第七章 光子的干涉测量 --- 第一部分

一、迈克尔逊星干涉仪和HBT实验（经典）

二、光子探测和量子关联函数

三、光子的二阶关联

四、光子的聚束和反聚束

五、光子计数和光子统计

* 干涉测量的重要性在于→实验上可观察到关联（量子）

$$\begin{cases} field-field\ interferometry \rightarrow amplitude\ correlation \rightarrow 1st\ order \\ photon-photon\ interferometry \rightarrow intensity\ correlation \rightarrow 2nd\ order \end{cases}$$

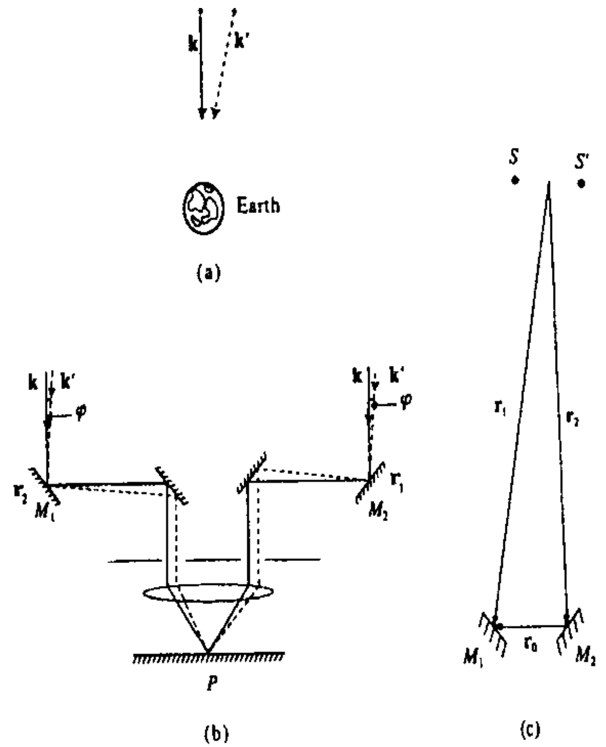
- **核心内容：** 2nd order of photons → HBT exp
- **实验上可观察到量子**

一、 Michelson Stellar interferometer and HBT exp.

1.1 问题：用Michelson Stellar干涉仪测双星$\vec{k},\vec{k'}$的夹角φ。

同一次天文事件中，$\vec{k}\sim\vec{k'}$接近，调节M_1P_1和M_2P_2使光程大致相等

测的是光场间的干涉



最后探测的是光电流

（光电效应原理）

$$I=\kappa\langle E^{*}E\rangle$$

$$E=E_k(e^{i\vec{k}\cdot\vec{r}_1}+e^{i\vec{k}\cdot\vec{r}_2})+E_{k'}(e^{i\vec{k}'\cdot\vec{r}_1}+e^{i\vec{k}'\cdot\vec{r}_2})$$

干涉仪探测到的光场可以表达为

$$E = E_{1\vec{k}} + E_{1\vec{k}'}$$

$$= E_{\vec{k}} \cdot \left(e^{i\vec{k}\cdot\vec{r}_1} + e^{i\vec{k}\cdot\vec{r}_2}\right) + E_{\vec{k}'} \cdot \left(e^{i\vec{k}'\cdot\vec{r}_1} + e^{i\vec{k}'\cdot\vec{r}_2}\right)$$

探测器上发生光电效应时，光电流正比于光强，于是探测到的光电流为

$$I = \kappa\langle E^* E\rangle$$

$$= \kappa\left\langle \left(E^*_{1\vec{k}} + E^*_{1\vec{k}'}\right) \cdot \left(E_{1\vec{k}} + E_{1\vec{k}'}\right)\right\rangle$$

$$= \kappa\left\langle \left|E_{1\vec{k}}\right|^2 + \left|E_{1\vec{k}'}\right|^2 + E_{1\vec{k}} \cdot E^*_{1\vec{k}'} + E^*_{1\vec{k}} \cdot E_{1\vec{k}'}\right\rangle$$

$$= \kappa\left\langle \left|E_{1\vec{k}}\right|^2\right\rangle + \kappa\left\langle \left|E_{1\vec{k}'}\right|^2\right\rangle + \kappa\left\langle E_{1\vec{k}} \cdot E^*_{1\vec{k}'}\right\rangle + \kappa\left\langle E^*_{1\vec{k}} \cdot E_{1\vec{k}'}\right\rangle$$

$E_{1\vec{k}}$和$E_{1\vec{k}'}$分别是来自两个双星的热光辐射场（来自独立的热光源）

$$\left\langle E_{1\vec{k}}\right\rangle = \left\langle E_{1\vec{k}'}\right\rangle = 0, \left\langle E_{1\vec{k}} \cdot E^*_{1\vec{k}'}\right\rangle = \left\langle E_{1\vec{k}'}\right\rangle \cdot \left\langle E_{1\vec{k}'}\right\rangle^* = 0$$

$$E_{1\vec{k}} = E_{\vec{k}} \cdot \left(e^{i\vec{k}\cdot\vec{r}_1} + e^{i\vec{k}\cdot\vec{r}_2}\right)$$

$$E_{1\vec{k}'} = E_{\vec{k}'} \cdot \left(e^{i\vec{k}'\cdot\vec{r}_1} + e^{i\vec{k}'\cdot\vec{r}_2}\right)$$

所以探测到的光电流和强度成正比，写作

$$I = \kappa\left\langle\left|E_{1\vec{k}}\right|^2\right\rangle + \kappa\left\langle\left|E_{1\vec{k}'}\right|^2\right\rangle$$

$$\begin{aligned}\left|E_{1\vec{k}}\right|^2 &= \left|E_{\vec{k}}\right|^2 \cdot \left|e^{i\vec{k}\cdot\vec{r}_1} + e^{i\vec{k}\cdot\vec{r}_2}\right|^2 \\ &= \left|E_{\vec{k}}\right|^2\left(2 + e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)} + e^{-i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)}\right) \\ &= \left|E_{\vec{k}}\right|^2\left\{2 + 2\cos\left[\vec{k}\cdot(\vec{r}_1-\vec{r}_2)\right]\right\}\end{aligned}$$

$$\begin{aligned}\left|E_{1\vec{k}'}\right|^2 &= \left|E_{\vec{k}'}\right|^2\left(2 + e^{i\vec{k}'\cdot(\vec{r}_1-\vec{r}_2)} + e^{-i\vec{k}'\cdot(\vec{r}_1-\vec{r}_2)}\right) \\ &= \left|E_{\vec{k}'}\right|^2\left\{2 + 2\cos\left[\vec{k}'\cdot(\vec{r}_1-\vec{r}_2)\right]\right\}\end{aligned}$$

如果

$$\left\langle \left| E_{\vec{k}} \right|^2 \right\rangle = \left\langle \left| E_{\vec{k}'} \right|^2 \right\rangle = I_0$$

并令$\vec{r}_0 = (\vec{r}_1 - \vec{r}_2)/2$，则

$$I = \kappa \left\langle \left| E_{1\vec{k}} \right|^2 \right\rangle + \kappa \left\langle \left| E_{1\vec{k}'} \right|^2 \right\rangle$$

$$= 2\kappa I_0 \{2 + \cos[\vec{k} \cdot (\vec{r}_1 - \vec{r}_2)] + \cos[\vec{k}' \cdot (\vec{r}_1 - \vec{r}_2)]\}$$

$$= 2\kappa I_0 \left\{ 2 + 2\cos\left[\frac{(\vec{k} + \vec{k}') \cdot (\vec{r}_1 - \vec{r}_2)}{2} \right] \times \cos\left[\frac{(\vec{k} - \vec{k}') \cdot (\vec{r}_1 - \vec{r}_2)}{2} \right] \right\}$$

$$= 4\kappa I_0 \{1 + \cos[(\vec{k} + \vec{k}') \cdot \vec{r}_0] \cdot \cos[(\vec{k} - \vec{k}') \cdot \vec{r}_0]\}$$

则 $I = 4\kappa I_0\{1+\cos[(\vec{k}+\vec{k'})\cdot\vec{r_0}]\cdot\cos[(\vec{k}-\vec{k'})\cdot\vec{r_0}]\}$

讨论：

a) 在$\vec{k}-\vec{k'}$中，大气层的扰动可以抵消

$\varphi = \quad (\vec{k}-\vec{k'})\cdot\vec{r_0}$，调节$\vec{r_0}$，可使$\varphi$达到$\pi$，从而看到干涉

b) 但$\cos(\vec{k}+\vec{k'})\cdot\vec{r_0}$项中大气扰动不能抵消

限制了这种方法的应用（没有达到目的）

必须考虑电场强度之间的干涉

即以下 HBT experiment

1.2 Handury-Brown-Twiss interferometer

1954到1957年间提出

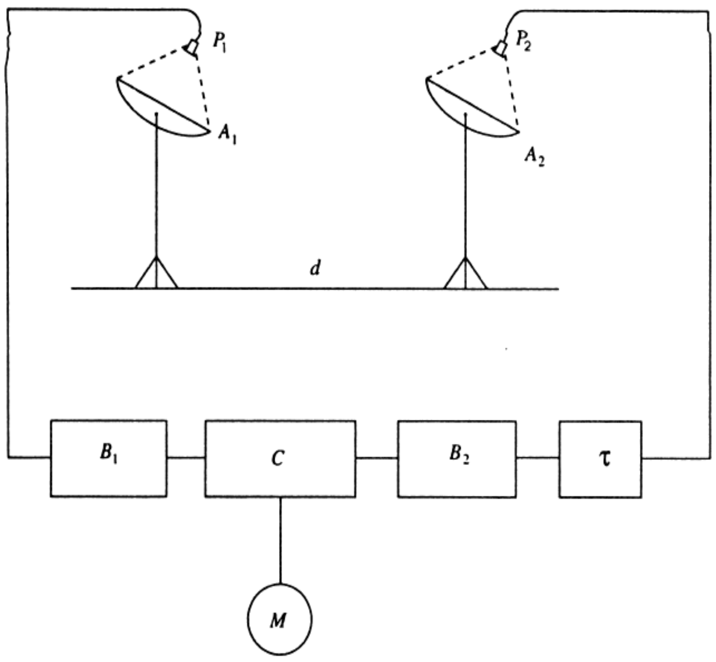
r_1和r_2点间强度干涉

photon

$$I(\vec{r}_1,t)=\kappa\left\langle E(\vec{r}_1,t)\,E^*(\vec{r}_1,t)\right\rangle$$

$$with\quad E(\vec{r}_1,t)=E_k e^{i\vec{k}\cdot\vec{r}_1}+E_{k'}e^{i\vec{k}'\cdot\vec{r}_1}$$

$$I(\vec{r}_1,t)=\kappa\{|E_k|^2+|E_{k'}|^2$$
$$+[E_k E_{k'}^{\ *}e^{i(\vec{k}-\vec{k}')\cdot\vec{r}_1}+c.c.]\}$$



$P_i(i=1,2)$探测器处的光场可以表达为

$$E(\vec{r}_i,t)=E_{\vec{k}}e^{i\vec{k}\cdot\vec{r}_i}+E_{\vec{k}'}e^{i\vec{k}'\cdot\vec{r}_i}$$

$P_i(i=1,2)$探测器处的光电流为

$$I(\vec{r}_i,t)=\kappa E^*E$$

$$=\kappa\left\{\left|E_{\vec{k}}\right|^2+\left|E_{\vec{k}'}\right|^2+\left[E_{\vec{k}}E_{\vec{k}'}^*e^{i(\vec{k}-\vec{k}')\cdot\vec{r}_i}+c.c.\right]\right\}$$

$$=\kappa[A+B(\vec{r}_i)]$$

对P_1和P_2两个探测器上的光电流（光强）进行关联

$$I(\vec{r}_1)I(\vec{r}_2)=\kappa^2[A+B(\vec{r}_1)]\cdot[A+B(\vec{r}_2)]$$

$$=\kappa^2\left[A^2+A\cdot B(\vec{r}_1)+A\cdot B(\vec{r}_2)+B(\vec{r}_1)\cdot B(\vec{r}_2)\right]$$

$$I(\vec{r}_1)I(\vec{r}_2) = \kappa^2\left[A^2 + A\cdot B(\vec{r}_1) + A\cdot B(\vec{r}_2) + B(\vec{r}_1)\cdot B(\vec{r}_2)\right]$$

$$\langle A^2\rangle = \left\langle\left(\left|E_{\vec{k}}\right|^2 + \left|E_{\vec{k}'}\right|^2\right)^2\right\rangle$$

$$\langle A\cdot B(\vec{r}_1)\rangle = \left\langle\left(\left|E_{\vec{k}}\right|^2 + \left|E_{\vec{k}'}\right|^2\right)\cdot\left[E_{\vec{k}}E^*_{\vec{k}'}e^{i(\vec{k}-\vec{k}')\cdot\vec{r}_1} + c.c.\right]\right\rangle$$

$$= \left\langle\left|E_{\vec{k}}\right|^2 E_{\vec{k}}E^*_{\vec{k}'}\right\rangle e^{i(\vec{k}-\vec{k}')\cdot\vec{r}_1} + \left\langle\left|E_{\vec{k}'}\right|^2 E^*_{\vec{k}'}E_{\vec{k}}\right\rangle e^{i(\vec{k}-\vec{k}')\cdot\vec{r}_1}$$

$$+c.c$$

对于来自双星的热态光场，$\langle E_{\vec{k}}\rangle = \langle E_{\vec{k}'}\rangle = 0$，且

$$\left\langle\left|E_{\vec{k}}\right|^2 E_{\vec{k}}E^*_{\vec{k}'}\right\rangle = \left\langle\left|E_{\vec{k}}\right|^2 E_{\vec{k}}\right\rangle\cdot\left\langle E^*_{\vec{k}'}\right\rangle = 0$$

$$\left\langle\left|E_{\vec{k}'}\right|^2 E^*_{\vec{k}'}E_{\vec{k}}\right\rangle = \left\langle\left|E_{\vec{k}'}\right|^2 E^*_{\vec{k}'}\right\rangle\cdot\left\langle E_{\vec{k}}\right\rangle = 0$$

所以

$$\langle A\cdot B(\vec{r}_1)\rangle = 0$$

类似地，有

$$\langle A\cdot B(\vec{r}_2)\rangle = 0$$

对于关联平均的最后一项

$$\begin{aligned}
&\langle B(\vec{r}_1)\cdot B(\vec{r}_2)\rangle \\
&= \left\langle \left[E_{\vec{k}}E^*_{\vec{k}'}e^{i(\vec{k}-\vec{k}')\cdot\vec{r}_1} + c.c.\right]\cdot\left[E_{\vec{k}}E^*_{\vec{k}'}e^{i(\vec{k}-\vec{k}')\cdot\vec{r}_2} + c.c.\right]\right\rangle \\
&= \left\langle E^2_{\vec{k}}E^{*2}_{\vec{k}'}\right\rangle e^{i(\vec{k}-\vec{k}')\cdot(\vec{r}_1+\vec{r}_2)} + \left\langle \left|E_{\vec{k}}\right|^2\cdot\left|E_{\vec{k}'}\right|^2\right\rangle e^{i(\vec{k}-\vec{k}')\cdot(\vec{r}_1-\vec{r}_2)} \\
&\qquad + c.c
\end{aligned}$$

热态光场$E_{\vec{k}}$和$E_{\vec{k}'}$的相位是随机的，$\left\langle E^2_{\vec{k}}\right\rangle = \left\langle E^2_{\vec{k}'}\right\rangle = 0$

$$\left\langle E^2_{\vec{k}}E^{*2}_{\vec{k}'}\right\rangle = \left\langle E^2_{\vec{k}}\right\rangle\cdot\left\langle E^{*2}_{\vec{k}'}\right\rangle = 0$$

所以

$$\langle B(\vec{r}_1)\cdot B(\vec{r}_2)\rangle = \left\langle \left|E_{\vec{k}}\right|^2\right\rangle \cdot \left\langle \left|E_{\vec{k}'}\right|^2\right\rangle e^{i(\vec{k}-\vec{k}')\cdot(\vec{r}_1-\vec{r}_2)} + c.c$$

最后得：

$\langle I(\overrightarrow{r_1},t)I(\overrightarrow{r_2},t)\rangle$

$$= \kappa^2\left\{\left\langle\left(|E_{\vec{k}}|^2+|E_{\overrightarrow{k'}}|^2\right)^2\right\rangle + \langle|E_{\vec{k}}|^2\rangle\langle|E_{\overrightarrow{k'}}|^2\rangle[e^{i(\vec{k}-\overrightarrow{k'})\cdot(\overrightarrow{r_1}-\overrightarrow{r_2})}+c.c.]\right\}$$

所以，测的是光强间的干涉*（光电效应原理）*

第二项表示的是intensity间的干涉，而没有（k＋k‘）项的存在

最后，仍然没有测得干涉条纹，即k和k‘的夹角。*但是：*

HBT实验是整个量子光学的奠基性实验

二、 photon detection and Quantum coherence functions

2.1 definitions of first-order correlation functions

* **photon detection theory**

$$E(\vec{r},t) = E^{+}(\vec{r},t) + E^{-}(\vec{r},t)$$

$$=\sum_{\vec{k}} \hat{e}_k \epsilon_k \, a_k e^{-i\nu_k t + i\vec{k}\cdot\vec{r}} + c.c.$$

光电效应中，*光子被吸收和打出电子是同时发出的*

→只有a_k有贡献，a_k^{+}不符合物理实际，略去

目前假设：探测效率 $\eta = 1$

*** 定义：counting rate（计数率）**

在$\vec{r}$点，$t \rightarrow t + dt$，探测器测到一个光子的几率为$w_1(t)$

$$\mathbf{w}_1(t) \propto |\langle f|E^+(\vec{r},t)|i\rangle|^2$$

$|i\rangle$ **: initial state;** $|f\rangle$ **: final state**

a. final state is never known

（所有状态都可能出现，是完备的）

$$\mathbf{w}_1(t) \sim \sum_f |\langle f|E^+(\vec{r},t)|i\rangle|^2$$

$$=\sum_f \langle i|E^-(\vec{r},t)|f\rangle \langle f|E^+(\vec{r},t)|i\rangle = \langle i|E^-(\vec{r},t)E^+(\vec{r},t)|i\rangle$$

b. never know initial state $|i\rangle$

$$\mathbf{w}_1(t) = \sum_i p_i \langle i|E^-(\vec{r},t)E^+(\vec{r},t)|i\rangle = Tr[\rho E^-(\vec{r},t)E^+(\vec{r},t)]$$

$E^- \rightarrow a^+, E^+ \rightarrow a$, **正规排列，可用P-表示**

$$\mathrm{w}_1(t) = Tr[\rho E^-(\vec{r},t)E^+(\vec{r},t)]$$

*** define: first-order correlation function** ***（时空关联）***

$$G^{(1)} = Tr[\rho \quad E^-(\overrightarrow{r_1},t_1)E^+(\overrightarrow{r_2},t_2)]$$

由于统计稳定性，每两个时间间隔上物理量相同，若 $t_2 - t_1 = \tau$

则有： $G^{(1)}(\overrightarrow{r_1},\overrightarrow{r_2},\tau) = G^{(1)}(\overrightarrow{r_1},\overrightarrow{r_2},t_2 - t_1)$

当$\tau = 0$**，** $\overrightarrow{r_1} = \overrightarrow{r_2} = \vec{r}$**时，**

$$G^{(1)}(\vec{r},\vec{r},0) = \mathrm{w}_1(t)$$

即first order correlation function退化到counting rate

也就是说, First-order correlation function和探测到一个光子的counting rate相联系

Interference → **meterage**

可观测

2.2 两个光子的联合几率与second-order correlation function

* **jointing counting rate**: 观测到一个光子(*相当于湮灭一个光子*)在$\overrightarrow{r_1}$，$t_1 \rightarrow t_1 + dt_1$，同时，另一个光子在$\overrightarrow{r_2}$，$t_2 \rightarrow t_2 + dt_2$的几率

$$\boldsymbol{w}_2(\overrightarrow{r_1}, t_1; \overrightarrow{r_2}, t_2) \sim |\langle f|E^+(\overrightarrow{r_2}, t_2)E^+(\overrightarrow{r_1}, t_1)|i\rangle|^2$$

summing over all final states and initial states（自己推一下）

$$\boldsymbol{w}_2(\overrightarrow{r_1}, t_1; \overrightarrow{r_2}, t_2) = Tr[\rho E^-(\overrightarrow{r_1}, t_1)E^-(\overrightarrow{r_2}, t_2)E^+(\overrightarrow{r_2}, t_2)E^+(\overrightarrow{r_1}, t_1)]$$

* **定义 second-order correlation function**

$$G^{(2)}(\overrightarrow{r_1}, \overrightarrow{r_2}, \overrightarrow{r_3}, \overrightarrow{r_4}; t_1, t_2, t_3, t_4) = Tr[\rho E^-(\overrightarrow{r_1}, t_1)E^-(\overrightarrow{r_2}, t_2)E^+(\overrightarrow{r_3}, t_3)E^+(\overrightarrow{r_4}, t_4)\rangle]$$

$$= \langle E^-(\overrightarrow{r_1}, t_1)E^-(\overrightarrow{r_2}, t_2)E^+(\overrightarrow{r_3}, t_3)E^+(\overrightarrow{r_4}, t_4)\rangle$$

$$G^{(2)}(\vec{r_1},\vec{r_2},\vec{r_3},\vec{r_4};t_1,t_2,t_3,t_4)=Tr[\rho E^-(\vec{r_1},t_1)E^-(\vec{r_2},t_2)E^+(\vec{r_3},t_3)E^+(\vec{r_4},t_4)\rangle]$$

$$=\langle E^-(\vec{r_1},t_1)E^-(\vec{r_2},t_2)E^+(\vec{r_3},t_3)E^+(\vec{r_4},t_4)\rangle$$

$E^-(r_1) \rightarrow a_1{}^+, E^+(\mathrm{r}_1) \rightarrow a_1, E^-(r_2) \rightarrow a_2{}^+, E^+(r_2) \rightarrow a_2$，自动**正规排列，可用P-表示**

$$\boldsymbol{w}_2(\vec{r_1},t_1;\vec{r_2},t_2) = Tr[\rho E^-(\vec{r_1},t_1)E^-(\vec{r_2},t_2)E^+(\vec{r_2},t_2)E^+(\vec{r_1},t_1)]$$

当$\vec{r_4}=\vec{r_1}$，$\vec{r_3}=\vec{r_2}$，$t_4=t_1$，$t_3=t_2$时，

$$G^{(2)}(\vec{r_1},\vec{r_2},\vec{r_2},\vec{r_1};t_1,t_2,t_2,t_1)=\boldsymbol{w}_2(\vec{r_1},t_1;\vec{r_2},t_2)$$

可推广到n^th order，但2nd是核心

说明： **correction function** $\underset{\text{联系}}{\longleftrightarrow}$ **photon detection**

2.3 关联度的定义 first and second order degree of coherence

从以上定义,我们看到 Correlation function<->detection probability紧密联系, 说明量子关联函数的理论是建立在光电效应 *（测量手段）* 的基础上的

* 在以下几个原则的基础上

a) Statistically stationary $\langle n_\tau \rangle = \langle a^+(t+\tau)a(t+\tau) \rangle = \langle n_t \rangle = \langle a^+(t)a(t) \rangle$

b) normal order $P(\alpha, \alpha^*)$

c) time ordering $t_1 < t_2 < t_3 < t_4$

定义了一阶关联度和二阶关联度（注意 r和 t的变化）

$$g^{(1)}(\vec{r},\tau)=\frac{\langle E^{-}(\vec{r},t)E^{+}(\vec{r},t+\tau)\rangle}{\sqrt{\langle E^{-}(\vec{r},t)E^{+}(\vec{r},t)\rangle\langle E^{-}(\vec{r},t+\tau)E^{+}(\vec{r},t+\tau)\rangle}}$$

$$g^{(2)}(\vec{r},\tau)=\frac{\langle E^{-}(\vec{r},t)E^{-}(\vec{r},t+\tau)E^{+}(\vec{r},t+\tau)E^{+}(\vec{r},t)\rangle}{\langle E^{-}(\vec{r},t)E^{+}(\vec{r},t)\rangle\langle E^{-}(\vec{r},t+\tau)E^{+}(\vec{r},t+\tau)\rangle}$$

代入算符得

$$g^{(1)}(\tau)=\frac{\langle a^{+}(t)a(t+\tau)\rangle}{\langle a^{+}a\rangle}$$

$$g^{(2)}(\tau)=\frac{\langle a^{+}(t)a^{+}(t+\tau)a(t+\tau)a(t)\rangle}{\langle a^{+}a\rangle^{2}}$$

均无量纲

2.4 一阶相干度 $g^{(1)}(0)$和二阶相干度$g^{(2)}(0)$的两个例子

a. for a thermal field

$$P(\alpha,\alpha^*) = \frac{1}{\pi\langle n\rangle} e^{-\frac{|\alpha|^2}{\langle n\rangle}}$$

$g^{(1)}(0) = 1$

$$g^{(2)}(0) = \frac{\int P(\alpha,\alpha^*)|\alpha|^4 \cdot d^2\alpha}{[\int P(\alpha,\alpha^*)|\alpha|^2 \cdot d^2\alpha]^2} = \frac{\int \frac{1}{\pi\langle n\rangle} e^{-\frac{|\alpha|^2}{\langle n\rangle}} |\alpha|^4 \cdot d^2\,\alpha}{\left[\int \frac{1}{\pi\langle n\rangle} e^{-\frac{|\alpha|^2}{\langle n\rangle}} |\alpha|^2 \cdot d^2\,\alpha\right]^2}$$

$$= \pi\langle n\rangle \frac{\int e^{-\frac{|\alpha|^2}{\langle n\rangle}} |\alpha|^4 \cdot d^2\alpha}{\left[\int e^{-\frac{|\alpha|^2}{\langle n\rangle}} |\alpha|^2 \cdot d^2\alpha\right]^2}$$

令$\alpha = \mathrm{x} + \mathrm{iy}, \quad d^2\alpha = \mathrm{dxdy} = \mathrm{rdrd}\theta \quad x = rcos\theta, y = rsin\theta$

$$\textbf{原式} = \pi\langle n\rangle \frac{\int e^{-\frac{x^2+y^2}{\langle n\rangle}} \left(x^2+y^2\right)^2 \cdot dxdy}{\left[\int e^{-\frac{x^2+y^2}{\langle n\rangle}} (x^2+y^2) \cdot dxdy\right]^2} = \frac{\pi\langle n\rangle \int e^{-\frac{r^2}{\langle n\rangle}} r^4 \cdot rdrd\theta}{\left[\int e^{-\frac{r^2}{\langle n\rangle}} r^2 \cdot rdrd\theta\right]^2}$$

$$\int_0^{+\infty} x^{2n+1} e^{-x^2} dx = \frac{1}{2} n!$$

$$\frac{\pi\langle n\rangle \int e^{-\frac{r^2}{\langle n\rangle}} r^4 \cdot r dr d\theta}{\left[\int e^{-\frac{r^2}{\langle n\rangle}} r^2 \cdot r dr d\theta\right]^2}$$

$$= \frac{\pi\langle n\rangle(\langle \boldsymbol{n}\rangle)^3 \int e^{-\left(\frac{r}{\sqrt{\langle n\rangle}}\right)^2} \left(\frac{r}{\sqrt{\langle n\rangle}}\right)^5 d\frac{r}{\sqrt{\langle n\rangle}} d\theta}{(\langle n\rangle)^4 \left[\int e^{-\left(\frac{r}{\sqrt{\langle n\rangle}}\right)^2} \left(\frac{r}{\sqrt{\langle n\rangle}}\right)^3 d\frac{r}{\sqrt{\langle n\rangle}} d\theta\right]^2} = \frac{\pi\frac{1}{2}\times 2!\times 2\pi}{(\frac{1}{2}\times 1!\times 2\pi)^2}$$

$$= 2$$

b. for a coherent state $|\alpha_0\rangle$

$$P(\alpha, \alpha^*) = \delta^2(\alpha - \alpha_0)$$

$$g^{(\mathrm{n})}(0) = \frac{\int P(\alpha,\alpha^*)|\alpha|^{\boldsymbol{2n}} \cdot d^2\alpha}{\left[\int P(\alpha,\alpha^*)|\alpha|^2 \cdot d^2\alpha\right]^{\boldsymbol{n}}}$$

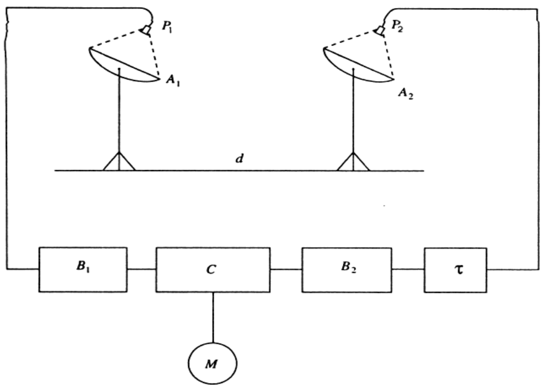
$$g^{(1)}(0) = g^{(2)}(0) \cdots g^{(n)}(0) = 1$$

以上看出，*量子和经典光场的光子关联不同*

三、 二阶关联

用HBT实验可测得$G^{(2)}$

*** HBT实验原理图**



原理：

1. **A_1和A_2探测器通过光电效应探测光子，并转化为光电流**
2. **经过光电倍增管B_1和B_2放大光电流**
3. **通过延时器调整 时间 （t_2-t_1）**
4. **通过混合器C测干涉**
5. **读出数据**

$$* \; G^{(2)}(\vec{r_1},\vec{r_2},;t,t) = \langle E^{-}(\vec{r_1},t)E^{-}(\vec{r_2},t)E^{+}(\vec{r_2},t)E^{+}(\vec{r_1},t)\rangle$$

即两个光子在不同位置、同一时刻测量（这节）

也可以在同一位置、不同时刻，$t_2 \neq t_1$，则有$t_2 - t_1 = \tau$，算符随时间变化（下节）

3.1 two-independent photon state: $|\varphi\rangle = |1_k 1_{k'}\rangle$

此时：两个光子在同一时刻，但不在同一位置

$$G^{(2)}(\vec{r_1}, \vec{r_2}; t, t) = \langle 1_k 1_{k'}|E^-(\vec{r_1}, t)E^-(\vec{r_2}, t)E^+(\vec{r_2}, t)E^+(\vec{r_1}, t)|1_k 1_{k'}\rangle$$

插入$\sum_{\{n\}} |\{n\}\rangle\langle\{n\}| = I$**，（自己推一下）**

$$G^{(2)}(\vec{r_1}, \vec{r_2}, ; t, t)$$

$$= \langle 1_k 1_{k'}|E^-(\vec{r_1}, t)E^-(\vec{r_2}, t) \sum_{\{n\}} |\{n\}\rangle\langle\{n\}| \; E^+(\vec{r_2}, t)E^+(\vec{r_1}, t)|1_k 1_{k'}\rangle$$

$$=\langle 1_k 1_{k'}|E^-(\vec{r_1}, t)E^-(\vec{r_2}, t)|0_k 0_{k'}\rangle\langle 0_k 0_{k'}|E^+(\vec{r_2}, t)E^+(\vec{r_1}, t)|1_k 1_{k'}\rangle$$

$$=|\langle 0|E^+(\vec{r_2}, t)E^+(\vec{r_1}, t)|1_k 1_{k'}\rangle|^2$$

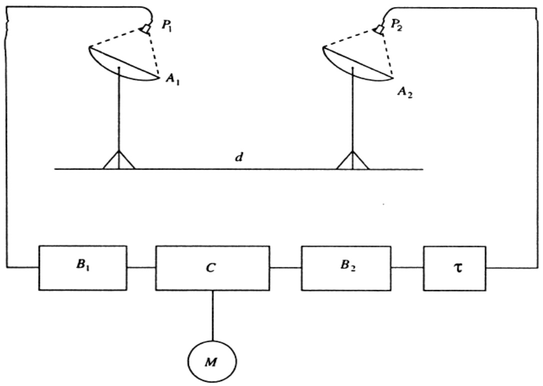
$$=\varphi^{(2)} \cdot \varphi^{(2)^*}$$ **其中：** $\varphi^{(2)} = \langle 0_k 0_{k'}|E^+(\vec{r_2}, t)E^+(\vec{r_1}, t)|1_k 1_{k'}\rangle$

计算：

$$\varphi^{(2)}=\langle 0_k 0_{k'}|E^{+}(\overrightarrow{r_2},t)E^{+}(\overrightarrow{r_1},t)|1_k 1_{k'}\rangle$$

$$E^{+}(\overrightarrow{r_1},t)=\epsilon_k(a_k e^{i\vec{k}\cdot\overrightarrow{r_1}}+a_{k'}e^{i\overrightarrow{k'}\cdot\overrightarrow{r_1}})$$

$$E^{+}(\overrightarrow{r_2},t)=\epsilon_k(a_k e^{i\vec{k}\cdot\overrightarrow{r_2}}+a_{k'}e^{i\overrightarrow{k'}\cdot\overrightarrow{r_2}})$$



则$\varphi^{(2)}=\epsilon_k{}^2[e^{i(kr_1+k'r_2)}+e^{i(k'r_1+kr_2)}]$

得$G^{(2)}=\varphi^{(2)}\cdot\varphi^{(2)^*}=2|\epsilon_k|^4[1+\cos(\vec{k}-\overrightarrow{k'})\cdot(\overrightarrow{r_1}-\overrightarrow{r_2})]$

k和k’接近，r_1和 r_2可调，可得干涉条纹

由HBT实验可测得*HBT term*

3.2. 光来源于两个独立的原子

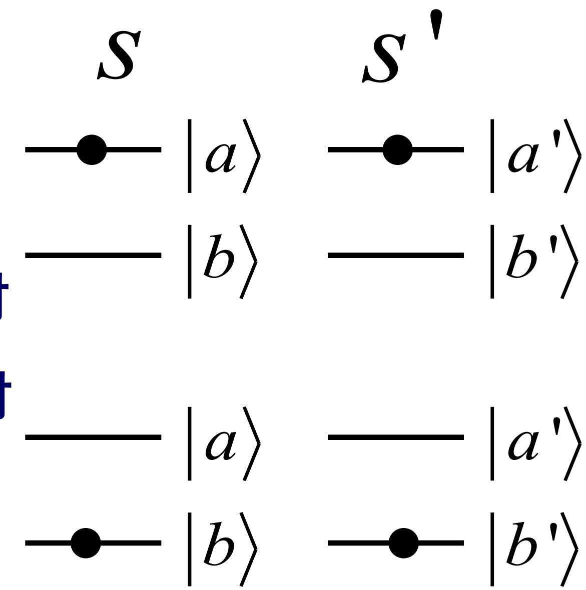
第一种情况,制备到激发态

两个原子S，S’，t=0时的波函数

$$|\psi(0)\rangle = |aa'\rangle \otimes |0\rangle$$

t=∞时，原子从高能级弛豫至低能级，同时放出两个光子γ，γ'，对应波矢k，k'，这时

$$|\psi(\infty)\rangle = |bb'\rangle \otimes |1_k 1_{k'}\rangle$$

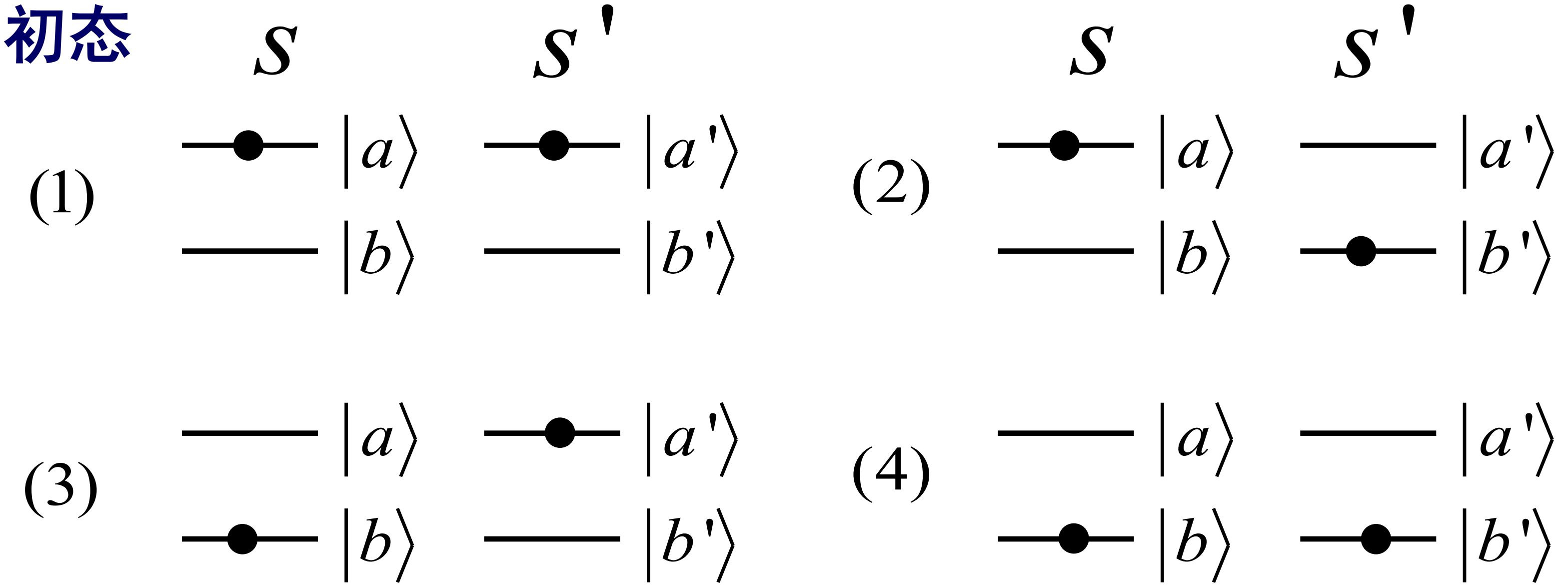


将 $\psi(\infty)$ 代入 $G^{(2)}$ 后, 发现

$$G^{(2)}(\vec{r}_1,\vec{r}_2;t,t) = 2\varepsilon_k{}^4\{1+\cos(\vec{k}-\vec{k}').(\vec{r}_1-\vec{r}_2)\}$$

第二种情况,未制备

初态



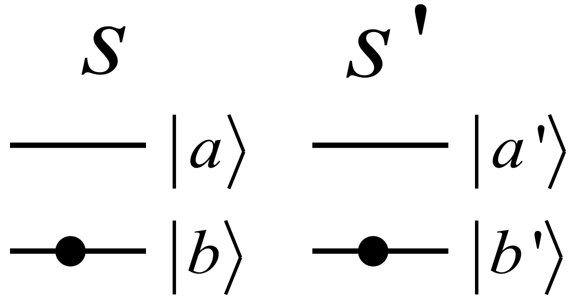
$$t = 0$$

$$\langle \psi(0)| = [|\alpha| e^{i\phi} |a, a'\rangle + |\beta| (e^{i\theta} |a, b'\rangle + e^{i\theta'} |b, a'\rangle)$$

$$+ |\gamma| |b, b'\rangle] \otimes |0\rangle$$

末态

$t = \infty$



$$\langle\psi(\infty)| = [|\alpha| e^{i\phi} |1_k, 1_{k'}\rangle + |\beta| (e^{i\theta} |1_k\rangle + e^{i\theta'} |1_{k'}\rangle) + |\gamma| |0\rangle] \otimes |b, b'\rangle$$

二阶关联，未制备（自己算一下）

$$G^{(2)}(\vec{r}_1,\vec{r}_2;t,t)=2|\alpha|^2\,\varepsilon_k{}^4\{1+\cos(\vec{k}-\vec{k}').(\vec{r}_1-\vec{r}_2)\}$$

二阶关联，制备

$$G^{(2)}(\vec{r}_1,\vec{r}_2;t,t)=2\varepsilon_k{}^4\{1+\cos(\vec{k}-\vec{k}').(\vec{r}_1-\vec{r}_2)\}$$

和刚才原子制备的情况相比，发现多了$|\alpha|^2$，

说明二阶关联函数能给出光子的信息

进而能推知原子（光源）的信息.

3.3. Fermi 子的二阶关联

我们知道Fermi 子有反对易关系

$$\{c, c^{+}\} = 1$$

如果测量原理相同，导致Fermi 子的二阶关联

$$G^{(2)}(\vec{r}_1, \vec{r}_2; t, t) = 2\varepsilon_k{}^4\{1 - \cos(\vec{k} - \vec{k}').(\vec{r}_1 - \vec{r}_2)\}$$

若HBT 实验的测量原理也可以用于Fermi 子

则关联函数的表达有所不同

四、photon antibunching, random and bunching

Quantum light　　　Coherent state　　　Thermal light

* 之前是同一时刻即$t_2=t_1$时，$\tau=0$的$G^{(2)}(\vec{r}_1,\vec{r}_2,0)$，来自不同光源

现在是$\vec{r_2}=\vec{r_1}$，但时间间隔不同时的$G^{(2)}(\tau)$，同一光源

* Schwarz不等式　（适用于经典情况）

$$|\langle a^* b\rangle|^2 \le \langle |a|^2\rangle\langle |b|^2\rangle$$

也就是说，若物理量之间的关联违背了Schwarz不等式，则是量子情况。

例如经典情况下：$a=I(\vec{r},t)$，$b=I(\vec{r},t+\tau)$，则

$$|\langle I^*(\vec{r},t)I(\vec{r},t+\tau)\rangle|^2 \le \langle |I(\vec{r},t)|^2\rangle \cdot \langle |I(\vec{r},t+\tau)|^2\rangle$$

$\tau=0$时，取等号

- **简单推广到量子**

这里：要测的两个光子在同一位置，但时间相差τ 

只需将算符做正规排列即可

$$|\langle : I(\vec{r},t)I(\vec{r},t+\tau): \rangle|^2 \le \langle : I^2(\vec{r},t): \rangle \cdot \langle : I^2(\vec{r},t+\tau): \rangle$$

$\tau = 0$**时，取等号**

For a well defined P-rep. *(closed to classical)*，**上式成立**

下面：将$I(\vec{r},t) \to a, a^+$，$I(\vec{r},t+\tau) \to a(t+\tau), a^+(t+\tau)$**代入**

上式，得

$$g^{(2)}(\tau)=\frac{\langle a^{+}(t)a^{+}(t+\tau)a(t+\tau)a(t)\rangle}{\langle a^{+}a\rangle^{2}}$$

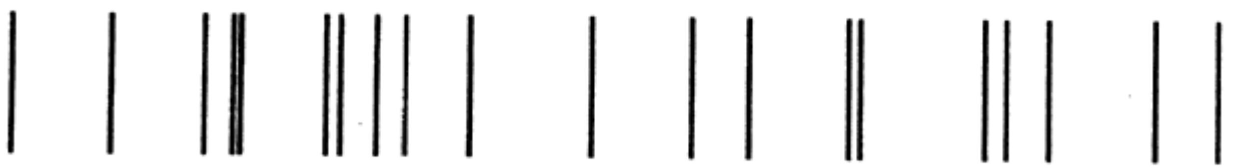
$$=\frac{|\langle :I(\vec{r},t)I(\vec{r},t+\tau):\rangle|}{\langle I\rangle^{2}}\leq\sqrt{\frac{\langle :I^{2}(\vec{r},t):\rangle}{\langle I\rangle^{2}}}\cdot\sqrt{\frac{\langle :I^{2}(\vec{r},t+\tau):\rangle}{\langle I\rangle^{2}}}$$

$\tau=0$**时，取等号。**

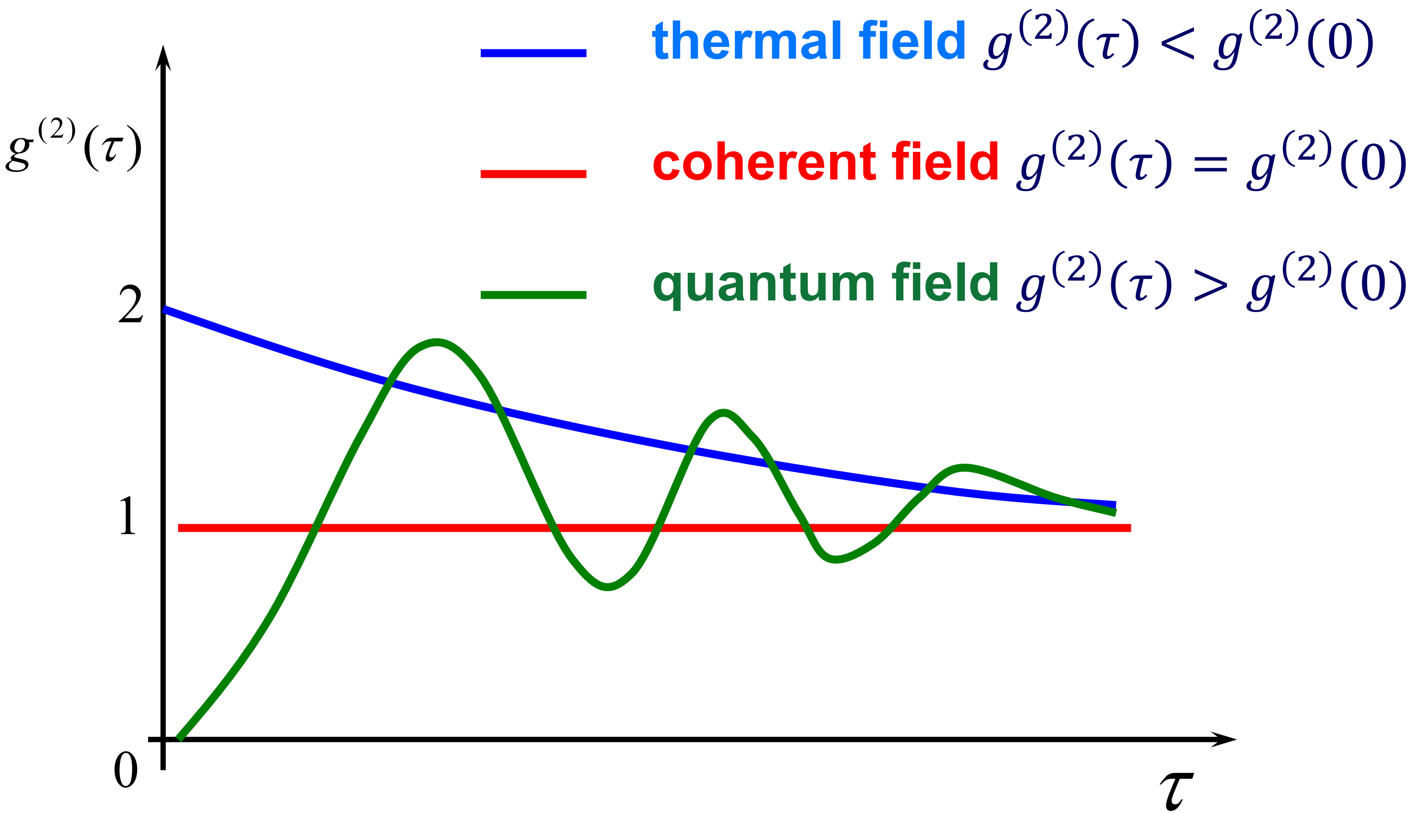
因此，经典情况下：$g^{(2)}(\tau)\leq g^{(2)}(0)$

a. 对于热光场， $g^{(2)}(\tau) < g^{(2)}(0)$ **Photon bunching**

b. 对于相干态， $g^{(2)}(\tau) = g^{(2)}(0)$ **Photon random**



c. 对于量子光场， $g^{(2)}(\tau) > g^{(2)}(0)$ **Photon antibunching**

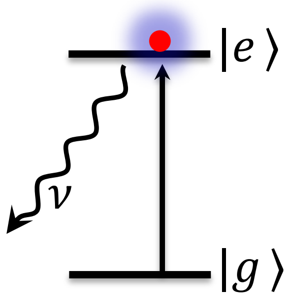


对于热光场， $g^{(2)}(\tau) > \mathbf{1}$, 说明在相隔$(\tau)$的时间内，再探测到一个光子的几率非常高，聚束效应

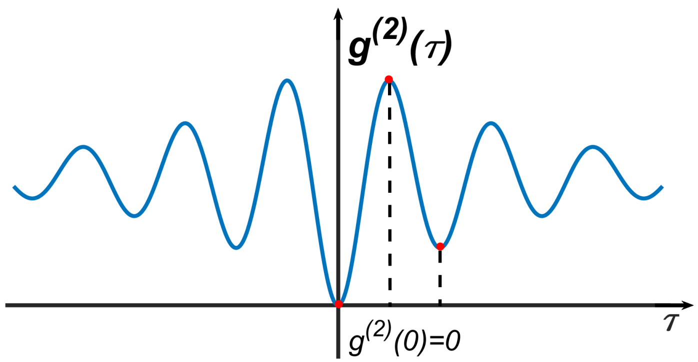
相反，对于量子光场，则有$g^{(2)}(\tau) < \mathbf{1}$,
即在相隔(τ)的时间内，再探测到一个光子的几率很低，反聚束效应

*** 一般的，以上关系可做为经典和量子光场的判据**

例子：单光子源



- 考虑一个二能级原子从上能级向下能级的跃迁（辐射光子）
- 一次（同时）最多只能有一个光子辐射
- $\tau = 0$(两个光子的辐射时间间隔是0)



振荡频率和外界驱动有关

五、 photon counting and photon statistics

光子如何计数P_m ⟶ 光子如何做统计$P(\alpha,\alpha^*)$

也就是说：$P_m \Longleftrightarrow G^{(1)}, G^{(2)}$

* 一个光子打到计数器上打出一个电子的几率 $\eta=\frac{N_e}{N_p}<1$
* 处于$|n\rangle$态的光子，打出m个电子的几率

$$P_m^{(n)} \propto \eta^m (1-\eta)^{n-m}$$

从n中挑出m个光子，有$\binom{n}{m}$种排法，所以

$$P_m^{(n)} = \binom{n}{m} \eta^m (1-\eta)^{n-m}, \quad \binom{n}{m} = \frac{n!}{m!\,(n-m)!}$$

目的：求P_m与$P(\alpha,\alpha^*)$的关系　　**先 * summing over all n**

$$P_m=\sum_n P_m^{(n)}\rho_{nn}$$

$P_m^{(n)}$上页已算出，下面算ρ_{nn}

$$\rho=\int P(\alpha,\alpha^*)|\alpha\rangle\langle\alpha|d^2\alpha$$

$$\rho_{nn}=\langle n|\int P(\alpha,\alpha^*)|\alpha\rangle\langle\alpha|\,d^2\alpha|n\rangle$$

$$=\int P(\alpha,\alpha^*)\,d^2\alpha\langle\alpha|n\rangle\langle n|\alpha\rangle$$

将$\langle n|\alpha\rangle=e^{-\frac{|\alpha|^2}{2}}\cdot\frac{\alpha^n}{(n!)^{\frac{1}{2}}}$代入，则 $\rho_{nn}=\int d^2\alpha\,P(\alpha,\alpha^*)\frac{|\alpha|^{2n}}{n!}e^{-|\alpha|^2}$

将ρ_{nn}，$P_m^{(n)}$代入P_m后得

$$P_m = \int d^2\alpha \, P(\alpha, \alpha^*) \cdot \frac{(\eta|\alpha|^2)^m}{m!} \cdot e^{-\eta|\alpha|^2}$$

测出P_m后，可反推$P(\alpha, \alpha^*)$，然后推得$G^{(1)}, G^{(2)}$

具体计算过程如下（自己算一下）：

$$P_m = \sum_{n=m}^{\infty} \binom{n}{m} \eta^m (1-\eta)^{n-m} \cdot \int P(\alpha, \alpha^*) \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2} \cdot d^2\alpha$$

$$= \int d^2\alpha P(\alpha, \alpha^*) \sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!} \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2} \eta^m (1-\eta)^{n-m}$$

令 $n = l + m$

$$= \int d^2\alpha P(\alpha, \alpha^*) \sum_{l=0}^{\infty} \frac{\eta^m (1-\eta)^l}{m!\,l!} |\alpha|^{2l+2m} e^{-|\alpha|^2}$$

$$= \int d^2\alpha P(\alpha, \alpha^*) \frac{\eta^m}{m!} e^{-|\alpha|^2} |\alpha|^{2m} \sum_{l=0}^{\infty} \frac{(1-\eta)^l}{l!} |\alpha|^{2l}$$

$$= \int d^2\alpha P(\alpha, \alpha^*) \frac{\eta^m |\alpha|^{2m}}{m!} e^{-|\alpha|^2} e^{(1-\eta)|\alpha|^2}$$

$$= \int d^2\alpha P(\alpha, \alpha^*) \frac{(\eta|\alpha|^2)^m}{m!} e^{-\eta|\alpha|^2}$$