than within its Compton wave length,

$$\Delta x \geq \frac{\hbar}{\Delta p} \geq \frac{\hbar}{mc} = \lambda_C \,. \tag{5.87}$$

Attempting to achieve better localization, the uncertainties of energy and momentum grow so much that the creation of new particles becomes energetically allowed and the problem loses its single-particle character. Then, one needs to use the full relativistic quantum field theory instead of quantum mechanics.

5.11 Spatial Quantization Revisited

As discussed in Section 1.8, the angular momentum of a quantum system is quantized as integer or half-integer (for spin) multiple of $\hbar$. We exclusively use the angular momentum in units of $\hbar$ [compare (4.34) and (4.68)]; we still denote the dimensionless orbital momentum vector as $\boldsymbol{\ell}$. If one experimentally singled out a direction z in space and measured the angular momentum projection $\ell_z = m$ onto this direction, for example, with the Stern–Gerlach set-up, there are $(2\ell + 1)$ possible results with values of m changing from $-\ell$ to $+\ell$. Due to the isotropy of space, the same quantization would be revealed with *any choice* of the quantization axis. Since the maximum possible projection is ℓ, it would be natural to assume that the length of the angular momentum was $\sqrt{\boldsymbol{\ell}^2} = \ell$. This however would be incompatible with the uncertainty relation.

Let us assume that the particles under study have no intrinsically specified direction. Applying the Stern–Gerlach fields with various orientations, we equiprobably obtain all possible values of the angular momentum projections (of course for each particle, only one given projection can be measured). An average value of $\boldsymbol{\ell}^2$ over a large number of measurements (here, the measurements refer to different experimental set-ups) is equal by virtue of isotropy to

$$\overline{\boldsymbol{\ell}^2} = \overline{\ell_x^2 + \ell_y^2 + \ell_z^2} = 3\overline{\ell_z^2} \,. \tag{5.88}$$

Since the possible results are quantized, this average value is

$$\overline{\boldsymbol{\ell}^2} = \frac{3}{2\ell + 1} \sum_{m=-l}^{\ell} m^2 = \frac{6}{2\ell + 1} \sum_{m=1}^{\ell} m^2 \,. \tag{5.89}$$

By calculating the sum of squares of integer numbers, we obtain

$$\overline{\boldsymbol{\ell}^2} = \frac{6}{2\ell + 1} \frac{\ell(\ell + 1)(2\ell + 1)}{6} = \ell(\ell + 1) \,. \tag{5.90}$$

We conclude that the average value of the "length" of the angular momentum vector equals $\sqrt{\ell(\ell + 1)}$ and thus, always *exceeds* the maximum projection of this vector onto any direction, Figure 5.13.

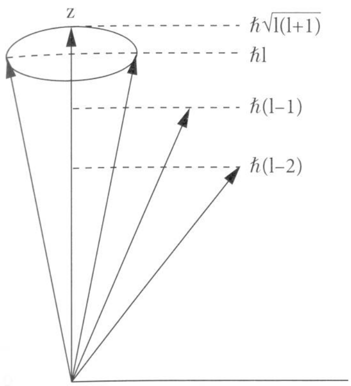

Figure 5.13 Spatial quantization of angular momentum and the precession picture.

This result (it is valid for half-integer quantization of angular momentum as well) may seem strange. It should be interpreted in the spirit of the uncertainty relation. If a state where the vector $\boldsymbol{\ell}$ is precisely aligned in a certain direction exists, one could choose this direction as the quantization axis and the measurement would provide the projection onto this direction equal to the maximum possible value coinciding with $\sqrt{\boldsymbol{\ell}^2}$. In reality, such a state does not exist. Here, we have a new pair of complementary variables, ℓ_z and the azimuthal angle φ. In classical mechanics, they are canonically conjugate, analogously to the linear momentum p_x and coordinate x. In quantum mechanics, we have the similarity between the linear momentum operator (4.29) and angular momentum operator ((4.34) and (4.68)). They both are presented as derivatives with respect to coordinates, linear and angular, respectively, and, in a deeper approach, they are generators of translations and rotations. Therefore, we expect that an uncertainty relation holds similar to $\Delta(\hbar\ell_z)\cdot\Delta\varphi \sim \hbar$. If so, the state with the angular momentum vector fully aligned along the quantization axis would be an analog to the plane wave state with precisely defined momentum p. In this state, the azimuthal angle would be uncertain. However, two interrelated circumstances make this surmise incorrect.

Firstly, the components ℓ_x and ℓ_y in this limiting case would also be fully determined (equal to zero), which would contradict the uncertainty principle for these components since the polar angle has a certain value $\theta = 0$ as well. Secondly, the uncertainty $\Delta\varphi$ cannot be infinite as required by a form of the uncertainty relation suggested above. Indeed, the angle is a *compact* coordinate defined only on the interval of 2π and it makes no sense to speak about its infinite uncertainty. Moreover, any single-valued function of φ has to be *periodic* with period of 2π, as we have already encountered in building the complete set of azimuthal functions (4.72). For all such functions, the uncertainty of the angle is surely finite (equal to $\pi/\sqrt{3}$ independently of the choice of interval of length 2π).

The difference in geometry or *topology* between the linear momentum with the infinite range of the Cartesian coordinate and the angular momentum with a compact domain of the conjugate angle is crucial. Formally, this is also seen in the fact that the operators $\hat{p}_k$ of components of the linear momentum commute, (4.57), while the components of the angular momentum $\hat{\ell}_k$ do not, (4.37). The result of two consecutive translations along arbitrary directions does not depend on the order of the operations. Contrary to that, the result of two rotations around different axis does depend on their order; the rotation group in three-dimensional space is *non-Abelian*. The numerous consequences of this will appear later in the course.

Since the complete alignment of the vector $\boldsymbol{\ell}$ is impossible and its "length" is always greater than its maximum projection, the closest classical analog of the situation is the picture of *precession* of the angular momentum vector around the quantization axis, Figure 5.13. The angle of the precession cone is fixed along with the projection ℓ_z. The transverse components are averaged to zero, $\langle \ell_x \rangle = \langle \ell_y \rangle = 0$, but their mean square fluctuations never vanish,

$$\langle \ell_x^2 \rangle = \langle \ell_y^2 \rangle = \frac{1}{2}\left[\langle \boldsymbol{\ell}^2 \rangle - \langle \ell_z^2 \rangle\right] = \frac{1}{2}[\ell(\ell+1) - m^2] > 0 \,. \tag{5.91}$$

Here, we jumped ahead by identifying the average over possible orientations discussed earlier with the quantum-mechanical expectation value. Later, we will consider the angular momentum algebra in more detail and strictly derive the properties loosely discussed here.

If one makes a formal limiting transition $\hbar \to 0$, the uncertainty relations cease to put any restrictions on the observables. The wave packet spreading (5.26) stops and all physical quantities can have certain values simultaneously. As mentioned, the Planck constant merely provides a numerical scale for manifestation of quantum laws based on symmetry. If this scale is much smaller than what is accessible to physical measurements, the uncertainties become insignificant. For the angular momentum, this limit is reached in the following way: one needs to go to large (macroscopic) quantum numbers, $\ell \to \infty$ when $\ell(\ell+1) \approx \ell^2 = (\ell_z^2)_{\max}$, and the classical alignment is restored. Then, we set $\hbar \to 0$ in such a way that $\hbar\ell$, the physical magnitude of angular momentum, is finite. The spin angular momentum of a particle does not have a macroscopic limit, and the spin vector $\hbar\mathbf{s}$ with magnitude $\hbar/2$ and

$$\mathbf{s}^2 = s(s+1) = \frac{3}{4} \tag{5.92}$$

vanishes in the classical limit. However, it survives, for example, in ferromagnetic alignment of a macroscopically large number of spins.

To conclude this lengthy chapter, we emphasize that quantum theory is based on the existence of complementary physical quantities and complementary classes of experiments. This complementarity, in turn, reflects specific symmetry properties of observables. The complementary experiments can be thought of as different projections of the same state of a microscopic object onto different physical situations. In some sense, this is analogous to different reference frames in relativity theory and, instead of the Lorentz transformations, there are certain rules of transformation of quantum amplitudes between various types of measurements. However, the interpretation of the results is probabilistic so that the full wave function can be explored only in a series of experiments under identical conditions. The uncertainty relations are numerical expressions for the complementarity principle, that is, interplay of fundamental symmetries on a quantum level.