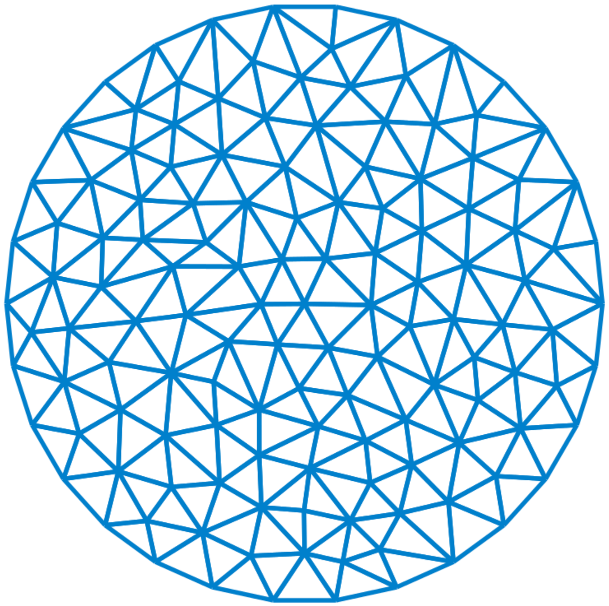
计算物理 第二部分
第7讲



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有限元方法（Finite Element Method）

简介

加权余量法

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变分方法

扩展

简介

有限元法是一种常用的计算方法，将系统网格节点统一编号，特别适用于不规则的系统，或任意边界形状的系统。有限元方法广泛应用于流体力学、结构力学等系统。

有限元法**早期是以变分原理为基础发展**起来的，所以广泛地应用于与泛函极值问题密切相关的拉普拉斯方程和泊松方程所描述的各类物理场中。

自1960年代以来，在流体力学中应用**加权余数法中的伽辽金(Galerkin)法或最小二乘法**等同样获得了有限元方程，从而使得有限元法可以应用于任何微分方程所描述的各类物理场中，而不再要求这类物理场和泛函的极值问题有所联系。

加权余量方法

Weighted Residual Method

在采用数值方法求解方程的近似解时,可用带有任意系数的一组线性无关函数来表示方程的近似解。将此近似解代入方程后，得到偏差，称为余量。这时问题就成为如何确定这组系数，使得余量最小。这就是加权余量法的基本思想。

先考虑一个简单的问题

$$\begin{cases} \dfrac{d^2u}{dx^2} - u = -x, \quad 0 < x < 1 \\ u(0) = 0, \quad u(1) = 0 \end{cases} \qquad (1)$$

A. 选近似解。先假设一个包含未知系数的近似试探解，**如$\tilde{u} = ax(1-x)$,** 满足边界条件。其中a为待定系数；

B. 求余量。将近似解代入 (1)得到余量。

$$R = \frac{d^2\tilde{u}}{dx^2} - \tilde{u} + x = -2a - ax(1-x) + x$$

加权余量方法：权函数

加权余量法的中心思想就是如何选择系数a，使得试探解是精确解得最好近似。为此，通常选择一个**权函数ω**使得问题的余量在求解区域的权平均（加权平均）为零。

$$I=\int_0^1 \omega R dx=\int_0^1 \omega\left(\frac{d^2\tilde{u}}{dx^2}-\tilde{u}+x\right)dx=0$$

根据权重函数的选择方法不同，加权余量法又可进一步的分类。

1. 配置法(collocation method)

选择狄拉克-δ 函数为权函数

$$w=\delta(x-x_i)$$

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例如取xi=0.5, 则有a=0.2222

$$\tilde{u}=0.2222x(1-x)$$

加权余量方法：　权函数

2. 最小二乘法(least squares method)

选择权函数为对余量所含的待求系数的导数

$$w = \frac{\mathrm{d}R}{\mathrm{d}a}$$

$$\frac{d}{da}\left[\int R^2 dx\right] = 0$$

$$\tilde{u} = 0.2305x(1-x)$$

3. 伽辽金法(Galerkin method)

选择权函数为对试探解所含待求系数的导数

$$w = \frac{\mathrm{d}\tilde{u}}{\mathrm{d}a}$$

$$\tilde{u} = 0.2272x(1-x)$$

加权余量方法：多个待定系数

为了改进近似解，可以选择含有多个待定系数的近似解。例如前面问题，假设近似解为

$$\tilde{u} = a_1 x(1-x) + a_2 x^2 (1-x)$$

这样代入原微分方程得余量

$$R = a_1(-2 - x + x^2) + a_2(2 - 6x - x^2 + x^3) + x$$

含有两个待确定的常数，需要同样数量的权函数。这样

配置法

$$w_1 = \delta(x - x_1), \quad w_2 = \delta(x - x_2)$$

最小二乘法

$$w_1 = -2 - x + x^2, \quad w_2 = 2 - 6x - x^2 + x^3$$

伽辽金法

$$w_1 = x(1-x), \quad w_2 = x^2(1-x)$$

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加权余量方法：与精确解的比较

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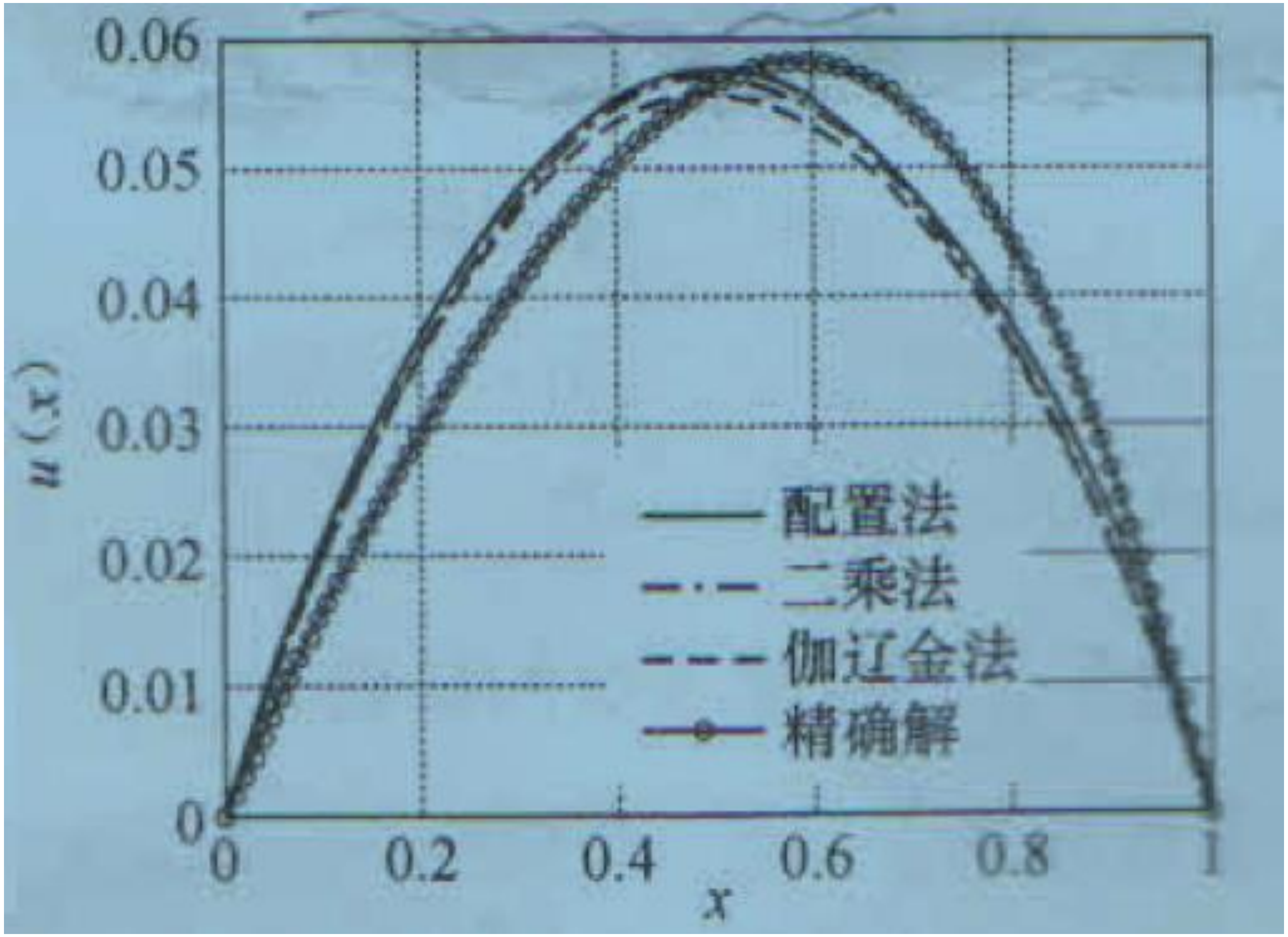

$$u(x) = -\frac{e^{x} - e^{-x}}{e - e^{-1}} + x$$

加权余量方法：弱形式

前面的简单例子要求试探的近似解必须能够两次可微，这是加权余量法的强形式。为了减少对试探函数的要求，可以先作分部积分：

$$I=\int_0^1 \omega R dx=\int_0^1 \omega\left(\frac{d^2\tilde{u}}{dx^2}-\tilde{u}+x\right)dx$$

$$=\int_0^1\left(-\frac{d\omega}{dx}\frac{d\tilde{u}}{dx}-\omega\tilde{u}+x\omega\right)dx+\left[\omega\frac{d\tilde{u}}{dx}\right]_0^1$$

这样，就不需要近似解必须能够两次可微。这种形式称为加权余量法的弱形式。（对权函数ω 的要变严格了）

分段连续函数

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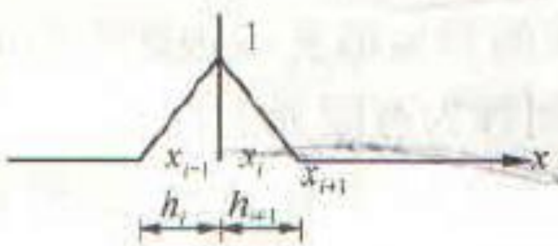
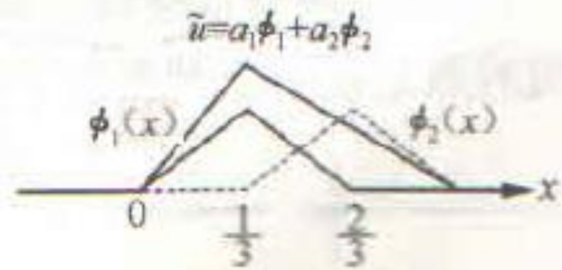
选择一个合适的试探解是很复杂的问题，特别是对于精确解在求解区域变化很大的情况，或是对于求解区域非常复杂的情况，或是求解区域的边界条件复杂的情况。为了克服这些问题，通常选取试探函数为分段连续的函数。

对于一维情况，假设考虑下面定义的分段线性函数

$$\phi_i(x)=\begin{cases}(x-x_{i-1})/h_i, & x_{i-1}<x<x_i\\(x_{i+1}-x)/h_{i+1}, & x_i<x<x_{i+1}\\0, & \text{其他}\end{cases}$$





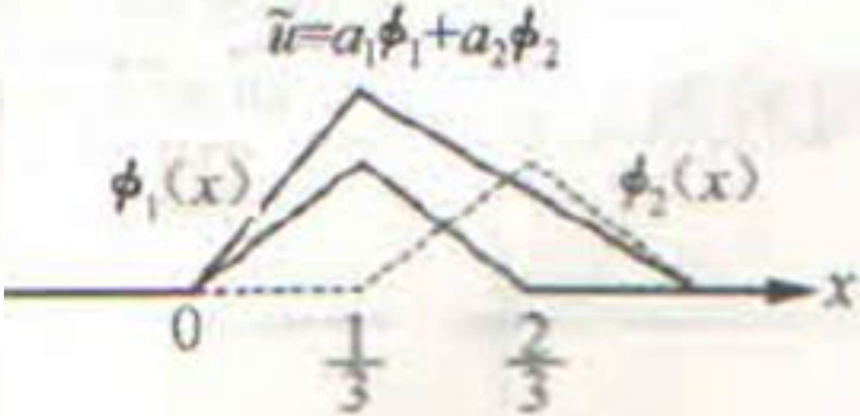
利用右图的试探函数（含两个待定系数）求解之前的例子

分段连续函数

$$\tilde{u}=a_1\phi_1(x)+a_2\phi_2(x)$$

$$\phi_1(x)=\begin{cases}3x, & 0\leqslant x\leqslant 1/3\\ 2-3x, & 1/3\leqslant x\leqslant 2/3\\ 0, & 2/3\leqslant x\leqslant 1\end{cases}$$

$$\phi_2(x)=\begin{cases}0, & 0\leqslant x\leqslant 1/3\\ 3x-1, & 1/3\leqslant x\leqslant 2/3\\ 3-3x, & 2/3\leqslant x\leqslant 1\end{cases}$$



$$\tilde{u}=\begin{cases}a_1(3x), & 0\leqslant x\leqslant 1/3\\ a_1(2-3x)+a_2(3x-1), & 1/3\leqslant x\leqslant 2/3\\ a_2(3-3x), & 2/3\leqslant x\leqslant 1\end{cases}$$

分段连续函数

$$w_1 = \phi_1(x), \quad w_2 = \phi_2(x)$$

平均的加权余量为

$$I_1 = \int_0^1 \left(-\frac{\mathrm{d}w_1}{\mathrm{d}x}\frac{\mathrm{d}\bar{u}}{\mathrm{d}x} - w_1\bar{u} + xw_1\right)\mathrm{d}x + \left[w_1\frac{\mathrm{d}\bar{u}}{\mathrm{d}x}\right]_0^1$$

$$= -6.222a_1 + 2.9444a_2 + 0.1111 = 0$$

$$I_2 = \int_0^1 \left(-\frac{\mathrm{d}w_2}{\mathrm{d}x}\frac{\mathrm{d}\bar{u}}{\mathrm{d}x} - w_2\bar{u} + xw_2\right)\mathrm{d}x + \left[w_2\frac{\mathrm{d}\bar{u}}{\mathrm{d}x}\right]_0^1$$

$$= 2.9444a_1 - 6.222a_2 + 0.2222 = 0$$

$a_1 = 0.0488$, $a_2 = 0.0569$, $\bar{u} = 0.0448\phi_1(x) + 0.0569\phi_2(x)$。

伽辽金有限元方法

前面看到，选取分段连续函数作为弱形式方程试探近似解是很好的方法。增加子区间的数量就可以利用简单的分段线性函数的和来构造复杂函数作为近似的试探解，这些子区间称为有限元。

考虑一维问题的一个有限元，如图所示的第i单元，其两个端点称为节点，坐标x_i和x_{i+1}对应的节点函数分别为u_i和u_{i+1}

如在$[x_i, x_{i+1}]$区间取近似解 $u = c_1 x + c_2$，
可得

$$c_i = \frac{u_{i+1} - u_i}{x_{i+1} - x_i}, \quad c_2 = \frac{u_i x_{i+1} - u_{i+1} x_i}{x_{i+1} - x_i}$$

$$\Rightarrow u = H_1^i(x) u_i + H_2^i(x) u_{i+1}$$

伽辽金有限元方法

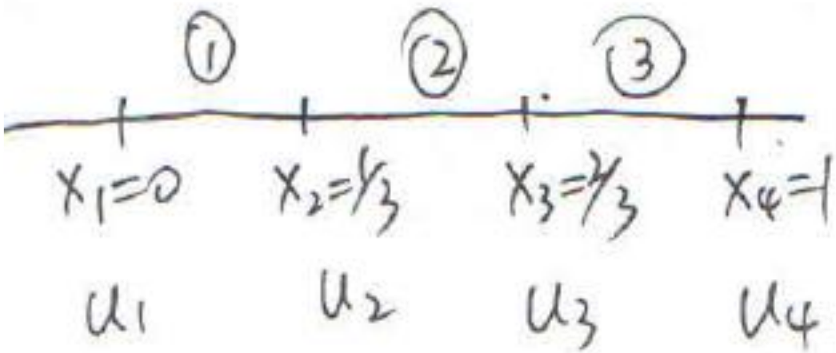
$$u = H_1^i(x)u_i + H_2^i(x)u_{i+1} = \left(H_1^i, H_2^i\right)\begin{pmatrix} u_i \\ u_{i+1} \end{pmatrix},$$

$$H_1^i(x) = \frac{x_{i+1} - x}{h_i}$$

$$H_2^i(x) = \frac{x - x_i}{h_i}, \qquad h_i = x_{i+1} - x_i$$

H函数称为型函数(Shape Function)

例1



$$H_1^i = \frac{x_{i+1} - x_i}{h} \qquad H_2^i = \frac{x - x_i}{h}$$

$$h = 1/3$$

$$\tilde{u} = u^{①} + \tilde{u}^{②} + \tilde{u}^{③}$$

$$\begin{cases} \tilde{u}^{①} = H_1^1 u_1 + H_2^1 u_2 \\ \tilde{u}^{②} = H_1^2 u_2 + H_2^2 u_3 \\ \tilde{u}^{③} = H_1^3 u_3 + H_2^3 u_4 \end{cases}$$

例1

$$H_1^1 = 1-3x$$

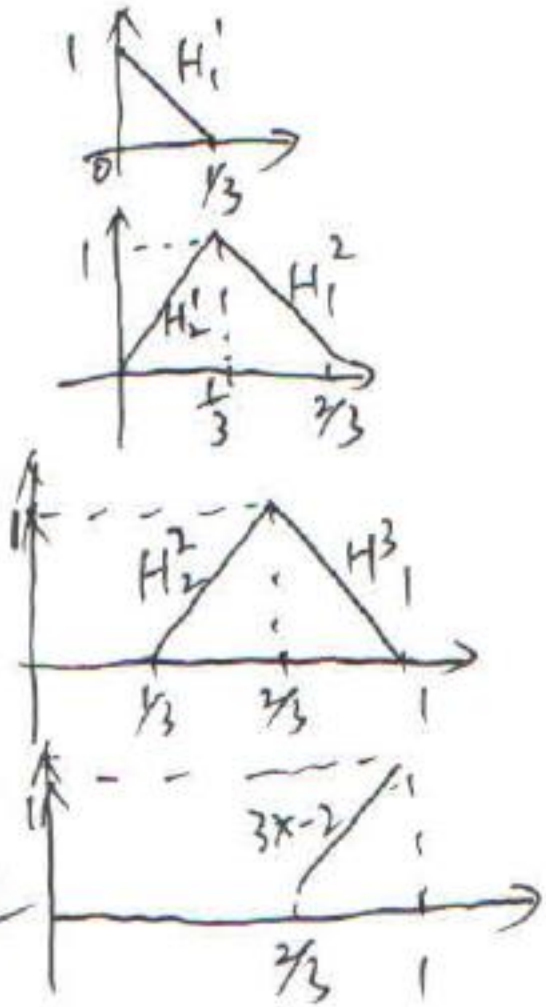
$$H_2^1 = 3x$$

$$H_1^2 = 2-3x$$

$$H_2^2 = 3x-1$$

$$H_1^3 = 3-3x$$

$$H_2^3 = 3x-2$$



$$\begin{cases} \omega_1 = \dfrac{\partial \tilde{u}}{\partial u_1} = H_1^1 \\ \omega_2 = \dfrac{\partial \tilde{u}}{\partial u_2} = H_2^1 + H_1^2 \\ \omega_3 = \dfrac{\partial \tilde{u}}{\partial u_3} = H_2^2 + H_1^3 \\ \omega_4 = \dfrac{\partial \tilde{u}}{\partial u_4} = H_2^3 \end{cases}$$

例1

$$\frac{d^2u}{dx^2} = 1-2x^2, \quad x_l = 0, \quad x_h = 1, \quad u(x_l) = 0, \quad u(x_h) = 0$$

解析解: $u(x) = -\frac{x}{3} + \frac{x^2}{2} - \frac{x^4}{6}$

$$I = \int_{x_i}^{x_{i+1}} w^i R \, dx = 0$$

第 ⓘ 段
$$\begin{cases} \tilde{u} = H_1^i u_i + H_2^i u_{i+1} \\ w = H_1^i, \ H_2^i \end{cases}$$

$$R = \frac{d^2u}{dx^2} - 1 + 2x^2$$

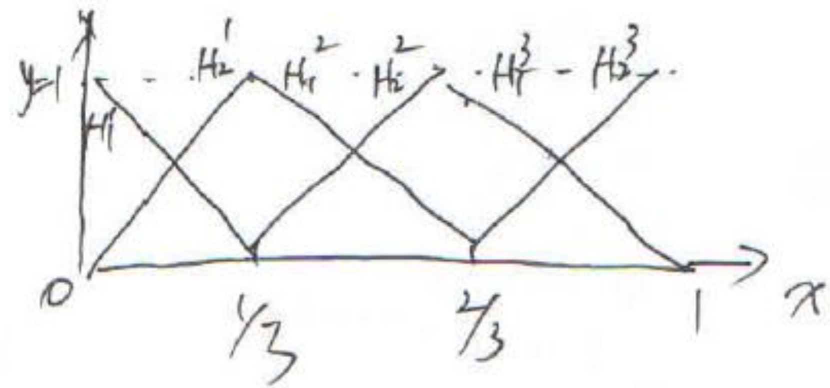
最后把所有element equation 粘合起来.

例1

① $$\int_0^{1/3}\left\{-\frac{d}{dx}\begin{pmatrix}H_1^1\\H_2^1\end{pmatrix}\frac{d}{dx}\left(H_1^1,\ H_2^1\right)\begin{pmatrix}u_1\\u_2\end{pmatrix}-\begin{pmatrix}H_1^1\\H_2^1\end{pmatrix}+2x^2\begin{pmatrix}H_1^1\\H_2^1\end{pmatrix}\right\}dx$$

$$+\begin{pmatrix}H_1^1\\H_2^1\end{pmatrix}\frac{du}{dx}\Bigg|_0^{1/3}$$

$$=\begin{pmatrix}0\\1\end{pmatrix}u'\!\left(\tfrac{1}{3}\right)-\begin{pmatrix}1\\0\end{pmatrix}u'(0)$$



例1

② $$\int_{\frac{1}{3}}^{\frac{2}{3}} \left\{ -\frac{d}{dx}\begin{pmatrix} H_1^2 \\ H_2^2 \end{pmatrix} \frac{d}{dx}\left(H_1^2, H_2^2\right)\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} - \begin{pmatrix} H_1^2 \\ H_2^2 \end{pmatrix} + 2x^2 \begin{pmatrix} H_1^2 \\ H_2^2 \end{pmatrix} \right\} dx$$

$$+ \begin{pmatrix} H_1^2 \\ H_2^2 \end{pmatrix} \frac{du}{dx}\Bigg|_{1/3}^{2/3} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} u'(2/3) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} u'(1/3)$$

③. $$\int_{\frac{2}{3}}^{1} \left\{ -\frac{d}{dx}\begin{pmatrix} H_1^3 \\ H_2^3 \end{pmatrix} \frac{d}{dx}\left(H_1^3, H_2^3\right)\begin{pmatrix} u_3 \\ u_4 \end{pmatrix} - \begin{pmatrix} H_1^3 \\ H_2^3 \end{pmatrix} + 2x^2 \begin{pmatrix} H_1^3 \\ H_2^3 \end{pmatrix} \right\} dx$$

$$+ \begin{pmatrix} H_1^3 \\ H_2^3 \end{pmatrix} \frac{du}{dx}\Bigg|_{2/3}^{1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} u'(1) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} u'\left(\frac{2}{3}\right)$$

例1

$$\int_0^{\frac{1}{3}} -\frac{d}{dx}\begin{pmatrix} H_1^1 \\ H_2^1 \end{pmatrix}\frac{d}{dx}(H_1^1, H_2^1)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}dx = \int_0^{\frac{1}{3}} \begin{pmatrix} 3 \\ -3 \end{pmatrix}(-3, 3)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}dx$$

$$= \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\int_{\frac{1}{3}}^{\frac{2}{3}} -\frac{d}{dx}\begin{pmatrix} H_1^2 \\ H_2^2 \end{pmatrix}\frac{d}{dx}(H_1^2, H_2^2)\begin{pmatrix} u_2 \\ u_3 \end{pmatrix}dx = \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} u_2 \\ u_3 \end{pmatrix}$$

$$\int_{\frac{2}{3}}^{1} -\frac{d}{dx}\begin{pmatrix} H_1^3 \\ H_2^3 \end{pmatrix}\frac{d}{dx}(H_1^3, H_2^3)\begin{pmatrix} u_3 \\ u_4 \end{pmatrix}dx = \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} u_3 \\ u_4 \end{pmatrix}$$

例1

$$\int_0^{\frac{1}{3}} (2x^2-1)\begin{pmatrix} H_1^1 \\ H_2^1 \end{pmatrix} dx = \int_0^{\frac{1}{3}} (2x^2-1)\begin{pmatrix} 1-3x \\ 3x \end{pmatrix} dx = \begin{pmatrix} -0.1605 \\ -0.1481 \end{pmatrix}$$

$$\int_{\frac{1}{3}}^{\frac{2}{3}} (2x^2-1)\begin{pmatrix} H_1^2 \\ H_2^2 \end{pmatrix} dx = \int_{\frac{1}{3}}^{\frac{2}{3}} (2x^2-1)\begin{pmatrix} 2-3x \\ 3x-1 \end{pmatrix} dx = \begin{pmatrix} -0.0988 \\ -0.0617 \end{pmatrix}$$

$$\int_{\frac{2}{3}}^{1} (2x^2-1)\begin{pmatrix} H_1^3 \\ H_2^3 \end{pmatrix} dx = \int_{\frac{2}{3}}^{1} (2x^2-1)\begin{pmatrix} 3-3x \\ 3x-2 \end{pmatrix} dx = \begin{pmatrix} 0.0370 \\ 0.0988 \end{pmatrix}$$

$$\begin{pmatrix} -u'(0) \\ u'(\frac{1}{3}) \end{pmatrix}, \quad \begin{pmatrix} -u'(\frac{1}{3}) \\ u'(\frac{2}{3}) \end{pmatrix}, \quad \begin{pmatrix} -u'(\frac{2}{3}) \\ u'(1) \end{pmatrix}$$

例1

$$\begin{cases}
\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0.1605 + u'(0) \\ 0.1481 - u'(\frac{1}{3}) \end{pmatrix} \\
\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0.0988 + u'(\frac{1}{3}) \\ 0.0617 - u'(\frac{2}{3}) \end{pmatrix} \\
\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} -0.0370 + u'(\frac{2}{3}) \\ -0.0988 - u'(1) \end{pmatrix}
\end{cases}$$

例1

$$\Rightarrow \begin{pmatrix} -3 & 3 & & \\ 3 & -3-3 & 3 & \\ & 3 & -3-3 & 3 \\ & & 3 & -3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0.1605 + u'(0) \\ 0.1481 + 0.0988 \\ 0.0617 \pm 0.0370 \\ -0.0988 - u'(1) \end{pmatrix}$$

边条件 $u_1 = 0,\ u_4 = 0$

$$\Rightarrow \begin{pmatrix} -6 & 3 \\ 3 & -6 \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0.2469 \\ 0.0247 \end{pmatrix}$$

$$u_3 \approx -0.0329$$

$$u_2 \approx -0.0576$$

例2

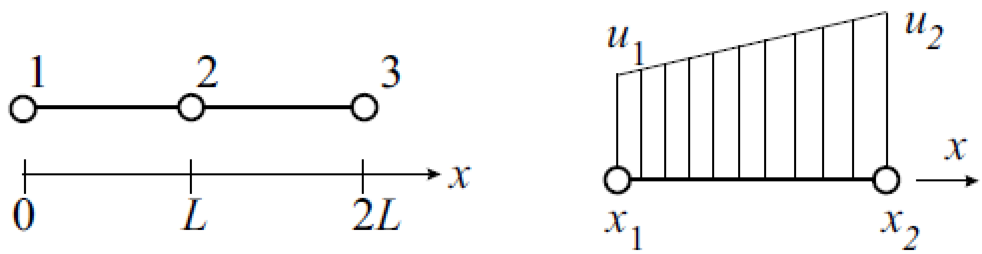


Figure 1.1: Two one-dimensional linear elements and function interpolation inside element.

Let us use simple one-dimensional example for the explanation of finite element formulation using the Galerkin method. Suppose that we need to solve numerically the following differential equation:

$$a\frac{d^2u}{dx^2} + b = 0, \qquad 0 \le x \le 2L \tag{1.1}$$

with boundary conditions

$$\begin{aligned} &u|_{x=0} = 0 \\ &a\frac{du}{dx}|_{x=2L} = R \end{aligned} \tag{1.2}$$

例2

where u is an unknown solution. We are going to solve the problem using two linear one-dimensional finite elements as shown in Fig. 1.1.

Fist, consider a finite element presented on the right of Figure. The element has two nodes and approximation of the function $u(x)$ can be done as follows:

$$\begin{aligned} &u = N_1 u_1 + N_2 u_2 = [N]\{u\} \\ &[N] = [N_1 \;\; N_2] \\ &\{u\} = \{u_1 \;\; u_2\} \end{aligned} \tag{1.3}$$

where N_i are the so called *shape functions*

$$N_1 = 1 - \frac{x - x_1}{x_2 - x_1}$$
$$N_2 = \frac{x - x_1}{x_2 - x_1}$$

例2

After substituting u expressed through its nodal values and shape functions, in the differential equation, it has the following approximate form:

$$a\frac{d^2}{dx^2}[N]\{u\} + b = \psi \tag{1.5}$$

where ψ is a nonzero residual because of approximate representation of a function inside a finite element. The Galerkin method provides residual minimization by multiplying terms of the above equation by shape functions, integrating over the element and equating to zero:

$$\int_{x_1}^{x_2} [N]^T a\frac{d^2}{dx^2}[N]\{u\}dx + \int_{x_1}^{x_2} [N]^T b dx = 0 \tag{1.6}$$

Use of integration by parts leads to the following discrete form of the differential equation for the finite element:

$$\int_{x_1}^{x_2} \left[\frac{dN}{dx}\right]^T a\left[\frac{dN}{dx}\right]dx\{u\} - \int_{x_1}^{x_2} [N]^T b dx - \left\{\begin{array}{c} 0 \\ 1 \end{array}\right\} a\frac{du}{dx}|_{x=x_2} + \left\{\begin{array}{c} 1 \\ 0 \end{array}\right\} a\frac{du}{dx}|_{x=x_1} = 0 \tag{1.7}$$

例2

Usually such relation for a finite element is presented as:

$$
\begin{aligned}
&[k]\{u\} = \{f\} \\
&[k] = \int_{x_1}^{x_2} \left[\frac{dN}{dx}\right]^T a \left[\frac{dN}{dx}\right] dx \\
&\{f\} = \int_{x_1}^{x_2} [N]^T b dx + \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right\} a \frac{du}{dx}|_{x=x_2} - \left\{ \begin{array}{c} 1 \\ 0 \end{array} \right\} a \frac{du}{dx}|_{x=x_1}
\end{aligned}
\tag{1.8}
$$

In solid mechanics $[k]$ is called *stiffness matrix* and $\{f\}$ is called *load vector*. In the considered simple case for two finite elements of length L stiffness matrices and the load vectors can be easily calculated:

$$
\begin{aligned}
&[k_1] = [k_2] = \tfrac{a}{L} \left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right] \\
&\{f_1\} = \tfrac{bL}{2} \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\}, \quad \{f_2\} = \tfrac{bL}{2} \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\} + \left\{ \begin{array}{c} 0 \\ R \end{array} \right\}
\end{aligned}
\tag{1.9}
$$

已经对所有element做了粘合

例2

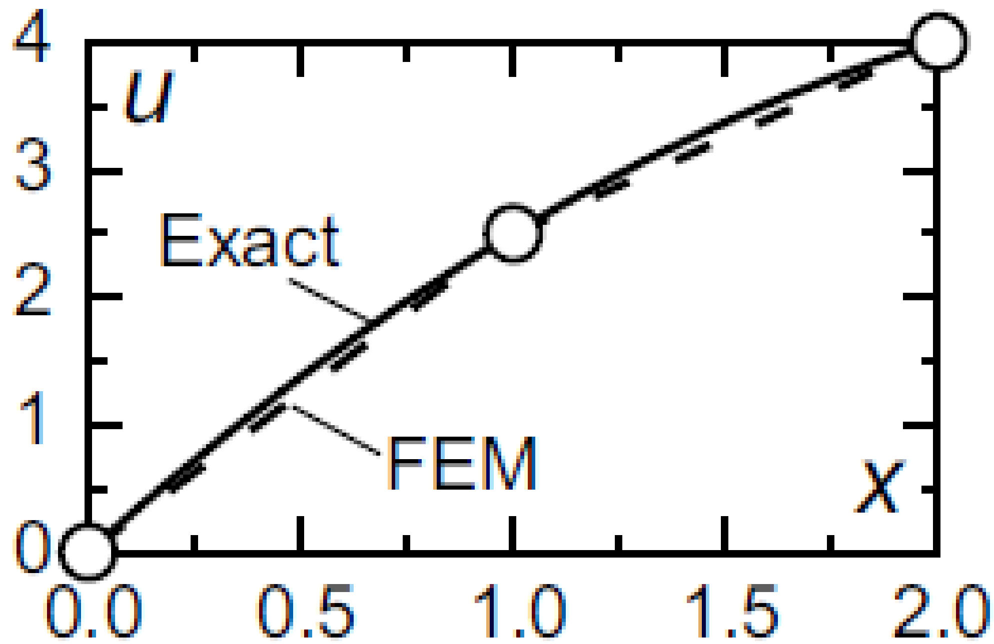
The above relations provide finite element equations for the two separate finite elements. A global equation system for the domain with 2 elements and 3 nodes can be obtained by an assembly of element equations. In our simple case it is clear that elements interact with each other at the node with global number 2. The assembled global equation system is:

$$\frac{a}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{bL}{2}\begin{Bmatrix} 1 \\ 2 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ R \end{Bmatrix} \tag{1.10}$$

After application of the boundary condition $u(x=0)=0$ the final appearance of the global equation system is:

$$\frac{a}{L}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{bL}{2}\begin{Bmatrix} 0 \\ 2 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ R \end{Bmatrix} \tag{1.11}$$

例2

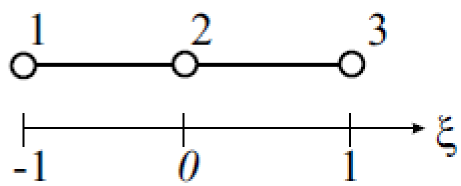


Nodal values u_i are obtained as results of solution of linear algebraic equation system. The value of u at any point inside a finite element can be calculated using the shape functions. The finite element solution of the differential equation is shown in Fig. 1.2 for $a = 1, b = 1, L = 1$ and $R = 1$.

二次型函数

Exact solution is a quadratic function. The finite element solution with the use of the simplest element is piece-wise linear. More precise finite element solution can be obtained increasing the number of simple elements or with the use of elements with more complicated shape functions. It is worth noting that at nodes the finite element method provides exact values of u (just for this particular problem). Finite elements with linear shape functions produce exact nodal values if the sought solution is quadratic. Quadratic elements give exact nodal values for the cubic solution *etc.*

Example. Obtain shape functions for the one-dimensional quadratic element with three nodes. Use local coordinate system $-1 \leq \xi \leq 1$.



Solution. With shape functions, any field inside element is presented as:

$$u(\xi) = \sum N_i u_i \ , \quad i = 1,\ 2,\ 3$$

At nodes the approximated function should be equal to its nodal value:

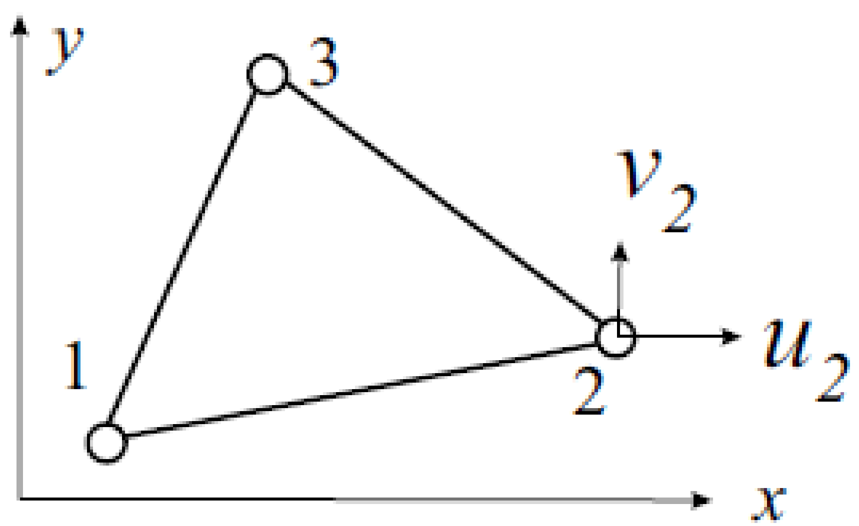
$$u(-1) = u_1$$
$$u(0) = u_2$$
$$u(1) = u_3$$

$$N_1 = \alpha_1 + \alpha_2 \xi + \alpha_3 \xi^2$$

二维情形

Triangular finite element was the first finite element proposed for continuous problems. Because of simplicity it can be used as an introduction to other elements. A triangular finite element in the coordinate system xy is shown in Fig. 4.1. Since the element has three nodes, linear approximation of displacements u and v is selected:

$$\begin{aligned} u(x,y) &= N_1u_1 + N_2u_2 + N_3u_3 \\ v(x,y) &= N_1v_1 + N_2v_2 + N_3v_3 \\ N_i &= \alpha_i + \beta_i x + \gamma_i y \end{aligned} \tag{4.1}$$



二维情形

Shape functions $N_i(x, y)$ can be determined from the following equation system:

$$u(x_i, y_i) = u_i \ , \quad i = 1,\ 2,\ 3$$

Shape functions for the triangular element can be presented as:

$$N_i = \frac{1}{2\Delta}(a_i + b_i x + c_i y)$$
$$a_i = x_{i+1}y_{i+2} - x_{i+2}y_{i+1}$$
$$b_i = y_{i+1} - y_{i+2}$$
$$c_i = x_{i+2} - x_{i+1}$$
$$\Delta = \frac{1}{2}(x_2y_3 + x_3y_1 + x_1y_2 - x_2y_1 - x_3y_2 - x_1y_3)$$

where Δ is the element area.

变分方法

$$a\frac{d^2u}{dx^2} + b = 0, \qquad 0 \le x \le 2L$$
$$u|_{x=0} = 0$$
$$a\frac{du}{dx}|_{x=2L} = R$$

with $a = EA$ has the following physical meaning in solid mechanics. It describes tension of the one dimensional bar with cross-sectional area A made of material with the elasticity modulus E and subjected to a distributed load b and a concentrated load R at its right end as shown in Fig 1.3.

Such problem can be formulated in terms of minimizing the potential energy functional Π:

$$\Pi = \int_L \frac{1}{2} a \left(\frac{du}{dx}\right)^2 dx - \int_L b u dx - R u|_{x=2L}$$
$$u|_{x=0} = 0$$

变分方法

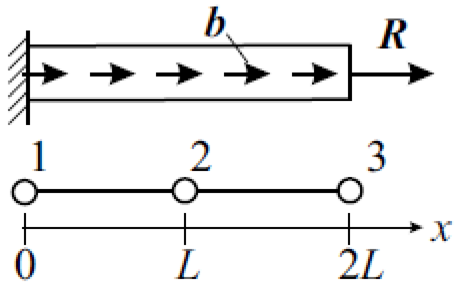


Figure 1.3: Tension of the one dimensional bar subjected to a distributed load and a concentrated load.

$$\Pi = \int_L \frac{1}{2} a \left(\frac{du}{dx}\right)^2 dx - \int_L b u dx - Ru|_{x=2L}$$

$$u|_{x=0} = 0$$

Using representation of $\{u\}$ with shape functions (1.3)-(1.4) we can write the value of potential energy for the second finite element as:

$$\Pi_e = \int_{x_1}^{x_2} \frac{1}{2} a \{u\}^T \left[\frac{dN}{dx}\right]^T \left[\frac{dN}{dx}\right] \{u\} dx$$

$$- \int_{x_1}^{x_2} \{u\}^T [N]^T b dx - \{u\}^T \left\{ \begin{array}{c} 0 \\ R \end{array} \right\}$$

变分方法

The condition for the minimum of Π is:

$$\delta\Pi = \frac{\partial\Pi}{\partial u_1}\delta u_1 + \ldots + \frac{\partial\Pi}{\partial u_n}\delta u_n = 0$$

which is equivalent to

$$\frac{\partial\Pi}{\partial u_i} = 0\,, \quad i = 1\ldots n$$

It is easy to check that differentiation of Π in respect to u_i gives the finite element equilibrium equation which is coincide with equation obtained by the Galerkin method:

$$\int_{x_1}^{x_2}\left[\frac{dN}{dx}\right]^T EA\left[\frac{dN}{dx}\right]dx\{u\} - \int_{x_1}^{x_2}[N]^T b\,dx - \left\{\begin{array}{c} 0 \\ R \end{array}\right\} = 0$$

Euler, Ritz, Galerkin, Courant: On the Road to the Finite Element Method

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Marseille, mars 2010

In collaboration with Gerhard Wanner

https://www.unige.ch/~gander/Preprints/RitzTalk.pdf

Summary

- **Euler (1744)** "invents" variational calculus **by piecewise linear discretization**.
- **Lagrange (1755)** puts it on a solid foundation.
- **Ritz (1908)** proposes and analyzes approximate solutions based on linear combinations of simple functions, and solves two difficult problems of his time.
- **Timoshenko (1913), Bubnov (1913) and Galerkin (1915)** realize the tremendous potential of Ritz' method and solve many difficult problems.
- **Courant (1941)** proposes to use piecewise linear functions on triangular meshes.
- **Clough et al. (1960)** name the method the **Finite Element Method**.

The mathematical development of the finite element method was however just to begin...

History of FEM

- In USA, etc – 1940s
  - A. Hrennikoff – discretized the domain by using a lattice analogy
  - R. Courant– divided the domain into finite triangular subregion – drawing a large body of earlier results for PDEs by Rayleigh, Ritz and Galerkin
- In USSR -- Leonard Oganesyan
- In China -- Kang Feng – proposed a systematic numerical method for solving PDEs – finite difference method based on variation principle
- FEM obtained its real impetus in 1960s &1970s
  - J. H. Argyris & co-workers, University of Stuttgart, Germany
  - R. W. Clough & co-workers, UC Berkeley; R. Gallagher, Cornell Univ, USA
  - O.C. Zienkiewicz & co-workers, University of Swanswa
  - P. G. Ciarlet, University of Paris 6, France, …..

https://en.wikipedia.org/wiki/Finite_element_method

History [edit]

While it is difficult to quote a date of the invention of the finite element method, the method originated from the need to solve complex elasticity and structural analysis problems in civil and aeronautical engineering. Its development can be traced back to the work by A. Hrennikoff[4] and R. Courant[5] in the early 1940s. Another pioneer was Ioannis Argyris. In the USSR, the introduction of the practical application of the method is usually connected with name of Leonard Oganesyan.[6] In China, in the later 1950s and early 1960s, based on the computations of dam constructions, K. Feng proposed a systematic numerical method for solving partial differential equations. The method was called the finite difference method based on variation principle, which was another independent invention of the finite element method. Although the approaches used by these pioneers are different, they share one essential characteristic: mesh discretization of a continuous domain into a set of discrete sub-domains, usually called elements.

Hrennikoff's work discretizes the domain by using a lattice analogy, while Courant's approach divides the domain into finite triangular subregions to solve second order elliptic partial differential equations (PDEs) that arise from the problem of torsion of a cylinder. Courant's contribution was evolutionary, drawing on a large body of earlier results for PDEs developed by Rayleigh, Ritz, and Galerkin.

The finite element method obtained its real impetus in the 1960s and 1970s by the developments of J. H. Argyris with co-workers at the University of Stuttgart, R. W. Clough with co-workers at UC Berkeley, O. C. Zienkiewicz with co-workers Ernest Hinton, Bruce Irons[7] and others at the University of Swansea, Philippe G. Ciarlet at the University of Paris 6 and Richard Gallagher with co-workers at Cornell University. Further impetus was provided in these years by available open source finite element software programs. NASA sponsored the original version of NASTRAN, and UC Berkeley made the finite element program SAP IV[8] widely available. In Norway the ship classification society Det Norske Veritas (now DNV GL) developed Sesam in 1969 for use in analysis of ships.[9] A rigorous mathematical basis to the finite element method was provided in 1973 with the publication by Strang and Fix.[10] The method has since been generalized for the numerical modeling of physical systems in a wide variety of engineering disciplines, e.g., electromagnetism, heat transfer, and fluid dynamics.[11][12]

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冯康的科学生涯（之三）

□冯　端

（上接8月12日第2版）

两次重大的科学突破

在科学上做出重大突破，往往是可遇而不可求的。眼光、能力和机遇，三者缺一不可。冯康在一生中实现了科学上的两次重大突破，是非常难能可贵的，值得大书一笔。一是1964－1965年间独立地开创有限元方法并奠定其数学基础；二是在1984年以后创建的哈密尔顿系统的辛几何算法及其发展。当前科学上创新的问题成为议论的焦点，不妨以冯康这两次突破作为科学上创新的案例，特别值得强调的是，这两次突破都是在中国土地上由中国科学家发现的。对之进行认真的案例分析，尚有待于行家来进行。我只能围绕这一课题，谈些外行话。

值得注意，这两次突破之所以能实现，不仅是得力于冯康的数学造诣，还和他精通经典物理学和通晓工程技术密切相关。科学上的突破常具有跨学科的特征。另一点需要强调的是在突破之前存在有长达数年的孕育期。需要厚积而发，急功近利的做法并不可取。开创有限元方法的契机来自国家的一项攻关任务，即刘家峡大坝设计中包括的计算问题。面对这样一个具体实际问题，冯康以敏锐的眼光发现了一个基础问题。他考虑到按常规来做，处理数学物理离散计算方法要分四步来进行：即(1)明确物理机制，(2)写出数学表述，(3)采用离散模型，(4)设计算法。但对几何和物理条件复杂的问题，常规的方法不一定奏效。因而他考虑是否可以越出常规，并不先写下描述物理现象的微分方程，而是从物理上的守恒定律或变分原理出发，直接和恰当的离散模型联系起来。在过去Euler、Rayleigh、Ritz、Polya等大师曾经考虑过这种做法，但这些都是在电子计算机出现之前。结合电子计算机计算的特点，将变分原理和差分格式直接联系起来，就形成了有限元方法，它具有广泛的适应性，特别适合于处理几何物理条件复杂的工程计算问题。这一方法的实施始于1964年，解决了具体的实际问题。1965年冯康发表了论文“基于变分原理的差分格式”，这篇论文是国际学术界承认我国独立发展有限元方法的主要依据。但是十分遗憾的是，对冯康这项重大贡献的评价姗姗来迟，而且不够充分。在70年代有限元方法重新从国外移植进来，有人公开在会议上大肆讥笑地说“居然有这样的奇谈怪论，说有限元方法是中国人发明的。”会上冯康只得噤口无语，这个事实是冯康亲口告诉我的。后来国际交往逐渐多起来了，来访的法国数学家Lions和美国数学家Lax都异口同声地承认冯康独立于国外发展有限元方法的功绩，坚冰总算打破了。但这项工作仅获得1982年国家自然科学二等奖。冯康得悉这一消息后非常难过，这是可以理解的，因为他对科学成果的估价具敏锐的眼光，曾打算将申请撤回，由于种种原因而未果。

文革以后，他虽然继续在和有限元有关的领域进行工作，也不乏出色的成果，例如间断有限元与边界归化方法等，但他也就开始在搜寻探索下一次突破的关口。他关注并进行了解处在数学与物理边界区域中的新动向，阅读了大量文献资料。有两篇介绍性的综述文章可以作为这一搜索过程的见证：“现代数理科学中的一些非线性问题”与“数学物理中的反问题”。文革后期一直到80年代中他经常和我谈论这方面的问题：诸如Thom的突变论，Prigogine的耗散结构，孤立子，Radon变换等。这种搜索过程，有点像老鹰在天空中盘旋，搜索目标，也可以比拟为“独上高楼，望尽天涯路”。70年代Arnold的“经典力学的数学问题”问世，阐述了哈密顿方程的辛几何结构，给他很大的启发，使他找到了突破口。他在计算数学中的长期实践，使他深深领悟到同一物理定律的不同的数学表述，尽管在物理上是等价的，但在计算上是不等价的（他的学生称之为冯氏大定理），这样经典力学的牛顿方程、拉格朗日方程和哈密顿方程，在计算上表现出不同的格局，由于哈密尔顿方程具有辛几何结构，他敏锐地察觉到如果在算法中能够保持辛几何的对称性，将可避免人为耗散性这类算法的缺陷，成为具有高保真性的算法。这样他就开拓了处理哈密尔顿系统计算问题的康庄大道，他戏称为哈密尔顿大道（The Hamiltonian way），在天体力学的轨道计算，粒子加速器中的轨道计算和分子动力学计算中得到广泛的应用。这项成果在1991年国家自然科学奖评议中评为三等奖，冯康获悉后撤回申请。直到1997年底，在冯康去世四年之后，终于授予了国家自然科学一等奖。

我在此提到冯康的成果评奖问题，并不是要非难评奖的机构或评委，而是强调对创新成果进行正确评价是一件极其困难的事情。我个人也多次参与国家自然科学奖的评议工作，也深深体会到评议者的难处。值得注意的是即使是享有盛誉的诺贝尔奖，也遭受许多人的议论。而时间也是一个重要因素，经过时间的淘洗，问题就看得清楚了：昔日曾获高奖的项目，今天看来，有些尚保留其价值，有些已有昨日黄花之感。“岁寒而知松柏之后凋也”，信然。

（待续）

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