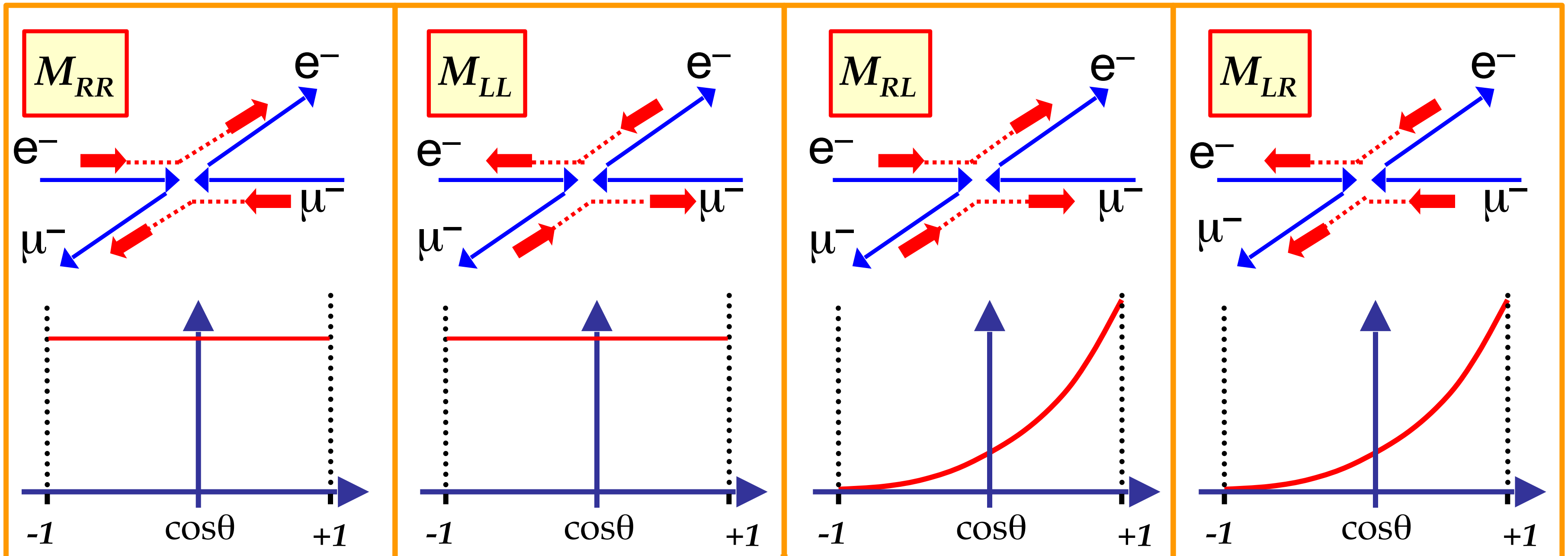
here $q^2 = (p_1 - p_3)^2 = E^2(1-\cos\theta)$

as $q^2 \rightarrow 0$ **the cross section tends to infinity.**

- What about the angular dependence of the **numerator**?

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{8\pi^2 s}\frac{\left[1+\frac{1}{4}(1+\cos\theta)^2\right]}{(1-\cos\theta)^2}$$

- The factor $1+\frac{1}{4}(1+\cos\theta)^2$ reflects helicity (really chiral) structure of QED
- Of the 16 possible helicity combinations only 4 are non-zero:



$S_z = 0$

$$\Rightarrow \frac{d\sigma}{d\Omega} \propto 1$$

i.e. no preferred polar angle

$S_z = +1$ $\qquad$ $S_z = -1$

$$\Rightarrow \frac{d\sigma}{d\Omega} \propto \frac{1}{4}(1+\cos\theta)^2$$

spin 1 rotation again

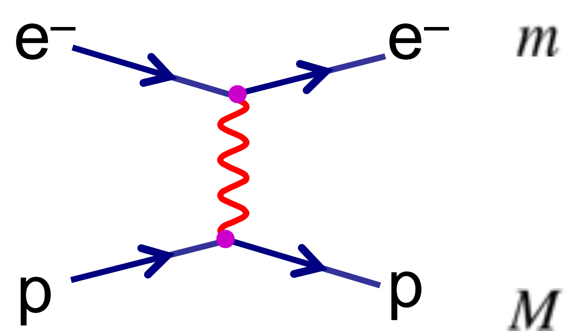
- The cross section calculated above is appropriate for the scattering of two spin half Dirac (i.e. point-like) particles in the ultra-relativistic limit (where both electron and muon masses can be neglected). In this case

$$\langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p_1.p_4)^2 + (p_1.p_2)^2}{(p_1.p_3)^2}$$

- We will use this again in the discussion of "Deep Inelastic Scattering" of electrons from the quarks within a proton.

- Before doing so we will consider the scattering of electrons from the composite proton - i.e. how do we know the proton isn't fundamental "point-like" particle ?

- In this discussion we will not be able to use the relativistic limit and require the general expression for the matrix element:

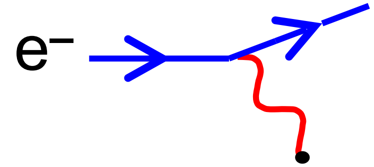


$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} \left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3) - (p_1.p_3)M^2 - (p_1.p_4)m^2 + 2m^2M^2 \right] \quad (3)$$

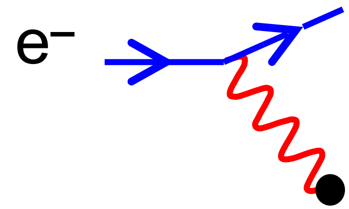
Probing the Structure of the Proton

★ In $e^-p \rightarrow e^-p$ scattering the nature of the interaction of the virtual photon with the proton depends strongly on wavelength

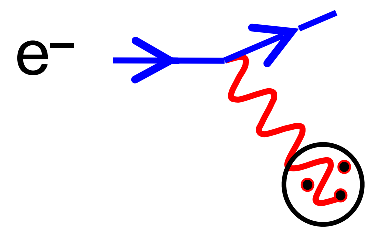
- ♦ At very low electron energies $\lambda \gg r_p$: the scattering is equivalent to that from a "point-like" spin-less object



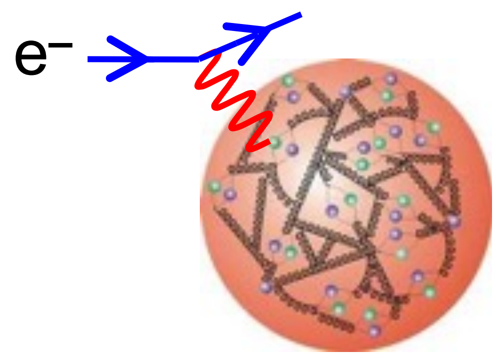
- ♦ At low electron energies $\lambda \sim r_p$: the scattering is equivalent to that from a extended charged object



- ♦ At high electron energies $\lambda < r_p$: the wavelength is sufficiently short to resolve sub-structure. Scattering from constituent quarks

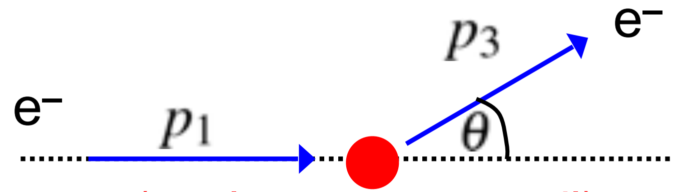


- ♦ At very high electron energies $\lambda \ll r_p$: the proton appears to be a sea of quarks and gluons.



Rutherford Scattering Revisited

★ **Rutherford scattering is the low energy limit where the recoil of the proton can be neglected and the electron is non-relativistic**



- **Start from RH and LH Helicity particle spinors**

$$u_\uparrow = N\begin{pmatrix} c \\ e^{i\phi}s \\ \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \end{pmatrix} \quad u_\downarrow = N\begin{pmatrix} -s \\ e^{i\phi}c \\ \frac{|\vec{p}|}{E+m}s \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}c \end{pmatrix} \quad \begin{aligned} N &= \sqrt{E+m}; \\ s &= \sin(\theta/2); \quad c = \cos(\theta/2) \end{aligned}$$

- **Now write in terms of:** $\alpha = \dfrac{|\vec{p}|}{E+m_e}$

Non-relativistic limit: $\alpha \rightarrow 0$

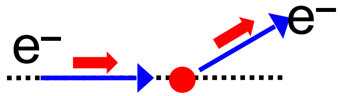
Ultra-relativistic limit: $\alpha \rightarrow 1$

$$\Rightarrow \quad u_\uparrow = N\begin{pmatrix} c \\ e^{i\phi}s \\ \alpha c \\ \alpha e^{i\phi}s \end{pmatrix} \quad u_\downarrow = N\begin{pmatrix} -s \\ e^{i\phi}c \\ \alpha s \\ -\alpha e^{i\phi}c \end{pmatrix}$$

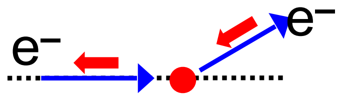
and the possible initial and final state electron spinors are:

$$u_\uparrow(p_1) = N_e\begin{pmatrix} 1 \\ 0 \\ \alpha \\ 0 \end{pmatrix} \quad u_\downarrow(p_1) = N_e\begin{pmatrix} 0 \\ 1 \\ 0 \\ -\alpha \end{pmatrix} \quad u_\uparrow(p_3) = N_e\begin{pmatrix} c \\ s \\ \alpha c \\ \alpha s \end{pmatrix} \quad u_\downarrow(p_3) = N_e\begin{pmatrix} -s \\ c \\ \alpha s \\ -\alpha c \end{pmatrix}$$

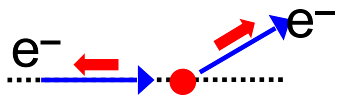
- Consider all four possible electron currents, i.e. Helicities R→R, L→L, L→R, R→L



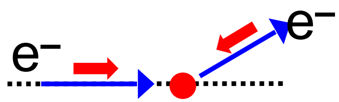
$$\bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = (E+m_e)\left[(\alpha^2+1)c, 2\alpha s, -2i\alpha s, 2\alpha c\right] \quad (4)$$



$$\bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = (E+m_e)\left[(\alpha^2+1)c, 2\alpha s, -2i\alpha s, 2\alpha c\right] \quad (5)$$



$$\bar{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) = (E+m_e)\left[(1-\alpha^2)s, 0, 0, 0\right] \quad (6)$$



$$\bar{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) = (E+m_e)\left[(\alpha^2-1)s, 0, 0, 0\right] \quad (7)$$

- In the relativistic limit ($\alpha = 1$), i.e. $E \gg m$

  (6) and (7) are identically zero; only R→R and L→L combinations non-zero

- In the non-relativistic limit, $|\vec{p}| \ll E$ we have $\alpha = 0$

$$\bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = \bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = (2m_e)\left[c, 0, 0, 0\right]$$

$$\bar{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) = -\bar{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) = (2m_e)\left[s, 0, 0, 0\right]$$

All four electron helicity combinations have non-zero Matrix Element

i.e. Helicity eigenstates ≠ Chirality eigenstates

- The initial and final state **proton** spinors (assuming no recoil) are:

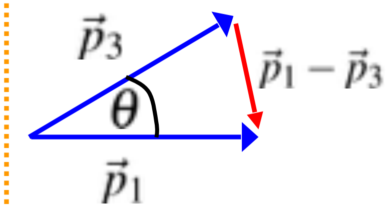
$$u_\uparrow(0) = \sqrt{2M_p}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad u_\downarrow(0) = \sqrt{2M_p}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

Solutions of Dirac equation for a particle at rest

giving the proton currents:

$$j_{p\uparrow\uparrow} = j_{p\downarrow\downarrow} = 2M_p\,(1,0,0,0)$$
$$j_{p\uparrow\downarrow} = j_{p\downarrow\uparrow} = 0$$

- The spin-averaged ME summing over the 8 allowed helicity states



$$\langle|M_{fi}^2|\rangle = \frac{1}{4}\frac{e^4}{q^4}(16M_p^2m_e^2)(4c^2+4s^2) = \frac{16M_p^2m_e^2e^4}{q^4}$$

where $q^2 = (p_1 - p_3)^2 = (0, \vec{p}_1 - \vec{p}_3)^2 = -4|\vec{p}|^2\sin^2(\theta/2)$

$$\langle|M_{fi}^2|\rangle = \frac{M_p^2m_e^2e^4}{|\vec{p}|^4\sin^4(\theta/2)}$$

Note: in this limit all angular dependence is in the propagator

- The formula for the differential cross-section in the lab. frame is

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M+E_1-E_1\cos\theta}\right)^2|M_{fi}|^2 \qquad (8)$$

- Here the electron is non-relativistic so $E \sim m_e \ll M_p$ and we can neglect E_1 in the denominator of equation (8)

$$\rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 M_p^2}|M_{fi}|^2 = \frac{m_e^2 e^4}{64\pi^2 |\vec{p}|^4 \sin^4(\theta/2)}$$

- Writing $e^2 = 4\pi\alpha$ and the kinetic energy of the electron as $E_K = p^2/2m_e$

$$\rightarrow \quad \left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} = \frac{\alpha^2}{16 E_K^2 \sin^4 \theta/2} \qquad (9)$$

★ This is the normal expression for the Rutherford cross section. It could have been derived by considering the scattering of a non-relativistic particle in the static Coulomb potential of the proton $V(\vec{r})$, without any consideration of the interaction due to the intrinsic magnetic moments of the electron or proton. From this we can conclude, that in this non-relativistic limit only the interaction between the electric charges of the particles matters.

The Mott Scattering Cross Section

- For Rutherford scattering we are in the limit where the target recoil is neglected and the scattered particle is non-relativistic $E_K \ll m_e$
- The limit where the target recoil is neglected and the scattered particle is **relativistic** (i.e. just neglect the electron mass) is called Mott Scattering
- In this limit the electron currents, equations **(4)** and **(6)**, become:

$$\bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = 2E\,[c, s, -is, c] \qquad \bar{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) = E\,[0,0,0,0]$$

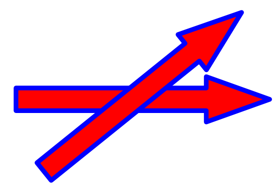
Relativistic ➡ Electron "helicity conserved"

- It is then straightforward to obtain the result:

$$\rightarrow \quad \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \frac{\alpha^2}{4E^2\sin^4\theta/2}\cos^2\frac{\theta}{2} \qquad \textbf{(10)}$$

Rutherford formula with $E_K = E \quad (E \gg m_e)$

Overlap between initial/final state electron wave-functions. Just QM of spin ½

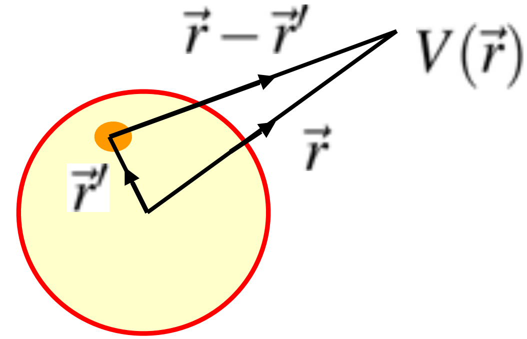


★ NOTE: we could have derived this expression from scattering of electrons in a static potential from a fixed point in space $V(\vec{r})$.
The interaction is **ELECTRIC** rather than magnetic (spin-spin) in nature.

★ Still haven't taken into account the charge distribution of the proton…..

Form Factors

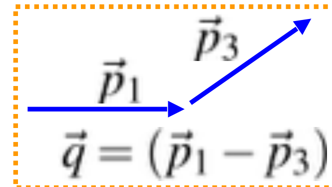
- Consider the scattering of an electron in the static potential due to an extended charge distribution.



- The potential at $\vec{r}$ from the centre is given by:

$$V(\vec{r}) = \int \frac{Q\rho(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|} d^3\vec{r}' \quad \text{with} \quad \int \rho(\vec{r}) d^3\vec{r} = 1$$

- In first order perturbation theory the matrix element is given by:



$$M_{fi} = \langle \psi_f | V(\vec{r}) | \psi_i \rangle = \int e^{-i\vec{p}_3.\vec{r}} V(\vec{r}) e^{i\vec{p}_1.\vec{r}} d^3\vec{r}$$

$$= \int\int e^{i\vec{q}.\vec{r}} \frac{Q\rho(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|} d^3\vec{r}' d^3\vec{r} = \int\int e^{i\vec{q}.(\vec{r}-\vec{r}')} e^{i\vec{q}.\vec{r}'} \frac{Q\rho(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|} d^3\vec{r}' d^3\vec{r}$$

- Fix $\vec{r}'$ and integrate over $d^3\vec{r}$ with substitution $\vec{R} = \vec{r} - \vec{r}'$

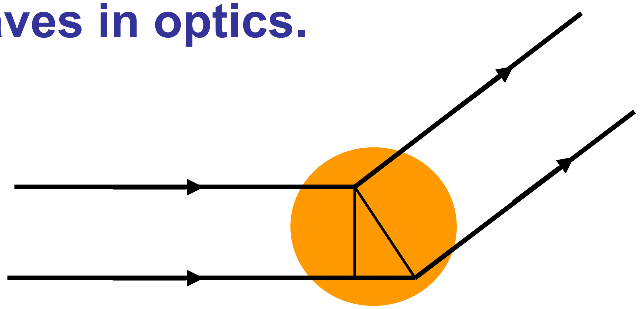
$$M_{fi} = \int e^{i\vec{q}.\vec{R}} \frac{Q}{4\pi|\vec{R}|} d^3\vec{R} \int \rho(\vec{r}') e^{i\vec{q}.\vec{r}'} d^3\vec{r}' = (M_{fi})_{point} F(\vec{q}^2)$$

★ The resulting matrix element is equivalent to the matrix element for scattering from a point source multiplied by the form factor

$$F(\vec{q}^2) = \int \rho(\vec{r}) e^{i\vec{q}.\vec{r}} d^3\vec{r}$$

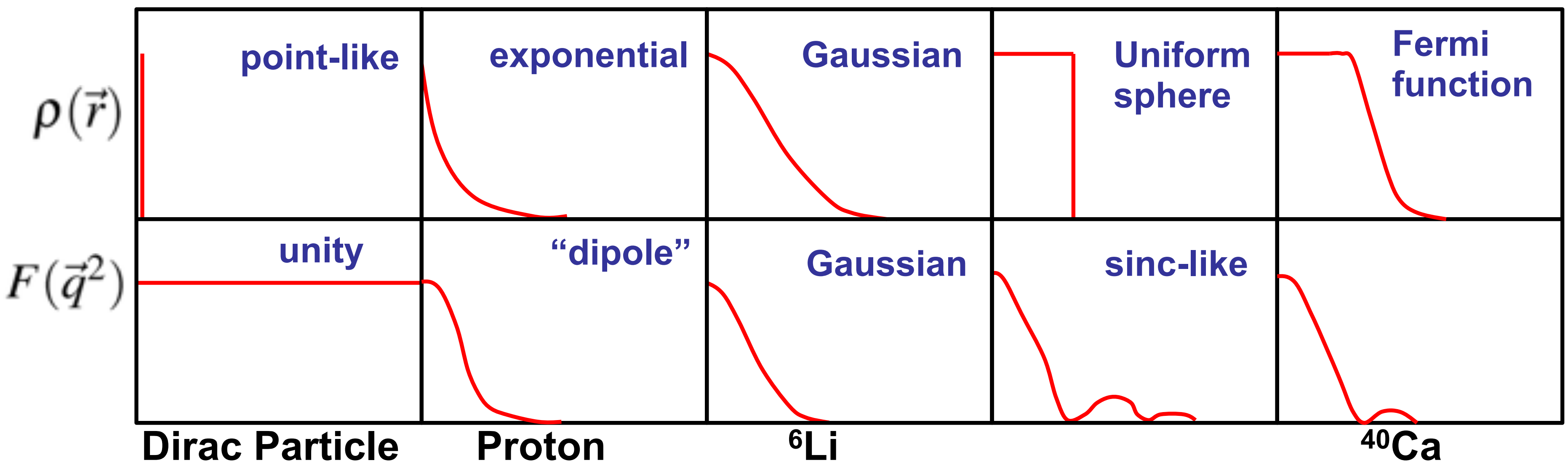
$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \rightarrow \frac{\alpha^2}{4E^2 \sin^4 \theta/2} \cos^2 \frac{\theta}{2} |F(\vec{q}^2)|^2$$

- There is nothing mysterious about form factors – similar to diffraction of plane waves in optics.



- The finite size of the scattering centre introduces a phase difference between plane waves "scattered from different points in space". If wavelength is long compared to size all waves in phase and $F(\vec{q}^2) = 1$

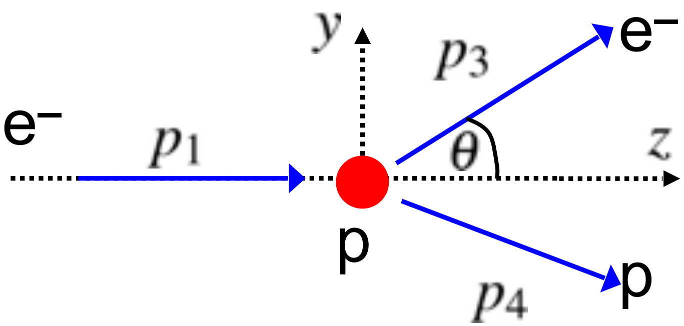
For example:



- **NOTE that for a point charge the form factor is unity.**

Point-like Electron-Proton Elastic Scattering

- So far have only considered the case we the proton does not recoil... For $E_1 \gg m_e$ the general case is



$$p_1 = (E_1, 0, 0, E_1)$$
$$p_2 = (M, 0, 0, 0)$$
$$p_3 = (E_3, 0, E_3 \sin\theta, E_3 \cos\theta)$$
$$p_4 = (E_4, \vec{p}_4)$$

- From Eqn. (2) with $m = m_e = 0$ the matrix element for this process is:

$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} \left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3) - (p_1.p_3)M^2 \right] \quad (11)$$

- Experimentally observe scattered electron so eliminate p_4
- The scalar products not involving p_4 are:

$$p_1.p_2 = E_1 M \qquad p_1.p_3 = E_1 E_3 (1 - \cos\theta) \qquad p_2.p_3 = E_3 M$$

- From momentum conservation can eliminate p_4 : $p_4 = p_1 + p_2 - p_3$

$$p_3.p_4 = p_3.p_1 + p_3.p_2 - p_3.p_3 = E_1 E_3 (1 - \cos\theta) + E_3 M$$
$$p_1.p_4 = p_1.p_1 + p_1.p_2 - p_1.p_3 = E_1 M - E_1 E_3 (1 - \cos\theta)$$

$$p_1.p_1 = E_1^2 - |\vec{p}_1|^2 = m_e^2 \approx 0$$ i.e. neglect m_e

- **Substituting these scalar products in Eqn. (11) gives**

$$\begin{aligned}\langle|M_{fi}|^2\rangle &= \frac{8e^4}{(p_1-p_3)^4}ME_1E_3\left[(E_1-E_3)(1-\cos\theta)+M(1+\cos\theta)\right] \\ &= \frac{8e^4}{(p_1-p_3)^4}2ME_1E_3\left[(E_1-E_3)\sin^2(\theta/2)+M\cos^2(\theta/2)\right]\end{aligned} \quad \textbf{(12)}$$

- **Now obtain expressions for** $q^4=(p_1-p_3)^4$ **and** (E_1-E_3)

$$q^2=(p_1-p_3)^2=p_1^2+p_3^2-2p_1.p_3=-2E_1E_3(1-\cos\theta) \quad \textbf{(13)}$$

$$=-4E_1E_3\sin^2\theta/2 \quad \textbf{(14)}$$

NOTE: $q^2<0$ **Space-like**

- **For** (E_1-E_3) **start from** $q.p_2=(p_1-p_3).p_2=M(E_1-E_3)$

  and use

$$\begin{aligned}(q+p_2)^2 &= p_4^2 \\ q^2+p_2^2+2q.p_2 &= p_4^2 \\ q^2+M^2+2q.p_2 &= M^2 \\ \Rightarrow\quad q.p_2 &= -q^2/2\end{aligned}$$

$$q=(p_1-p_3)=(p_4-p_2)$$

- **Hence the energy transferred to the proton:**

$$E_1 - E_3 = -\frac{q^2}{2M} \qquad \textbf{(15)}$$

Because q^2 is always negative $E_1 - E_3 > 0$ and the scattered electron is always lower in energy than the incoming electron

- **Combining equations (11), (13) and (14):**

$$\begin{aligned}\langle|M_{fi}|^2\rangle &= \frac{8e^4}{16E_1^2E_3^2\sin^4\theta/2}2ME_1E_3\left[M\cos^2\theta/2 - \frac{q^2}{2M}\sin^2\theta/2\right]\\ &= \frac{M^2e^4}{E_1E_3\sin^4\theta/2}\left[\cos^2\theta/2 - \frac{q^2}{2M^2}\sin^2\theta/2\right]\end{aligned}$$

- **For $E \gg m_e$ we have**

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2|M_{fi}|^2$$

$$\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\theta/2}\frac{E_3}{E_1}\left(\cos^2\theta/2 - \frac{q^2}{2M^2}\sin^2\theta/2\right) \qquad \textbf{(16)}$$

Interpretation

- So far have derived the differential cross-section for $e^-p \rightarrow e^-p$ **elastic** scattering assuming point-like Dirac spin ½ particles. How should we interpret the equation?

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2} \frac{E_3}{E_1}\left(\cos^2\theta/2 - \frac{q^2}{2M^2}\sin^2\theta/2\right)$$

- Compare with $$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \frac{\alpha^2}{4E^2 \sin^4\theta/2}\cos^2\frac{\theta}{2}$$

the important thing to note about the Mott cross-section is that it is equivalent to scattering of spin-½ electrons in a fixed **electro-static** potential. Here the term E_3/E_1 is due to the proton recoil.

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2} \frac{E_3}{E_1}\left(\cos^2\theta/2 - \underbrace{\frac{q^2}{2M^2}\sin^2\theta/2}\right)$$

- the new term: $\propto \sin^2\frac{\theta}{2}$ ⟺ **Magnetic interaction : due to the spin-spin interaction**

- The above differential cross-section depends on a single parameter. For an electron scattering angle θ , both q^2 and the energy, E_3 , are fixed by kinematics

• Equating (13) and (15)	• Substituting back into (13):
$-2M(E_1 - E_3) = -2E_1E_3(1-\cos\theta)$ $\rightarrow \dfrac{E_3}{E_1} = \dfrac{M}{M+E_1(1-\cos\theta)}$	$\rightarrow q^2 = -\dfrac{2ME_1^2(1-\cos\theta)}{M+E_1(1-\cos\theta)}$

- e.g. e⁻p → e⁻p at E_{beam}= 529.5 MeV, look at scattered electrons at θ = 75°

For elastic scattering expect:

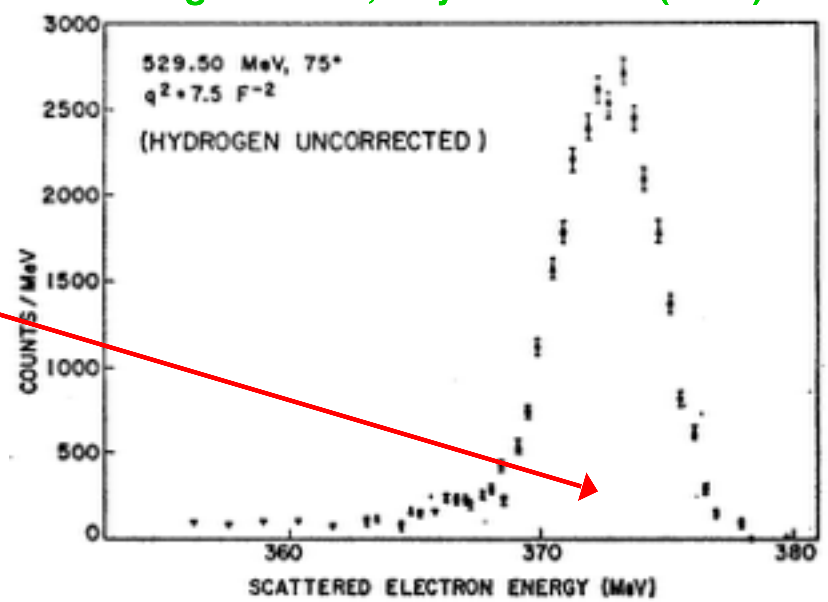
$$E_3 = \frac{ME_1}{M+E_1(1-\cos\theta)}$$

$$E_3 = \frac{938\times 529}{938+529(1-\cos 75^\circ)} = 373\text{ MeV}$$

The energy identifies the scatter as elastic.
Also know squared four-momentum transfer

$$|q^2| = \frac{2\times 938\times 529^2(1-\cos 75^\circ)}{938+529(1-\cos 75^\circ)} = 294\,\text{MeV}^2$$

E.B.Hughes et al., Phys. Rev. 139 (1965) B458



Elastic Scattering from a Finite Size Proton

★ **In general** the finite size of the proton can be accounted for by introducing **two structure functions**. One related to the **charge distribution** in the proton, $G_E(q^2)$ and the other related to the distribution of the **magnetic moment** of the proton, $G_M(q^2)$

- It can be shown that equation (16) generalizes to the **ROSENBLUTH FORMULA**.

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2} \frac{E_3}{E_1} \left(\frac{G_E^2 + \tau G_M^2}{(1+\tau)} \cos^2\frac{\theta}{2} + 2\tau G_M^2 \sin^2\frac{\theta}{2} \right)$$

with the Lorentz Invariant quantity: $\tau = -\frac{q^2}{4M^2} > 0$

- Unlike our previous discussion of form factors, here the form factors are a function of q^2 rather than $\vec{q}^2$ and cannot simply be considered in terms of the FT of the charge and magnetic moment distributions.

But $q^2 = (E_1 - E_3)^2 - \vec{q}^2$ and from eq (15) obtain

$$\Rightarrow \quad -\vec{q}^2 = q^2 \left[1 - \left(\frac{q}{2M} \right)^2 \right]$$

So for $\frac{q^2}{4M^2} \ll 1$ we have $q^2 \approx -\vec{q}^2$ and $G(q^2) \approx G(\vec{q}^2)$

- Hence in the limit $q^2/4M^2 \ll 1$ we can interpret the structure functions in terms of the Fourier transforms of the charge and magnetic moment distributions

$$G_E(q^2) \approx G_E(\vec{q}^2) = \int e^{i\vec{q}.\vec{r}} \rho(\vec{r}) \mathrm{d}^3\vec{r}$$

$$G_M(q^2) \approx G_M(\vec{q}^2) = \int e^{i\vec{q}.\vec{r}} \mu(\vec{r}) \mathrm{d}^3\vec{r}$$

- Note in deriving the Rosenbluth formula we assumed that the proton was a spin-half Dirac particle, i.e.

$$\vec{\mu} = \frac{e}{M}\vec{S}$$

- However, the experimentally measured value of the proton magnetic moment is larger than expected for a point-like Dirac particle:

$$\vec{\mu} = 2.79\frac{e}{M}\vec{S}$$

So for the proton expect

$$G_E(0) = \int \rho(\vec{r}) \mathrm{d}^3\vec{r} = 1 \qquad G_M(0) = \int \mu(\vec{r}) \mathrm{d}^3\vec{r} = \mu_p = +2.79$$

- Of course the anomalous magnetic moment of the proton is already evidence that it is not point-like !

Measuring $G_E(q^2)$ and $G_M(q^2)$

- Express the Rosenbluth formula as:

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_0 \left(\frac{G_E^2 + \tau G_M^2}{(1+\tau)} + 2\tau G_M^2 \tan^2\frac{\theta}{2}\right)$$

where

$$\left(\frac{d\sigma}{d\Omega}\right)_0 = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2}\frac{E_3}{E_1}\cos^2\frac{\theta}{2}$$

i.e. the Mott cross-section including the proton recoil. It corresponds to scattering from a spin-0 proton.

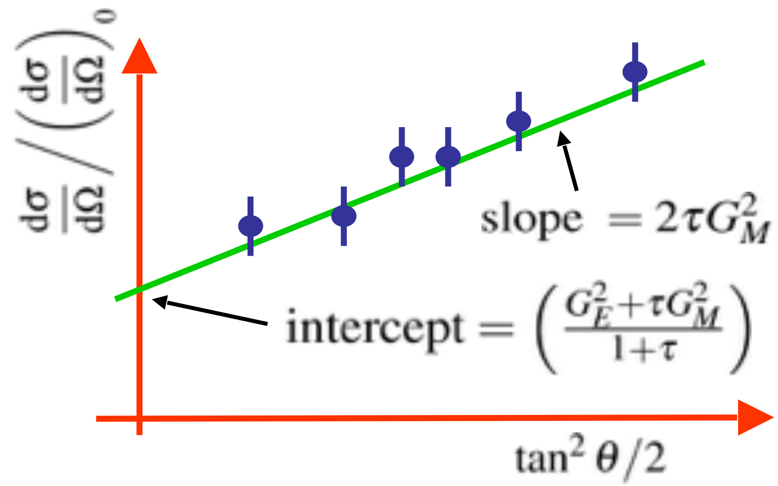
- At very low q^2: $\tau = -q^2/4M^2 \approx 0$

$$\frac{d\sigma}{d\Omega} \Big/ \left(\frac{d\sigma}{d\Omega}\right)_0 \approx G_E^2(q^2)$$

- At high q^2: $\tau \gg 1$

$$\frac{d\sigma}{d\Omega} \Big/ \left(\frac{d\sigma}{d\Omega}\right)_0 \approx \left(1 + 2\tau\tan^2\frac{\theta}{2}\right) G_M^2(q^2)$$

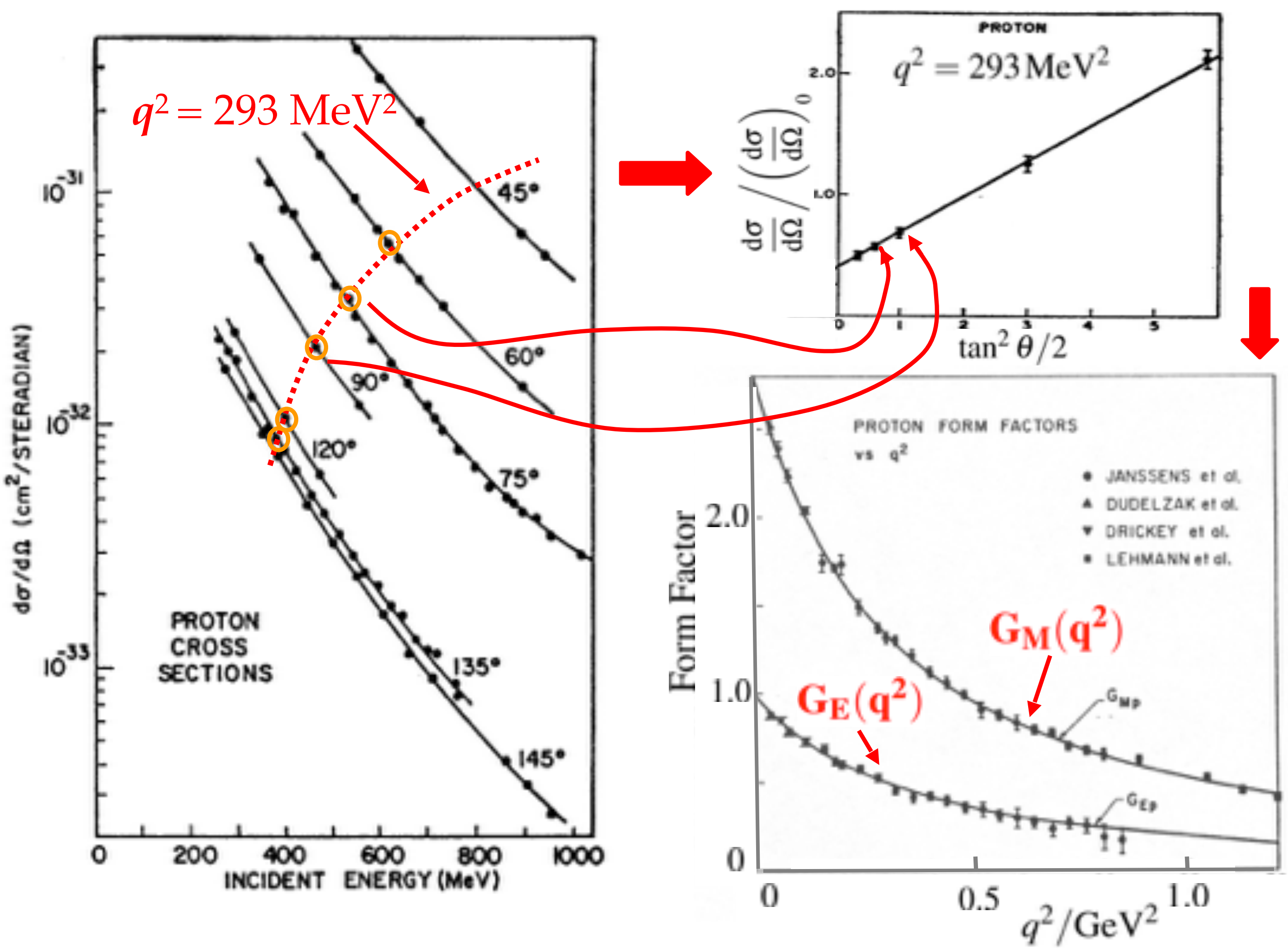
- In general we are sensitive to both structure functions! These can be resolved from the angular dependence of the cross section at **FIXED** q^2



EXAMPLE: $e^-p \rightarrow e^-p$ at E_{beam} = 529.5 MeV

- Electron beam energies chosen to give certain values of q^2
- Cross sections measured to 2-3 %

E.B.Hughes et al., Phys. Rev. 139 (1965) B458



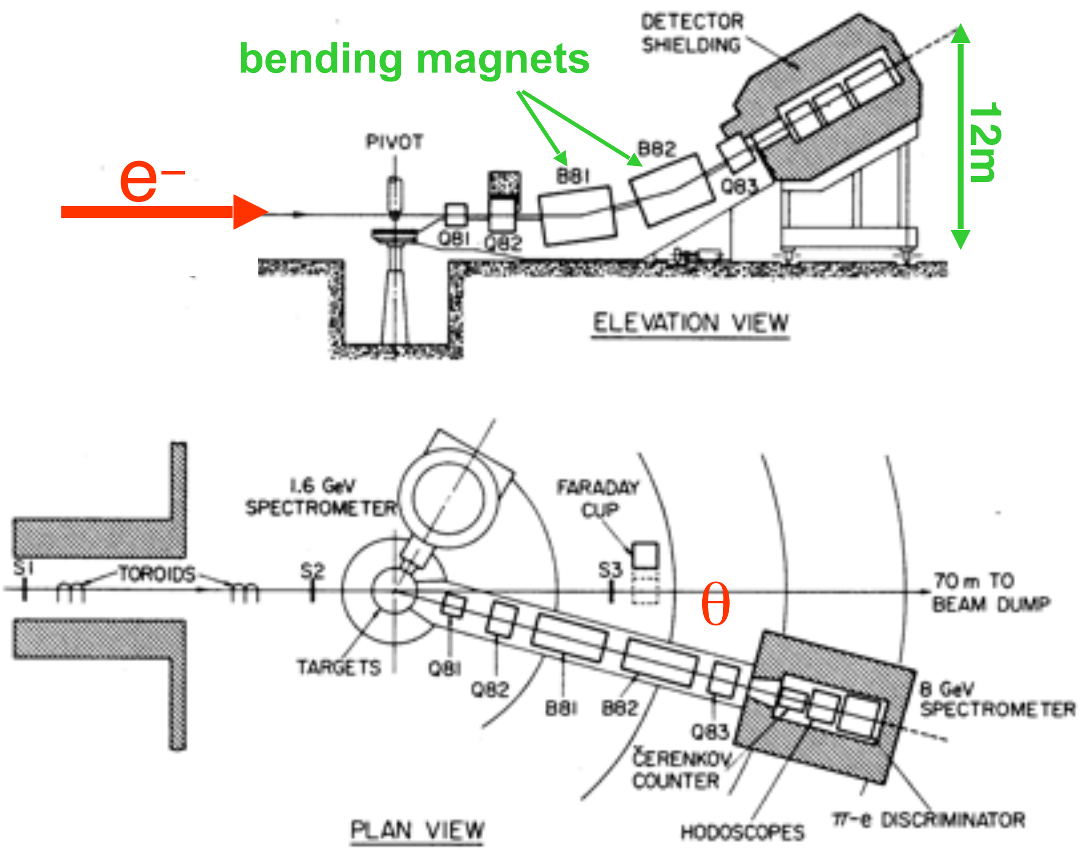
NOTE

Experimentally find $G_M(q^2) = 2.79 G_E(q^2)$, i.e. the electric and and magnetic form factors have same distribution

Higher Energy Electron-Proton Scattering

★ Use electron beam from SLAC LINAC: $5 < E_{beam} < 20$ GeV

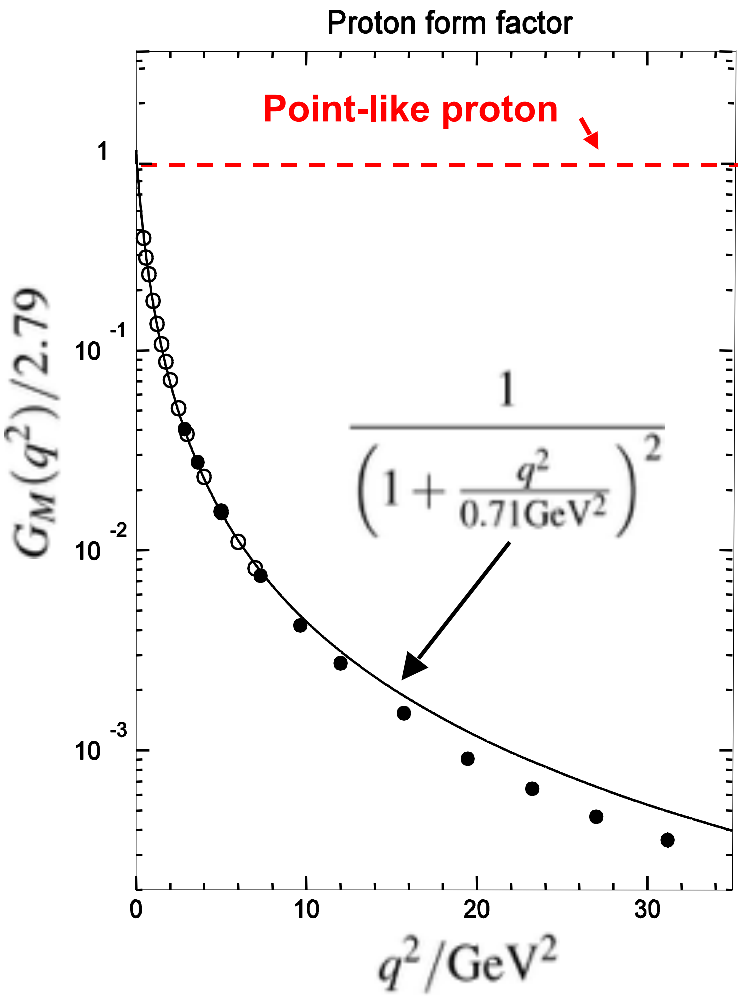
- Detect scattered electrons using the "8 GeV Spectrometer"



High q^2 → Measure $G_M(q^2)$

P.N.Kirk et al., Phys Rev D8 (1973) 63

High q^2 Results



R.C.Walker et al., Phys. Rev. D49 (1994) 5671
A.F.Sill et al., Phys. Rev. D48 (1993) 29

★ Form factor falls rapidly with q^2

- Proton is not point-like
- Good fit to the data with "dipole form":

$$G_E^p(q^2) \approx \frac{G_M^p}{2.79} \approx \frac{1}{(1+q^2/0.71\mathrm{GeV}^2)^2}$$

★ Taking FT find spatial charge and magnetic moment distribution

$$\rho(r) \approx \rho_0 e^{-r/a}$$

with $a \approx 0.24$ fm

Corresponds to a rms charge radius

$$r_{rms} \approx 0.8 \text{ fm}$$

★ Although suggestive, does not imply proton is composite !

★ **Note:** so far have only considered ELASTIC scattering; Inelastic scattering is the subject of next handout

Summary: Elastic Scattering

★For elastic scattering of relativistic electrons from a point-like Dirac proton:

$$\frac{d\sigma}{d\Omega} = \underbrace{\frac{\alpha^2}{4E_1^2 \sin^4\theta/2}}_{\text{Rutherford}} \underbrace{\frac{E_3}{E_1}}_{\text{Proton recoil}} \left(\underbrace{\cos^2\frac{\theta}{2}}_{\text{Electric/Magnetic scattering}} - \underbrace{\frac{q^2}{2M^2}\sin^2\frac{\theta}{2}}_{\text{Magnetic term due to spin}} \right)$$

★For elastic scattering of relativistic electrons from an extended proton:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2} \frac{E_3}{E_1} \left(\frac{G_E^2 + \tau G_M^2}{(1+\tau)} \cos^2\frac{\theta}{2} + 2\tau G_M^2 \sin^2\frac{\theta}{2} \right)$$

Rosenbluth Formula

★Electron elastic scattering from protons demonstrates that the proton is an extended object with rms charge radius of ~0.8 fm