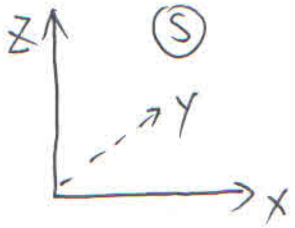
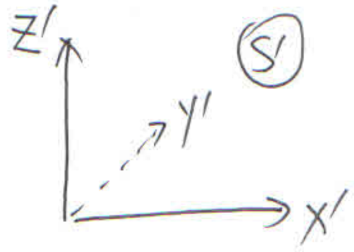




两惯性系 S, S'

在 $t=0$，有同样的原点 $x=y=z=0$

S'相对于S沿x轴以速度v右移

Lorentz变换

$$\begin{cases} x' = \gamma(x - vt) \\ y' = y \\ z' = z \\ t' = \gamma\left(t - \dfrac{vx}{c^2}\right) \end{cases}$$

逆变换

$$\begin{cases} x = \gamma(x' + vt') \\ y = y' \\ z = z' \\ t = \gamma\left(t' + \dfrac{vx'}{c^2}\right) \end{cases}$$

$$\boxed{\gamma \equiv \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} > 1 \qquad \beta \equiv \frac{v}{c} < 1}$$

当 $\beta \to 0$, $\gamma \to 1$：$\begin{cases} t' \to t \\ x' \to x - vt \end{cases}$

四维时空坐标 $x^\mu = (ct, x, y, z)$，$\mu = 0, 1, 2, 3$

则Lorentz变换可看作"转动"变换

$$x'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu_{\ \nu} x^\nu \qquad \Lambda^\mu_{\ \nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & \\ 0 & 0 & & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

由于 $\gamma^2-(-\gamma\beta)^2=1$，该变换使得如下四维距离不变 ②

$$(ct')^2-x'^2-y'^2-z'^2=(ct)^2-x^2-y^2-z^2$$

闵氏时空度规：$g^{\mu\nu}=\begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix}=g_{\mu\nu}$

$$x^\mu=(ct,x,y,z),\quad x_\mu=g_{\mu\nu}x^\nu=(ct,-x,-y,-z)$$

$$g^{\mu\nu}dx_\mu dx_\nu=g_{\mu\nu}dx^\mu dx^\nu$$

4-矢量　协变 $A^\mu=(A^0,A^1,A^2,A^3)$　　$A^\mu=g^{\mu\nu}A_\nu$

　　　逆变 $A_\mu=(A^0,-A^1,-A^2,-A^3)$

4维标积：$A^2=A^\mu A_\mu=g_{\mu\nu}A^\mu A^\nu=(A^0)^2-(A^1)^2-(A^2)^2-(A^3)^2$

$$A\cdot B=g_{\mu\nu}A^\mu B^\nu=A^0B^0-A^1B^1-A^2B^2-A^3B^3$$

$$x'^\mu=\Lambda^\mu_\nu x^\nu$$

$$x^\mu=(ct,x,y,z)$$

$$\Lambda^\mu_\nu=\begin{pmatrix}\gamma, & -\gamma\beta, & 0, & 0\\ -\gamma\beta, & \gamma, & 0, & 0\\ 0, & 0, & 1, & 0\\ 0, & 0, & 0, & 1\end{pmatrix}$$

$$g_{mn}x'^m x'^n$$

$$=g_{mn}\Lambda^m_\mu x^\mu\Lambda^n_\nu x^\nu$$

$$=\underbrace{\left(g_{mn}\Lambda^m_\mu\Lambda^n_\nu\right)}_{g_{\mu\nu}}x^\mu x^\nu$$

对任何 A^μ，如其与 x^μ 类似变化，

$\Rightarrow A^\mu A_\mu$ 洛伦兹不变

时间延长： 时钟在 S' 系静止，x' 一定

$$\begin{cases} t_1=\gamma(t_1'+\frac{v}{c^2}x_1') \\ t_2=\gamma(t_2'+\frac{v}{c^2}x_2') \end{cases} \quad x_1'=x_2' \Rightarrow \underbrace{t_2-t_1}_{\Delta t} = \gamma(t_2'-t_1') > \underbrace{t_2'-t_1'}_{\Delta t'}$$

$$\Delta t > \Delta t'$$

类似的，若时钟在 S 系静止，则从 S' 看，$\Delta t' > \Delta t$

长度收缩： 尺子在 S' 系中静止，则任何时刻 t' 测长度均可

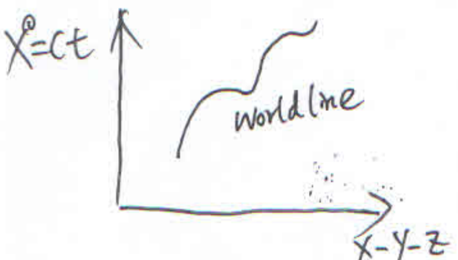
S 系中同时刻 t 测尺子两端 x_1, x_2

$$\begin{cases} x_1'=\gamma(x_1-vt_1) \\ x_2'=\gamma(x_2-vt_1) \end{cases} \Rightarrow (x_2'-x_1') = \gamma(x_1-x_2) > x_1-x_2$$

反之，若尺子在 S 系中静止，则从 S' 系看来，$\Delta x' < \Delta x$，尺子缩短

4-速度

间隔不变量 $(ds)^2=\sum_{\mu=0}^{3} dx^\mu dx_\mu = (cdt)^2-(dx)^2-(dy)^2-(dz)^2$
$= c^2(d\tau)^2$



$d\tau = ds/c$ 粒子随动惯性系中的时间间隔 ⇒ "原时"

通常粒子速度（3-速度）：$\vec{u}=\frac{d\vec{r}}{dt}$ $\left(u_x=\frac{dx}{dt}, \cdots\right)$

则有 $c^2 d\tau^2 = c^2 dt^2 - u^2 dt^2 \Rightarrow dt = \frac{d\tau}{\sqrt{1-(\frac{u}{c})^2}} = \gamma d\tau > dt$

"钟慢"效应

定义 4-速度 $u^\mu = \frac{dx^\mu}{d\tau} = \left(c\frac{dt}{d\tau}, \gamma\frac{d\vec{r}}{dt}\right) = (\gamma c, \gamma\vec{u})$

可得 $u^\mu u_\mu$ 是不变量：

$$\gamma^2c^2-\gamma^2u^2 = c^2$$

$u'^{\mu} = \Lambda^{\mu}_{\nu} u^{\nu}$　为了简单，假设速度沿 X 方向，相对速度为 v

$$\begin{pmatrix} \gamma_{u'} c \\ \gamma_{u'} u'_x \\ \gamma_{u'} u'_y \\ \gamma_{u'} u'_z \end{pmatrix} = \begin{pmatrix} \gamma_v & -\gamma_v \beta & 0 & 0 \\ -\gamma_v \beta & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix}$$

注意区分 u 与 v

$$\Rightarrow \begin{cases} ① \ \dfrac{c}{\sqrt{1-\beta_{u'}^2}} = \dfrac{\gamma_u c - \gamma_u u_x \beta_v}{\sqrt{1-\beta_v^2}} = \dfrac{c - u_x \beta_v}{\sqrt{(1-\beta_v^2)(1-\beta_u^2)}} \\[2ex] ② \ \dfrac{u'_x}{\sqrt{1-\beta_{u'}^2}} = \dfrac{-v\gamma_u + \gamma_u u_x}{\sqrt{1-\beta_v^2}} = \dfrac{u_x - v}{\sqrt{1-\beta_v^2}\sqrt{1-\beta_u^2}} \end{cases}$$

$$②/① \Rightarrow \frac{u'_x}{c} = \frac{u_x - v}{c - u_x \beta_v} \Rightarrow u'_x = \frac{u_x - v}{1 - \frac{u_x}{c}\beta_v} = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}$$

$$\left.\begin{aligned} &\gamma_{u'} u'_y = \gamma_u u_y \Rightarrow u'_y = \frac{\gamma_u}{\gamma_{u'}} u_y \\ &① \Rightarrow \frac{\gamma_{u'}}{\gamma_u} = \gamma_v\left(1 - \frac{u_x \beta_v}{c}\right) \end{aligned}\right\} \Rightarrow u'_y = \frac{u_y}{\gamma_v\left(1 - \frac{v u_x}{c^2}\right)}$$

当 $v \to 0$ 时：$\begin{cases} u'_x \to u_x - v \\ u'_y \to u_y \end{cases}$

定义"静止质量"：m_0 为粒子在随动惯性系中的"惯性质量"

4-动量 $p^\mu = m_0 u^\mu = \left(\frac{m_0 c}{\sqrt{1-\beta_u^2}}, \frac{m_0 \vec{u}}{\sqrt{1-\beta_u^2}}\right)$

引入"动质量" $m = m_0/\sqrt{1-\beta^2}$.

$\Rightarrow p^\mu = (p^0, \vec{p}) = (mc, m\vec{u})$

$\vec{p} = m\vec{u} = \frac{m_0}{\sqrt{1-\beta^2}}\vec{u}$

$$\boxed{\text{定义"相对论性力"为 } \vec{F} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{u})}{dt}}$$

单纯从时空的角度出发，4-动量 p^μ 的各个分量对应于四维时空的时空平移不变性的守恒量。

$\begin{cases} m\vec{u}：\text{空间平移不变性的守恒} \\ mc：\text{时间平移不变} \cdots \end{cases}$

p^0 的经典对应 $\Rightarrow$ 能量

定义 $E = mc^2 = p^0 \cdot c = \frac{m_0 c^2}{\sqrt{1-\beta_u^2}} = m_0 c^2 + \frac{1}{2} m_0 \vec{u}^2 + \cdots$

$p^\mu p_\mu = \left(\frac{E}{c}\right)^2 - \vec{p}\cdot\vec{p} = E_0^2/c^2 = m_0^2 c^2$ $\xrightarrow{\text{对 } t \text{ 求导}}$ $2\frac{E}{c^2}\frac{dE}{dt} - 2\vec{p}\cdot\frac{d\vec{p}}{dt} = 0$

$\vec{p} = m\vec{u} = \frac{E}{c^2}\vec{u}$

$\Rightarrow \frac{dE}{dt} = \vec{F}\cdot\vec{u}$

"$X-X'$" 转动下

$$\begin{pmatrix} E'/c \\ p_1' \\ p_2' \\ p_3' \end{pmatrix} = \begin{pmatrix} \gamma_v, & -\gamma_v\beta_v, & 0, & 0 \\ -\gamma_v\beta, & \gamma_v, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \end{pmatrix} \begin{pmatrix} E/c \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

$$\begin{cases} \frac{E'}{c} = \frac{\frac{E}{c} - \beta_v p_1}{\sqrt{1-\beta_v^2}} \\ p_1' = \frac{p_1 - \frac{E}{c}\beta}{\sqrt{1-\beta^2}} \end{cases} \qquad \begin{cases} p_2' = p_2 \\ p_3' = p_3 \end{cases}$$

以光子为例 $E_r = \hbar\omega$, $\vec{p}_r = \hbar\vec{k}$, $|\vec{k}| = \frac{\omega}{c}$, $m_0 = 0$

$$\frac{\hbar\omega'}{c} = \frac{\frac{\hbar\omega}{c} - \beta\hbar k}{\sqrt{1-\beta^2}} = \frac{\hbar\omega}{c}\sqrt{\frac{1-\beta}{1+\beta}}$$

即光波的（纵向）多普勒频移公式.

力的变换

$$F'_x = \frac{dp'_x}{dt'} = \frac{d(\gamma p_x - \frac{\gamma E}{c}\beta)}{d(\gamma t - \frac{\gamma\beta}{c}x)} = \frac{d(p_x - \frac{E}{c}\beta)}{d(t - \frac{\beta}{c}x)} = \frac{\frac{dp_x}{dt} - \frac{\beta}{c}\frac{dE}{dt}}{1 - \frac{\beta}{c}\frac{dx}{dt}}$$ ⑥

由前 $\frac{dE}{dt} = \vec{F}\cdot\vec{u}$

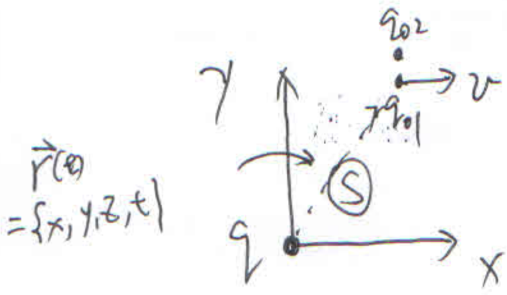
$$\begin{cases} F'_x = \dfrac{F_x - \beta(\vec{u}\cdot\vec{F})/c}{1 - \beta u_x/c} \\ F'_y = \dfrac{dp'_y}{dt'} = \dfrac{dp_y}{\gamma(dt - \frac{\gamma\beta}{c}dx)} = \dfrac{F_y}{\gamma(1 - \frac{\beta}{c}u_x)} \\ F'_z = F_z / \gamma(1 - \beta u_x/c) \end{cases}$$

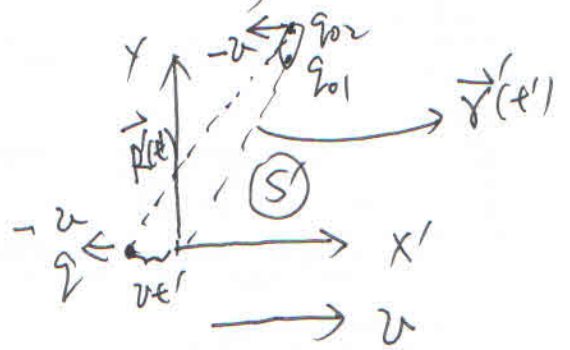
特例：$\vec{u} = 0$，粒子相对 S 系静止

$$\vec{F}'_{\parallel} = \vec{F}_{\parallel}, \quad \vec{F}'_{\perp} = \vec{F}_{\perp}/\gamma$$

（"∥"与"⊥"是相对于变换速度 v 方向而言）

电磁场变换：$\vec{F} = q(\vec{E} + \vec{u}\times\vec{B})$，$q$ 为洛仑兹不变量





S系：$\vec{r}(t)$ 处两个试探电荷

q_{01}：$\vec{v}$ 动

q_{02}：静

O 处静止源电荷 q

场：$\vec{B} = 0$，$\vec{E} = \frac{q\vec{r}}{4\pi\varepsilon_0 r^3}$

S′系：$\vec{r}'(t')$ 处

q_{01}：静

q_{02}：$-\vec{v}$ 动

源电荷：$(-vt', 0, 0, 0)$ 处，$-v$ 运动。

$\vec{E}'$？ $\vec{B}' = ?$

先看 q_{01}：　$\vec{F}_1 = q_{01}\vec{E}$　　$\vec{F}_1' = q_{01}\vec{E}'$

由前力的变换：$F_{1\parallel} = F'_{1\parallel}$，$F'_{1\perp} = \gamma F_{1\perp}$

$$\Rightarrow E'_{\parallel} = E_{\parallel},\quad E'_{\perp} = \gamma E_{\perp}$$

$$E_x' = E_x = \frac{q}{4\pi\varepsilon_0}\frac{x}{(x^2+y^2+z^2)^{3/2}} = \frac{q}{4\pi\varepsilon_0}\frac{\gamma(x'+vt')}{[\gamma^2(x'+vt')^2+y'^2+z'^2]^{3/2}}$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{\gamma R_x'}{[\gamma^2\cos^2\theta+\sin^2\theta]^{3/2}R'^3}$$

$$E_y' = \frac{q}{4\pi\varepsilon_0}\frac{\gamma y}{[x^2+y^2+z^2]^{3/2}} = \frac{q}{4\pi\varepsilon_0}\frac{\gamma R_y'}{[\gamma^2\cos^2\theta+\sin^2\theta]^{3/2}R'^3}$$

$$E_z' = \cdots$$

$$\therefore \vec{E}' = \frac{q}{4\pi\varepsilon_0}\frac{\gamma}{[\gamma^2\cos^2\theta+\sin^2\theta]^{3/2}}\frac{\vec{R}'}{R'^3}$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{(1-v^2/c^2)}{\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^{3/2}}\frac{\vec{R}'}{R'^3}$$

再看 q_{02}，（在 S' 系中~~静止~~运动）　$\vec{F}_2 = q_{02}\vec{E}_2$，　$\vec{F}_2' = q_{02}(\vec{E}' - \vec{v}\times\vec{B}')$

$$\Rightarrow \vec{F}_{2\parallel} = \vec{F}'_{2\parallel},\quad \vec{F}_{2\perp} = \gamma \vec{F}'_{2\perp}$$

$$\Rightarrow \vec{E}'_{\parallel} = \vec{E}_{\parallel},\quad E_{\perp}/\gamma = \vec{E}'_{\perp} - (v\times\vec{B}')_{\perp}$$

（$\vec{v}\times\vec{B}$ 是 $\perp$ 方向）

由上而 $E'_{\perp} = \gamma E_{\perp}$ $\Rightarrow (\vec{v}\times\vec{B}')_{\perp} = \vec{E}'_{\perp} - E'_{\perp}/\gamma^2 = \frac{v^2}{c^2}\vec{E}'_{\perp}$

用 $\vec{v}$ 叉乘两边　$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$

$$(\vec{v}\cdot\vec{B}')\vec{v} - v^2\vec{B}' = \frac{v^2}{c^2}\vec{v}\times\vec{E}'$$

可证 $\vec{v}\cdot\vec{B}'=0$

$$\Rightarrow \vec{B}' = -\frac{1}{c^2}\vec{v}\times\vec{E}' = \frac{1}{c^2}\vec{u}\times\vec{E}'$$

$\vec{u}=-\vec{v}$ 为粒子在S′系中速度

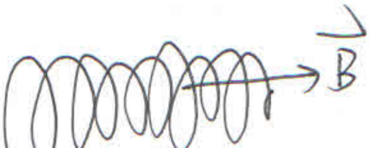
参见刘俊明 P28

$\vec{E}\approx\frac{1}{4\pi\varepsilon_0}\frac{Q\vec{r}}{r^3}$, $\vec{B}=\frac{\mu_0}{4\pi}\frac{Q\vec{v}\times\vec{r}}{r^3}$

$\Rightarrow \vec{B}=\mu_0\varepsilon_0\,\vec{v}\times\vec{E}$

$\Rightarrow c=\frac{1}{\sqrt{\varepsilon_0\mu_0}}$

$\vec{v}\cdot\vec{B}'=0$ 　　 $B_{//}$ 不变，S系中 $B_{//}=0$，则S′系中 $B'_{//}=0$.



$B=\mu_0 n I$

S → S′系：钟慢 $I'=dq/dt'=\frac{dq}{\gamma dt}=I/\gamma$

尺缩 $n'=dN/dl'=\gamma\, dN/dl=\gamma n$

$\Rightarrow B'=\mu_0 n' I'=B$

$\Rightarrow B'_{//}=B_{//}$

电磁场变换

对于运动点电荷

S系：$\vec{E}$，$\vec{B}=0$

S′系：$\vec{E}'_{//}=\vec{E}_{//}$，$\vec{E}'_{\perp}=\gamma\vec{E}_{\perp}$

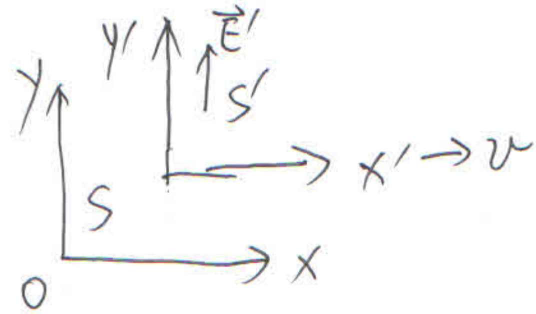
$\vec{B}_{//}=\vec{B}'_{//}=0$，$\vec{B}'_{\perp}=-\frac{\gamma}{c^2}\vec{v}\times\vec{E}$

$=-\frac{1}{c^2}\vec{v}\times\vec{E}'$

一般情形：$\vec{E}'_{//}=\vec{E}_{//}$，$\vec{E}'_{\perp}=\gamma(\vec{E}_{\perp}+\vec{v}\times\vec{B})$

$\vec{B}'_{//}=\vec{B}_{//}$，$\vec{B}'_{\perp}=\gamma(\vec{B}_{\perp}-\frac{1}{c^2}\vec{v}\times\vec{E})$

例：S'系，均匀带电线(λ')位于x'轴.



S'系：$\vec{E'} = \frac{\lambda'}{2\pi\varepsilon_0 y'}\vec{j}$，$\vec{B'} = 0$

S系：$\vec{E}_{\parallel} = \vec{E'}_{\parallel} = 0$

$$\vec{E}_{\perp} = \gamma \vec{E'}_{\perp} = \frac{\gamma\lambda'}{2\pi\varepsilon_0 y}\vec{j} \quad (y = y')$$

$$\vec{B} = \vec{B}_{\perp} = \frac{1}{c^2}\underbrace{\vec{v}\times\vec{E}}_{(S'\to S)} = \frac{\gamma\lambda' v}{2\pi\varepsilon_0 c^2 y}\vec{j} \quad ①$$

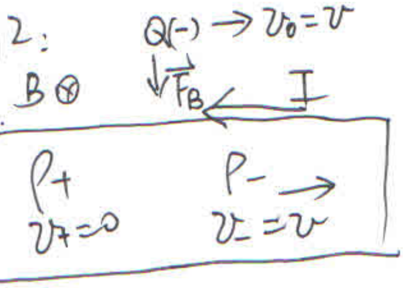
换个角度理解：尺缩 $\lambda = \gamma\lambda' \Rightarrow \vec{E} = \frac{\lambda}{2\pi\varepsilon_0 y}\vec{j} = \frac{\gamma\lambda'}{2\pi\varepsilon_0 y}\vec{j}$

S系中，带电线运动 $I = \frac{dq}{dt} = \frac{\lambda v dt}{dt} = \lambda v$

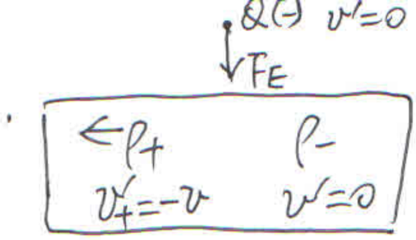
$$\Rightarrow \vec{B} = \frac{\mu_0\lambda v}{2\pi y}\vec{j} \quad ②$$

对比①、② $\Rightarrow \mu_0 = \frac{1}{c^2\varepsilon_0} \Rightarrow c^2 = \frac{1}{\varepsilon_0\mu_0}$

例2:



S系



S′系

$$\begin{cases} \text{正电荷尺度收缩，} \lambda^+ \uparrow & \dfrac{1}{\sqrt{1-v^2/c^2}} \\ \text{负电荷尺度放大，} \lambda^- \downarrow & \sqrt{1-v^2/c^2} \, . \end{cases}$$

$$\vec{B} = \frac{\mu_0 I}{2\pi r^2} \vec{r}$$

$$\vec{F_B} = Q\vec{v} \times \vec{B} = \frac{\mu_0}{4\pi} \frac{2IQv}{r^2} \vec{r}$$

$$I = \rho_+ v S = -\rho_- v S$$

$$\Rightarrow \vec{F_B} = \frac{\mu_0}{4\pi} \frac{2Q\rho_+ S v^2}{r^2} \vec{r}$$

$$\rho' = \rho'_+ + \rho'_- = \rho_+ \frac{v^2/c^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$(|\rho_+| = |\rho_-|)$$

$$\vec{E}' = \frac{\rho' S}{2\pi\varepsilon_0} \frac{\vec{r}}{r^2}$$

$$\vec{F}_E' = \frac{Q \rho_+ S \, v^2/c^2}{2\pi\varepsilon_0 \sqrt{1-\frac{v^2}{c^2}}} \frac{\vec{r}}{r^2} .$$

$$\Downarrow \quad c^2 = \frac{1}{\mu_0 \varepsilon_0}$$

$$\vec{F}_E' \sim \vec{F}_B'$$