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第六章 PQW-表示

一、P-表示和Q-表示存在的必要性

二、P-表示（ coherent state -representation）

三、Q-表示（ squeezed state -representation）

四、Wigner 函数

五、思考题

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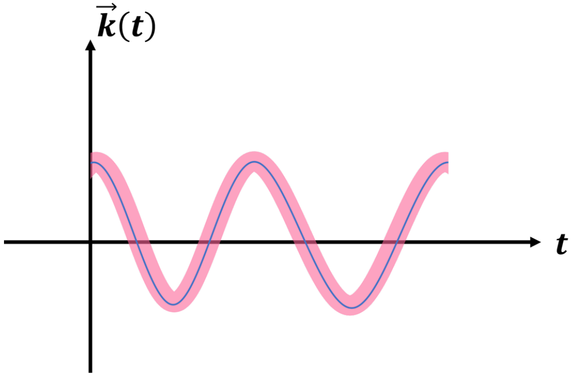
一、P表示和Q表示存在的必要性

1. 经典中定义的很好的物理量（如电场$\vec{E}$）在量子体系中有涨落。

- 由于$\hat{E} = \hat{\varepsilon}\varepsilon a e^{-i\nu t} sinkz + H.C.$，算符$\hat{E}$只有在具体的量子态中平均，才能给出物理量。而量子态不一定是$\hat{E}$的本征态，并且有一定分布，所以$\hat{E}$的涨落不可避免



- 所以准确知道物理量的涨落情况，必须用到密度算符$\hat{\rho}$，通常$\hat{\rho}$很复杂。后来，人们发现，计算$\hat{O}(a, a^+)$的期望值时，

只需将① $\hat{O}$做正规排列($\hat{O} \rightarrow \hat{O}_N$)或反正规排列($\hat{O} \rightarrow \hat{O}_{AN}$)

② 并将算符变成C数，即$\hat{O}_N(a, a^+) \rightarrow O_N(\alpha, \alpha^*)$，$\hat{O}_{AN}(a, a^+) \rightarrow O_{AN}(\alpha, \alpha^*)$

③ 再乘以仅对角元存在的分布函数$P(\alpha, \alpha^*)$或$Q(\alpha, \alpha^*)$即可

这是一个由量子对应到经典的过程

2. 密度算符$\hat{\rho}$在各个表象中的表示

宏观物理量（用算符表示）$\hat{O}$在态$|\Psi\rangle$上的平均为

$$\langle\hat{O}\rangle_{QM}=\langle\psi|\hat{O}|\psi\rangle$$——量子平均，

同时$|\Psi\rangle$又有一定的几率分布P_{ψ}——统计平均，因此有

$$\left\langle\langle\hat{O}\rangle_{QM}\right\rangle_{ensemble}=\sum_{\psi}P_{\psi}\langle\psi|\hat{O}|\psi\rangle$$

插入单位算符$\sum_n|n\rangle\langle n|=1$，得

$$\langle\langle O\rangle\rangle=\sum_n\sum_{\psi}P_{\psi}\langle\psi|\hat{O}|n\rangle\langle n|\psi\rangle$$

$$=\sum_n\sum_{\psi}P_{\psi}\langle n|\psi\rangle\langle\psi|\hat{O}|n\rangle=\sum_n\langle n|\sum_{\psi}P_{\psi}\psi\rangle\langle\psi|\hat{O}|n\rangle$$

而$\sum_{\psi}P_{\psi}|\psi\rangle\langle\psi|=\hat{\rho}$，故$\langle O\rangle=tr\left[\hat{\rho}\hat{O}\right]=tr[\hat{O}\hat{\rho}]$

$\langle\hat{O}\rangle = \mathrm{tr}[\hat{\rho}\hat{O}] = \mathrm{tr}[\hat{O}\hat{\rho}]$**的证明** **（自己推一下）**

$$
\begin{aligned}
\mathrm{tr}[\hat{\rho}\hat{O}] &= \sum_n \langle n|\hat{\rho}\hat{O}|n\rangle \\
&= \sum_n \sum_m \langle n|\hat{\rho}|m\rangle\langle m|\hat{O}|n\rangle \\
&= \sum_n \sum_m \langle m|\hat{O}|n\rangle\langle n|\hat{\rho}|m\rangle \\
&= \sum_m \langle m|\hat{O}\hat{\rho}|m\rangle \\
&= \mathrm{tr}[\hat{O}\hat{\rho}]
\end{aligned}
$$

进一步的：$\mathrm{tr}[\hat{A}\hat{O}] = \mathrm{tr}[\hat{O}\hat{A}]$

$\hat{\rho}$在Fock态中的展开形式

插入两次单体算符：

$$\rho = \sum_n \sum_m |n\rangle\langle n|\rho|m\rangle\langle m| = \sum_{nm} \rho_{nm} |n\rangle\langle m|$$

$\hat{\rho}$在相干态中的展开形式

插入两次单体算符：

$$\rho = \iint \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi} |\alpha\rangle\langle\alpha|\rho|\beta\rangle\langle\beta|$$

$$= \iint \frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi} |\alpha\rangle\rho_{\alpha\beta}\langle\beta|$$

此时非对角元存在，计算复杂。

3. P表示和Q表示的简单解释

正规排列和反正规排列

通常情况下，算符$\hat{O}$中，a和a^+不是按一定顺序排列。

正规排列：每一项中，a都在a^+的右边

$$\hat{O} \equiv \hat{O}_N(a, a^+) = \sum_n \sum_m C_{nm} a^{+n} a^m$$

反正规排列：每一项中，a都在a^+的左边

$$\hat{O} \equiv \hat{O}_{AN}(a, a^+) = \sum_n \sum_m D_{nm} a^n a^{+m}$$

通过$[a, a^+] = 1$调换。

例子：$\hat{O}$在相干态中平均

$\langle \alpha | \hat{O} | \alpha \rangle = \langle \alpha | O_N | \alpha \rangle = \langle \alpha | a^{+n} a^m | \alpha \rangle = \alpha^{*n} \alpha^m$，计算简单

P表示和Q表示的核心：就是将密度算符$\hat{\rho}$（算符的分布函数）变成仅对角元项存在的$P(\alpha,\alpha^*)$或$Q(\alpha,\alpha^*)$（C数的分布函数）同时，对应宏观物理量的算符$\hat{O}$需做正规排列或反正规排列

$$\langle O\rangle = Tr\left[\hat{\rho}\hat{O}\right] = \int P(\alpha,\alpha^*)O_N(\alpha,\alpha^*)\, d^2\alpha$$

$$\langle O\rangle = Tr\left[\hat{\rho}\hat{O}\right] = \int Q(\alpha,\alpha^*)O_{AN}(\alpha,\alpha^*)\, d^2\alpha$$

P表示：适用于偏“经典”的量子态，如热光场、相干态

Q表示：适用于偏“量子”的量子态，如压缩态、Fock态

二、P-representation（coherent state rep）

1. P-rep的定义

求算符$\hat{O}(a, a^{+})$期望值

$$\langle \hat{O}_N(a, a^{+}) \rangle = Tr[\hat{\rho}\hat{O}_N(a, a^{+})] = \sum_n \sum_m C_{nm} Tr[\rho a^{+n} a^m]$$

引入δ算符

$$\delta(\alpha^* - a^{+})\delta(\alpha - a) = \frac{1}{\pi^2}\int \exp[-\beta(\alpha^* - a^{+})] \exp[\beta^*(\alpha - a)]\, d^2\beta$$

$$= \frac{1}{\pi^2}\int \exp[-i\beta(\alpha^* - a^{+})] \exp[-i\beta^*(\alpha - a)]\, d^2\beta$$

原式=$\int d^2\alpha \sum_n \sum_m C_{nm} Tr[\rho\delta(\alpha^* - a^{+})\delta(\alpha - a)\, a^{+n} a^m]$

$$= \int d^2\alpha \sum_n \sum_m C_{nm} Tr[\rho\delta(\alpha^* - a^{+})\delta(\alpha - a)\alpha^{*n}\alpha^m]$$

$$= \int d^2\alpha\ P(\alpha, \alpha^*) O_N(\alpha, \alpha^*)$$

故$P(\alpha, \alpha^*) = Tr[\rho\delta(\alpha^* - a^{+})\delta(\alpha - a)]$ ①

故 $P(\alpha,\alpha^*) = Tr[\rho\delta(\alpha^* - a^+)\delta(\alpha - a)]$ ①

$$\rho = \int P(\alpha,\alpha^*)|\alpha\rangle\langle\alpha|d^2\alpha$$ ②

$$\int P(\alpha,\alpha^*)\,d^2\alpha = \int Tr\,[\rho\delta(\alpha^* - a^+)\delta(\alpha - a)]\,d^2\alpha$$

$$= Tr\left[\rho\int \delta(\alpha^* - a^+)\delta(\alpha - a)\,d^2\alpha\right]$$

$$= Tr[\rho] = 1$$

需证明①↔️②：(自己会推导)

下面，将②带入①

$$P(\alpha,\alpha^*) = Tr[\rho\delta(\alpha^* - a^+)\delta(\alpha - a)] \quad ①$$

证明①↔️②:

$$\rho = \int P(\alpha,\alpha^*)|\alpha\rangle\langle\alpha|d^2\alpha \quad ②$$

将②带入①（自己推一下）

$$P(\alpha,\alpha^*) = Tr[\int P(\beta,\beta^*)|\beta\rangle\langle\beta|d^2\beta\, \delta(\alpha^* - a^+)\delta(\alpha - a)]$$

$$= \int d^2\beta\, P(\beta,\beta^*) Tr[|\beta\rangle\langle\beta|\delta(\alpha^* - a^+)\delta(\alpha - a)]$$

$$= \int d^2\beta\, P(\beta,\beta^*)\langle\beta|\, \delta(\alpha^* - a^+)\delta(\alpha - a)|\beta\rangle$$

$$= \int d^2\beta\, P(\beta,\beta^*)\delta(\alpha^* - \beta^*)\langle\beta|\beta\rangle\delta(\alpha - \beta)$$

$$= \int d^2\beta\, P(\beta,\beta^*)\delta(\alpha^* - \beta^*)\, \delta(\alpha - \beta)$$

$$= P(\alpha,\alpha^*)$$

另一方面

已知： $\rho = \int P(\alpha,\alpha^*)|\alpha\rangle\langle\alpha| d^2\alpha$ **（自己推一下）**

$$\langle a^{+n}a^m\rangle = Tr[\rho a^{+n}a^m]$$

$$= Tr\left[\int P(\alpha,\alpha^*)|\alpha\rangle\langle\alpha| d^2\alpha \;\; a^{+n}a^m\right]$$

$$= \int P(\alpha,\alpha^*)\, d^2\alpha\, Tr[|\alpha\rangle\langle\alpha| a^{+n}a^m]$$

$$= \int P(\alpha,\alpha^*)\, d^2\alpha \sum_n \langle n|\alpha\rangle\langle\alpha|a^{+n}a^m|n\rangle$$

$$= \int P(\alpha,\alpha^*)\, d^2\alpha\langle\alpha|a^{+n}a^m|\alpha\rangle$$

$$= \int P(\alpha,\alpha^*)\, \alpha^{*n}\alpha^m\langle\alpha|\alpha\rangle d^2\alpha$$

$$= \int P(\alpha,\alpha^*)\, \alpha^{*n}\alpha^m d^2\,\alpha$$

2. P-rep和密度算符$\hat{\rho}$之间的关系

前面已知，$\rho = \int P(\alpha,\alpha^*)|\alpha\rangle\langle\alpha|d^2\alpha$，那么

$$\langle -\beta|\rho|\beta\rangle = \int P(\alpha,\alpha^*)\langle -\beta|\alpha\rangle\langle\alpha|\beta\rangle d^2\alpha$$

$$\langle\alpha|\beta\rangle = e^{-\frac{1}{2}|\alpha|^2+\beta\alpha^*-\frac{1}{2}|\beta|^2}$$

故$\langle -\beta|\rho|\beta\rangle = e^{-|\beta|^2}\int P(\alpha,\alpha^*)\,e^{-|\alpha|^2}e^{\beta\alpha^*-\beta^*\alpha}d^2\alpha$

坐标变换$\begin{cases} \alpha = x_\alpha + iy_\alpha,\ \ \beta = x_\beta + iy_\beta \\ d^2\alpha = dx_\alpha dy_\alpha,\quad \beta\alpha^* - \beta^*\alpha = 2i(y_\beta x_\alpha - x_\beta y_\alpha) \end{cases}$

得

$$\langle -\beta|\rho|\beta\rangle e^{|\beta|^2} = \iint [P(x_\alpha,y_\alpha)e^{-({x_\alpha}^2+{y_\alpha}^2)}]\cdot e^{2i(y_\beta x_\alpha - x_\beta y_\alpha)}\cdot dx_\alpha dy_\alpha$$

$$\langle -\beta|\rho|\beta\rangle e^{|\beta|^2} = \iint [P(x_\alpha, y_\alpha) e^{-(x_\alpha{}^2+y_\alpha{}^2)}]\cdot e^{2i(y_\beta x_\alpha - x_\beta y_\alpha)} \cdot dx_\alpha dy_\alpha$$

做傅立叶变换：

$$P(\alpha, \alpha^*)$$

$$= \frac{1}{\pi^2} e^{(x_\alpha{}^2+y_\alpha{}^2)} \iint \langle -\beta|\rho|\beta\rangle \, e^{(x_\beta{}^2+y_\beta{}^2)} \cdot e^{-2i(y_\beta x_\alpha - x_\beta y_\alpha)} dx_\beta dy_\beta$$

$$P(\alpha, \alpha^*) = \frac{1}{\pi^2} e^{|\alpha|^2} \int \langle -\beta|\rho|\beta\rangle \, e^{|\beta|^2} \cdot e^{-\beta\alpha^* + \beta^*\alpha} d^2\beta \qquad \bigstar$$

知道了$\hat{\rho}$**即可求出**$P(\alpha, \alpha^*)$ **（★ 式要会用）**

3. P-rep的几个例子 （了解核心结论）

- 热光场下，$\hat{\rho}$是玻尔兹曼分布

$$\hat{\rho} = \frac{\exp[-\mathcal{H}/k_B T]}{Tr[\exp(-\mathcal{H}/k_B T)]}$$

其中$\mathcal{H} = \hbar\nu(a^{+}a + 1/2)$

$$Tr\left[\exp\left(-\frac{\mathcal{H}}{k_B T}\right)\right] = \sum_n \langle n|\ e^{-\frac{\hbar\nu(\hat{n}+\frac{1}{2})}{k_B T}}|n\rangle$$

$$= \sum_n e^{-\frac{\hbar\nu(n+\frac{1}{2})}{k_B T}} = e^{-\frac{\hbar\nu}{2k_B T}}\frac{1}{1-e^{-\frac{\hbar\nu}{k_B T}}}$$

故 $\hat{\rho} = \dfrac{\sum_n \exp\left(-\frac{\mathcal{H}}{k_B T}\right)|n\rangle\langle n|}{e^{-\frac{\hbar\nu}{2k_B T}}\frac{1}{1-e^{-\frac{\hbar\nu}{k_B T}}}}$

$$= \sum_n \left(1 - e^{-\frac{\hbar\nu}{k_B T}}\right) \exp(-\frac{\hbar\nu\hat{n}}{k_B T})|n\rangle\langle n|$$

计算： $<\mathbf{n}> = Tr[a^+ a\rho]$

$$= \sum_n \langle n|\hat{n}(1 - e^{-\frac{\hbar\nu}{k_B T}})\exp(-\frac{\hbar\nu\hat{n}}{k_B T})|n\rangle$$

$$\langle n\rangle = (1 - e^{-\frac{\hbar\nu}{k_B T}})\sum_n n exp(-\frac{n\hbar\nu}{k_B T})$$

由 $\sum_n ne^{-nx} = \frac{e^{-x}}{(1-e^{-x})^2}$， 令 $x = \frac{\hbar\nu}{k_B T}$，

则 $\langle n\rangle = (1-e^{-x})\frac{e^{-x}}{(1-e^{-x})^2} = \frac{e^{-x}}{1-e^{-x}} = \frac{1}{e^{x}-1} = \frac{1}{e^{\frac{\hbar\nu}{k_B T}}-1}$

所以 $\hat{\rho} = \sum_n \frac{<n>^n}{(1+<n>)^{n+1}}|n\rangle\langle n|$

代入 $\langle -\beta|\rho|\beta\rangle = \sum_n \frac{<n>^n}{(1+<n>)^{n+1}}\langle -\beta|n\rangle\langle n|\beta\rangle$

利用 $|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_m \frac{\alpha^m}{\sqrt{m!}}|m\rangle$，

可得 $\langle -\beta|\rho|\beta\rangle = \frac{e^{-|\beta|^2}}{1+\langle n\rangle}\exp[-|\beta|^2/(1+\frac{1}{\langle n\rangle})]$

$P(\alpha,\alpha^*) = \frac{1}{\pi^2}e^{|\alpha|^2}\int\langle -\beta|\rho|\beta\rangle e^{|\beta|^2}\cdot e^{-\beta\alpha^*+\beta^*\alpha}d^2\beta$ ★

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$\langle -\beta|\rho|\beta\rangle = \sum_n \frac{\langle n\rangle^n}{(1+\langle n\rangle)^{n+1}} \langle -\beta|n\rangle\langle n|\beta\rangle$$

$$= \frac{e^{-|\beta|^2}}{1+\langle n\rangle} \sum_{n=0}^{\infty} \frac{(-|\beta|^2)^n}{n!} \left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^n$$

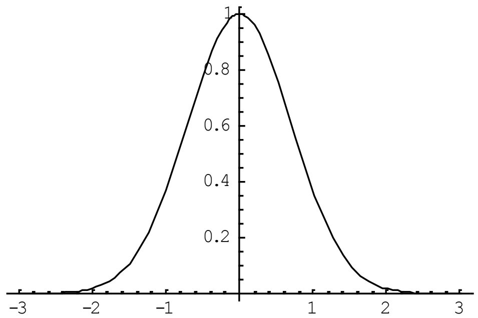
$$= \frac{e^{-|\beta|^2}}{1+\langle n\rangle} \exp\left[-|\beta|^2 \Big/ \left(1+\frac{1}{\langle n\rangle}\right)\right]$$

$$P(\alpha,\alpha^*) = \frac{1}{\pi^2} e^{|\alpha|^2} \int \langle -\beta|\rho|\beta\rangle\, e^{|\beta|^2} \cdot e^{-\beta\alpha^* + \beta^*\alpha} d^2\beta \quad ★$$

最后得

$$P(\alpha,\alpha^*) = \frac{e^{|\alpha|^2}}{\pi^2(1+\langle n\rangle)}\int \exp[-|\beta|^2/(1+\frac{1}{\langle n\rangle})]\, e^{-\beta\alpha^*+\beta^*\alpha} d^2\beta$$

$$= \frac{1}{\pi\langle n\rangle} e^{-\frac{|\alpha|^2}{\langle n\rangle}}$$



热光场的态分布是高斯分布

经典中解释得很好的物理，在量子中*很复杂*

若无“新”的物理考虑，不会有“新”的物理现象，无需用量子代替经典重新推导，其结果应该是一致的

- **相干态的P表示 （记住）**

$$\rho = |\alpha_0\rangle\langle\alpha_0|$$

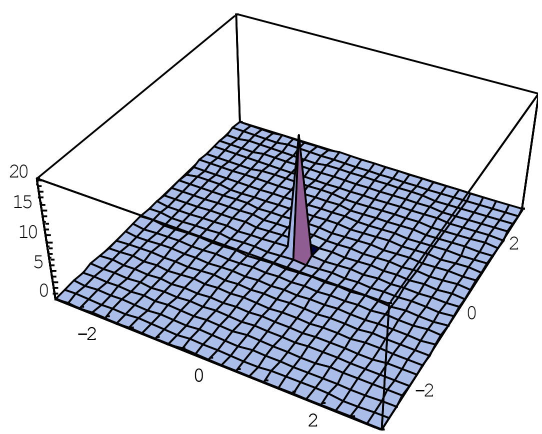
$$\langle-\beta|\rho|\beta\rangle = \langle-\beta|\alpha_0\rangle\langle\alpha_0|\beta\rangle$$

$$= \exp[-|\alpha_0|^2 - |\beta|^2 + \beta\alpha_0{}^* - \beta^*\alpha_0]$$

带入★式， $P(\alpha,\alpha^*) = \frac{1}{\pi^2} e^{|\alpha|^2} \int \langle-\beta|\rho|\beta\rangle \, e^{|\beta|^2} \cdot e^{-\beta\alpha^* + \beta^*\alpha} d^2\beta$

得 $P(\alpha,\alpha^*) = \frac{1}{\pi^2} e^{|\alpha|^2 - |\alpha_0|^2} \int e^{-\beta(\alpha^* - \alpha_0{}^*) + \beta^*(\alpha - \alpha_0)} \, d^2\beta$

$$= \delta^2(\alpha - \alpha_0)$$



$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

● $|n\rangle$态下的P表示

P-rep有一定的局限性，$|n\rangle$态下P出现负值

$$\rho = |n\rangle\langle n|$$

$$\langle -\beta|\rho|\beta\rangle = \langle -\beta|n\rangle\langle n|\beta\rangle = \exp[-|\beta|^2]\frac{(-1)^n|\beta|^{2n}}{n!}$$

代入★式，有

$$P(\alpha,\alpha^*) = \frac{1}{n!\,\pi^2}(-1)^n e^{|\alpha|^2}\int |\beta|^{2n}\, e^{\beta\alpha^*-\beta^*\alpha} d^2\beta$$

$$= \frac{e^{|\alpha|^2}}{n!}\frac{\partial^{2n}}{\partial\alpha^n\alpha^{*n}}\delta^2(\alpha)$$

当$n = 1,3,5,7,\cdots\cdots$时，$P(\alpha,\alpha^*) < 0$，出现了无意义的结果，不适用。

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三、 Q –representation

从与P-rep比较的角度研究 Q-rep

●对于normally ordered operator $\hat{O}_N$

$$\hat{O}_N(a,a^+) = \sum_n \sum_m C_{nm} a^{+n} a^m$$

$$P(\alpha,\alpha^*) = Tr[\rho\delta(\alpha^* - a^+)\delta(\alpha - a)]$$

而对于antinormally ordered operator $\hat{O}_{AN}$

$$\hat{O}_{AN}(a,a^+) = \sum_n \sum_m D_{nm} a^n a^{+m}$$

$$Q(\alpha,\alpha^*) = Tr[\rho\delta(\alpha - a)\delta(\alpha^* - a^+)]$$

$$Q(\alpha,\alpha^*) = Tr[\rho\delta(\alpha - a)\delta(\alpha^* - a^+)]$$

- **Q-rep和密度算符$\hat{\rho}$之间的关系** **（这几页自己会推导）**

$$Q(\alpha,\alpha^*) = Tr\left\{\frac{1}{\pi}\int d^2\alpha' \,[\rho\delta(\alpha - a)|\alpha'\rangle\langle\alpha'|\delta(\alpha^* - a^+)]\right\}$$

$$= \frac{1}{\pi}Tr\{\int d^2\alpha' \,[\rho\delta(\alpha - \alpha)|\alpha'\rangle\langle\alpha'|\delta(\alpha^* - \alpha^+)]\}$$

$$= \frac{1}{\pi}Tr[\rho|\alpha\rangle\langle\alpha|] = \frac{1}{\pi}\sum_n \langle n|\rho|\alpha\rangle\langle\alpha|n\rangle$$

$$= \frac{1}{\pi}\sum_n \langle\alpha|n\rangle\langle n|\rho|\alpha\rangle = \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle$$

Q-rep和密度算符$\hat{\rho}$之间的关系简单

$Q(\alpha,\alpha^*) = \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle$

- **用$Q(\alpha,\alpha^*)$算反正规排列$\hat{O}_{AN}(a,a^+)$的期望值**

$$\langle\hat{O}_{AN}(a,a^+)\rangle = Tr[\hat{O}_{AN}(a,a^+)\hat{\rho}]$$

$$= \sum_n \sum_m D_{nm} Tr(a^n a^{+m} \rho)$$

$$= \sum_n \sum_m D_{nm} Tr(\frac{1}{\pi}\int d^2\alpha\, a^n |\alpha\rangle\langle\alpha| a^{+m} \rho)$$

$$= \sum_n \sum_m D_{nm} Tr(\frac{1}{\pi}\int d^2\alpha \alpha^n \alpha^{*m} |\alpha\rangle\langle\alpha| \rho)$$

$$= \int \hat{O}_{AN}(\alpha,\alpha^*)\, Q(\alpha,\alpha^*) d^2\alpha$$

与P-rep中$\hat{O}_N(a,a^+) = \int \hat{O}_N(\alpha,\alpha^*)\, P(\alpha,\alpha^*) d^2\alpha$ **类似**

● Q-rep的性质

将$\rho = \sum_{\psi} P_{\psi} |\psi\rangle\langle\psi|$**代入** $Q = \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle$**，**

$$Q(\alpha,\alpha^*) = \frac{1}{\pi}\sum_{\psi} P_{\psi} |\langle\psi|\alpha\rangle|^2$$

则 $0 < Q(\alpha,\alpha^*) < \frac{1}{\pi}$**，正定有界，与P-rep中不同**

● P-rep和Q-rep之间的关系

$$Q(\alpha,\alpha^*)=\frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle$$

而$\rho=\int P(\alpha,\alpha^*)|\alpha\rangle\langle\alpha|d^2\alpha$，得

$$Q(\alpha,\alpha^*)=\frac{1}{\pi}\int\langle\alpha|\alpha'\rangle\ P(\alpha',\alpha'^*)\langle\alpha'|\alpha\rangle d^2\alpha'$$

$$=\frac{1}{\pi}\int P(\alpha',\alpha'^*)e^{-|\alpha-\alpha'|^2}\,d^2\alpha'$$

● Fock态下的Q-rep

$$\rho = |n\rangle\langle n|$$

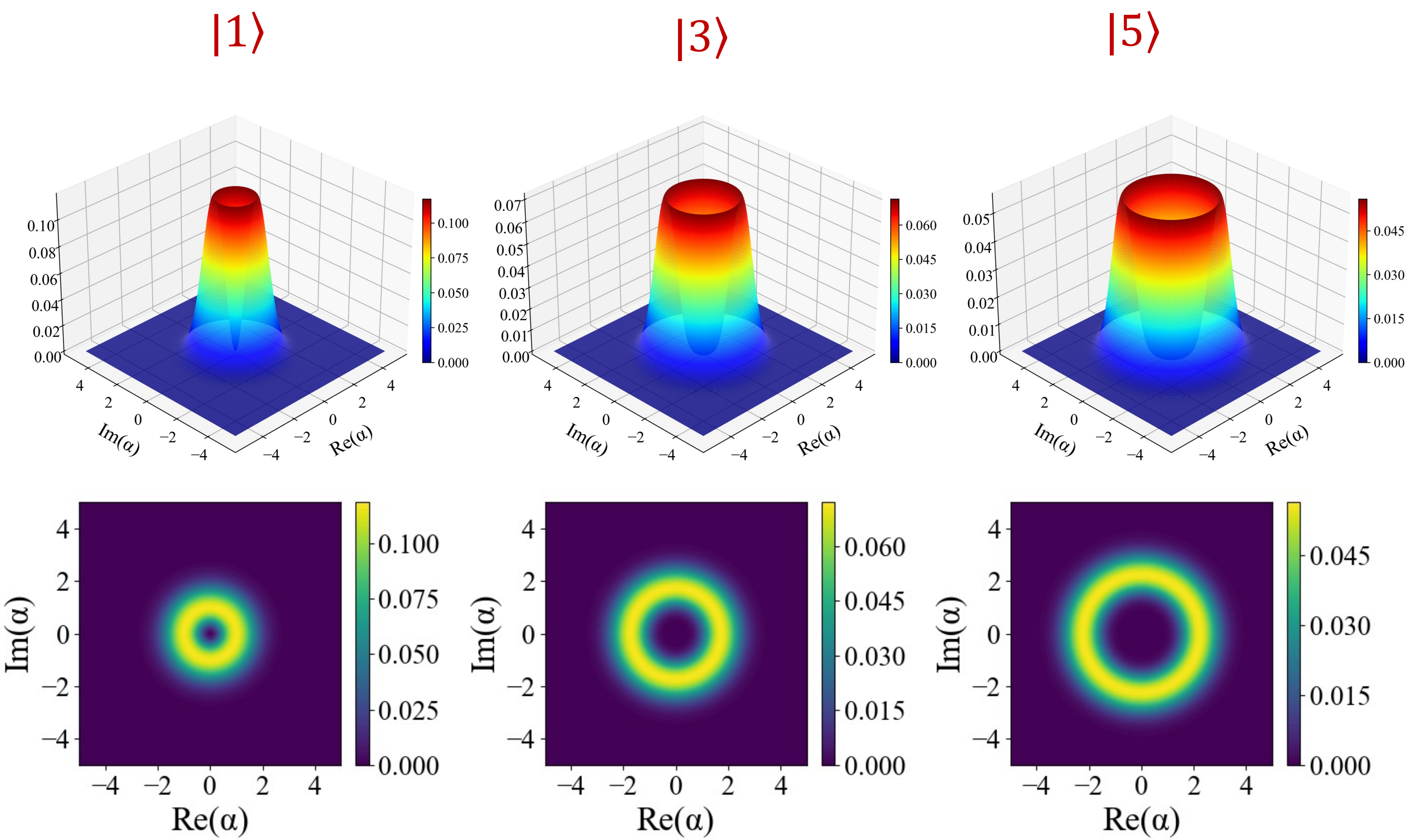
由 $Q(\alpha,\alpha^*) = \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle = \frac{1}{\pi}|\langle n|\alpha\rangle|^2 = \frac{|\alpha|^{2n}e^{-|\alpha|^2}}{\pi n!}$ 可 知， 在 Fock态中$Q(\alpha,\alpha^*)$定义良好。

$|n\rangle$态是非常靠近量子的态，符合$Q(\alpha,\alpha^*)$适用范围。

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

- **Fock态$|n\rangle$的Q表示**

$$Q(\alpha,\alpha^*)=\frac{|\alpha|^{2n}e^{-|\alpha|^2}}{\pi n!}$$



可以看出：对称分布，n越大，分布越分散

● **相干态下的Q-rep （自己推一下）**

$$\rho = |\alpha_0\rangle\langle\alpha_0|$$

$$Q(\alpha,\alpha^*) = \frac{1}{\pi}\langle\alpha|\alpha_0\rangle\langle\alpha_0|\alpha\rangle$$

$$= \frac{1}{\pi}e^{-|\alpha-\alpha_0|^2}$$

$$\langle\alpha|\alpha'\rangle = \exp\left(-\frac{1}{2}|\alpha|^2 + \alpha^*\alpha' - \frac{1}{2}|\alpha'|^2\right)$$

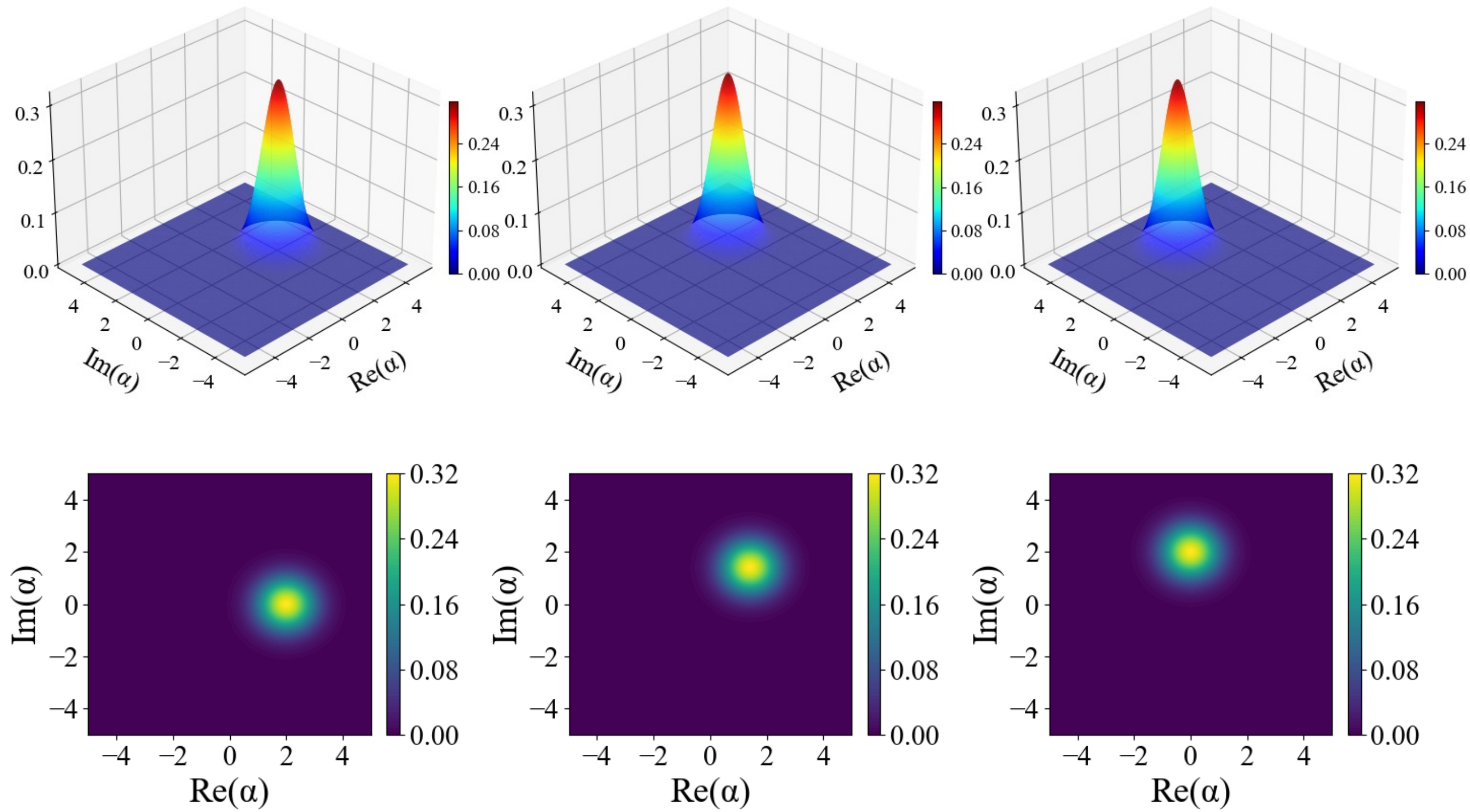
$$Q(\alpha,\alpha^*) = \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle$$

- **相干态的Q表示**

$|\alpha_0\rangle$ $\alpha_0 = 2\, e^{i\theta}$

$$Q(\alpha, \alpha^*) = \frac{1}{\pi} e^{-|\alpha - \alpha_0|^2}$$

$\theta = 0$ $\theta = \pi/4$ $\theta = \pi/2$



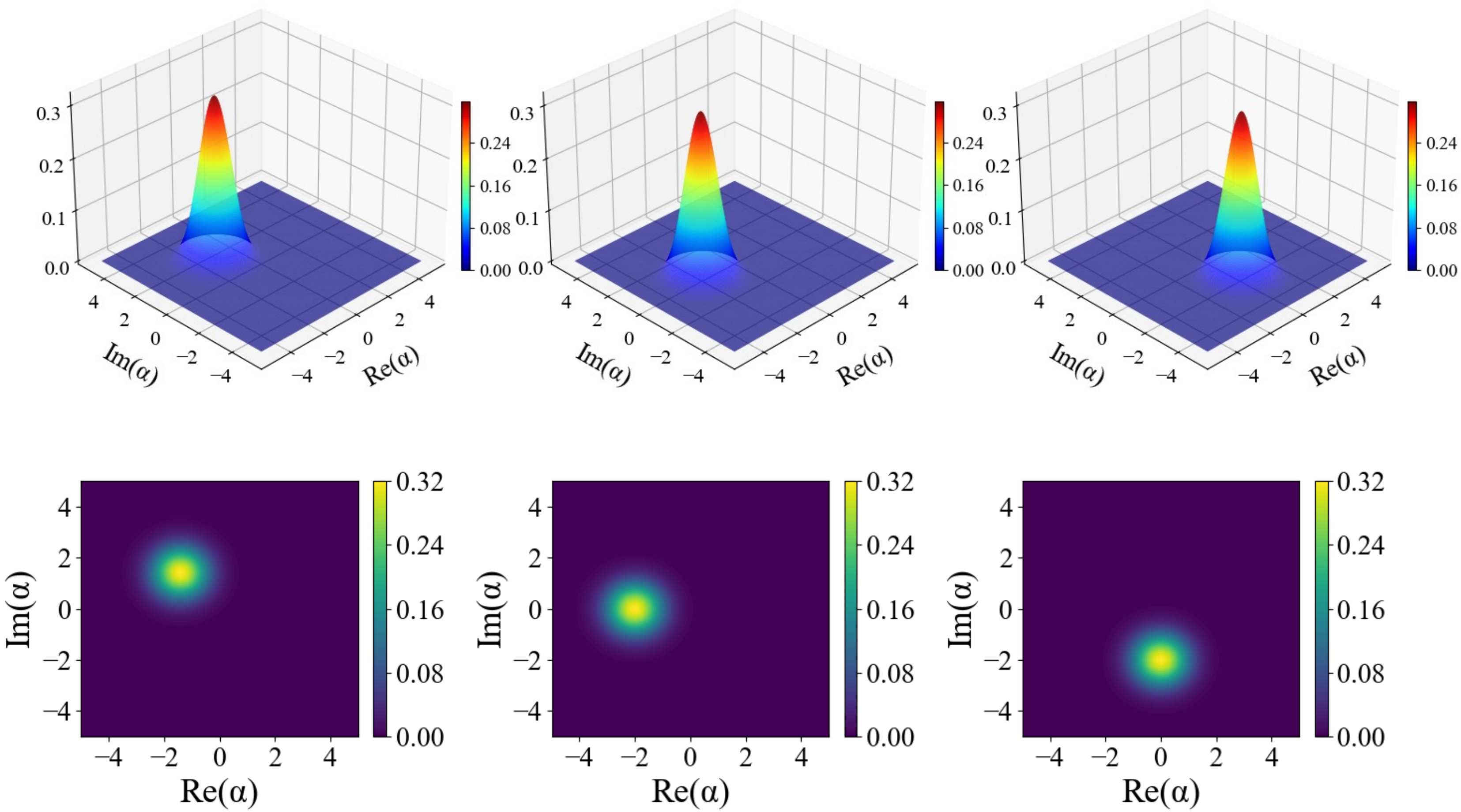
$|\alpha_0\rangle$

- **相干态的Q表示**

$\alpha_0 = 2\, e^{i\theta}$

$$Q(\alpha,\alpha^*) = \frac{1}{\pi} e^{-|\alpha-\alpha_0|^2}$$

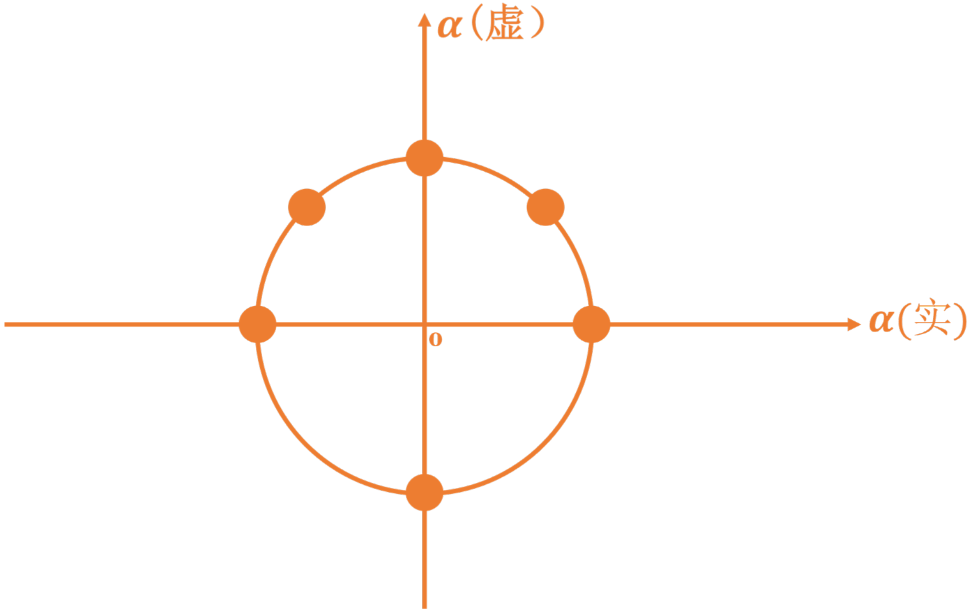
$\theta = 3\pi/4$ $\theta = \pi$ $\theta = 3\pi/2$



相干态的Q表示特点：

1. 高斯分布

2. 相空间中绕 α 的绝对值圆周分布



- **猫态的Q表示：** $|\psi\rangle = N(|\alpha_0\rangle + e^{i\phi}|-\alpha_0\rangle)|\psi\rangle$　**N为归一化常数**

$$\rho = |\psi\rangle\langle\psi|$$

$$= N^2\left(|\alpha_0\rangle + e^{i\phi}|-\alpha_0\rangle\right)\left(\langle\alpha_0| + \langle-\alpha_0|e^{-i\phi}\right)$$

$$= N^2\left(|\alpha_0\rangle\langle\alpha_0| + e^{-i\phi}|\alpha_0\rangle\langle-\alpha_0| + e^{i\phi}|-\alpha_0\rangle\langle\alpha_0| + |-\alpha_0\rangle\langle-\alpha_0|\right)$$

由此得到猫态的Q-分布函数为 (自己算一下）

$$Q_{cat} = \frac{N^2}{\pi}[e^{-|\alpha-\alpha_0|^2} + e^{-|\alpha+\alpha_0|^2} + e^{-i\phi}e^{-|\alpha|^2-|\alpha_0|^2+\alpha^*\alpha_0-\alpha_0^*\alpha} + e^{i\phi}e^{-|\alpha|^2-|\alpha_0|^2-\alpha^*\alpha_0+\alpha_0^*\alpha}]$$

$$Q(\alpha,\alpha^*) = \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle$$　相干态：$Q(\alpha,\alpha^*) = \frac{1}{\pi}e^{-|\alpha-\alpha_0|^2}$

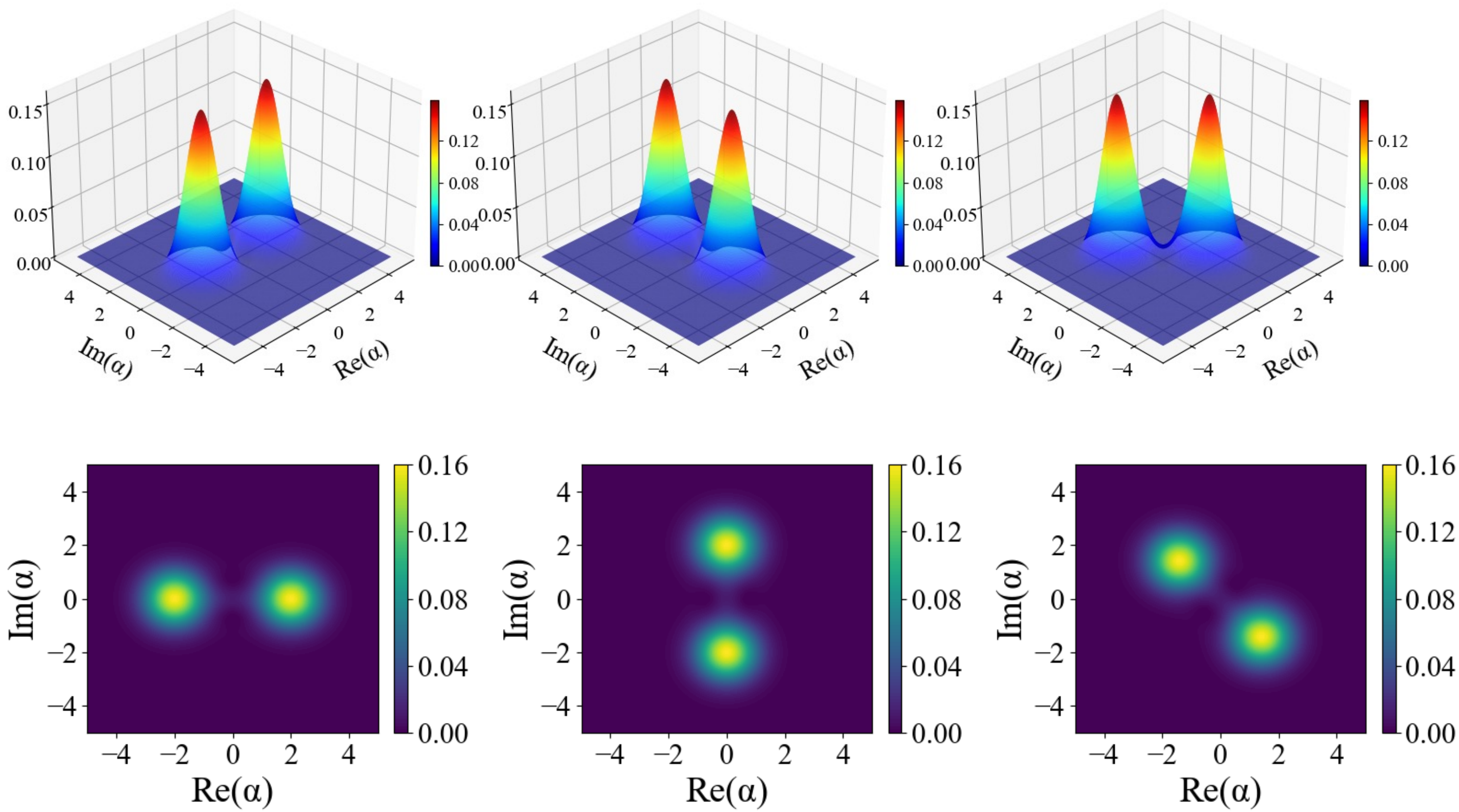
$$\langle\alpha|\alpha'\rangle = \exp\left(-\frac{1}{2}|\alpha|^2 + \alpha^*\alpha' - \frac{1}{2}|\alpha'|^2\right)$$

偶猫态的Q-分布

$\alpha_0 = 2\,e^{i\theta}$ $\qquad$ $e^{i\phi} = 1$

$$Q_{cat}(\alpha) = \frac{N_e^{\,2}}{\pi}\left[e^{-|\alpha-\alpha_0|^2} + e^{-|\alpha+\alpha_0|^2} + e^{-|\alpha|^2-|\alpha_0|^2+\alpha^*\alpha_0-\alpha_0^*\alpha} + e^{-|\alpha|^2-|\alpha_0|^2-\alpha^*\alpha_0+\alpha_0^*\alpha}\right]$$

$\theta = 0$ $\qquad$ $\theta = \pi/2$ $\qquad$ $\theta = 3\pi/4$

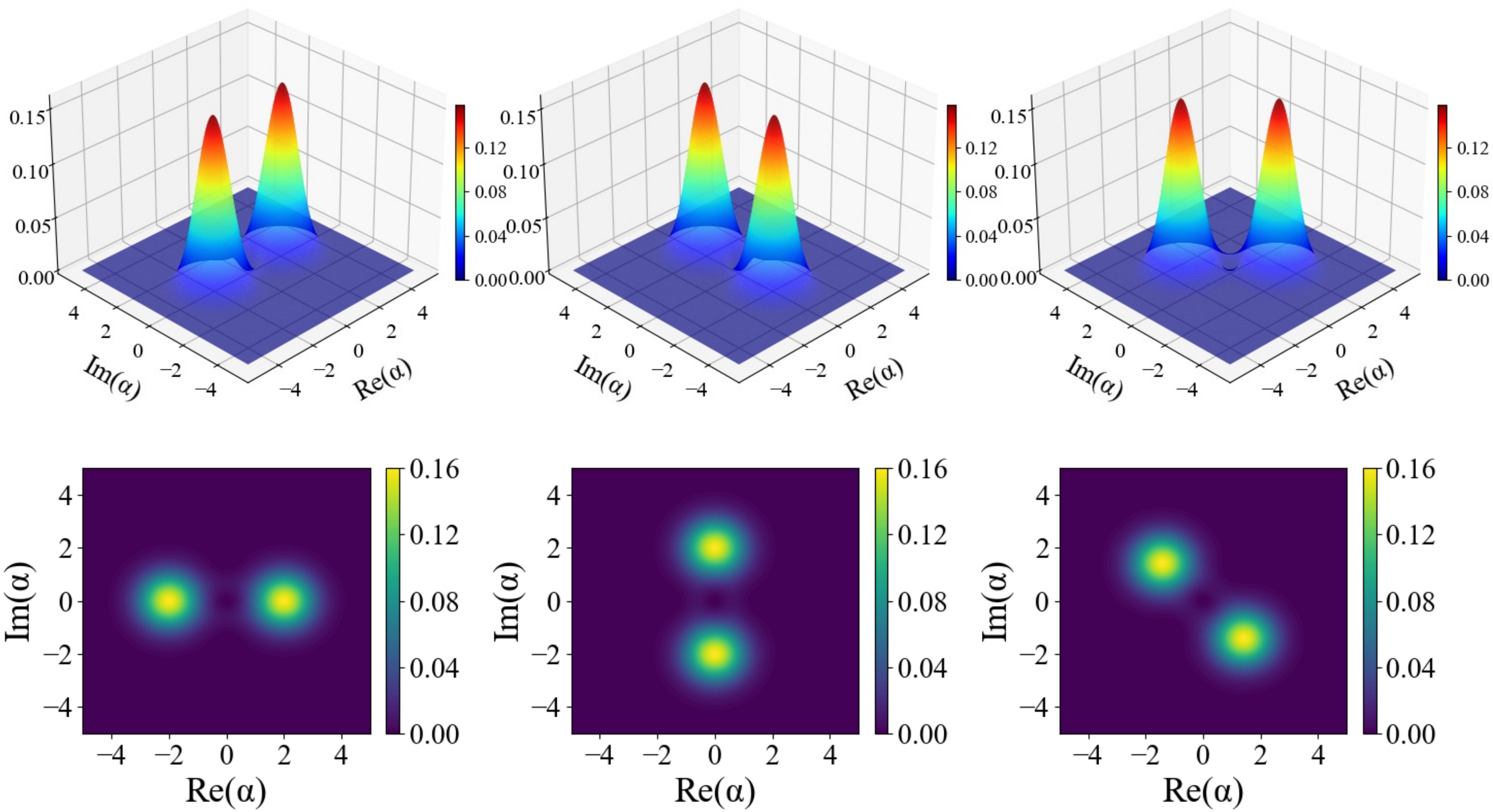


奇猫态的Q-分布

$\alpha_0 = 2\, e^{i\theta}$ $e^{i\phi} = -1$

$$Q_{cat}(\alpha) = \frac{N_o{}^2}{\pi}\left[e^{-|\alpha-\alpha_0|^2} + e^{-|\alpha+\alpha_0|^2} - e^{-|\alpha|^2-|\alpha_0|^2+\alpha^*\alpha_0-\alpha_0^*\alpha} - e^{-|\alpha|^2-|\alpha_0|^2-\alpha^*\alpha_0+\alpha_0^*\alpha}\right]$$

$\theta = 0$ $\theta = \pi/2$ $\theta = 3\pi/4$



$$Q(\alpha, \alpha^*) = \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle$$

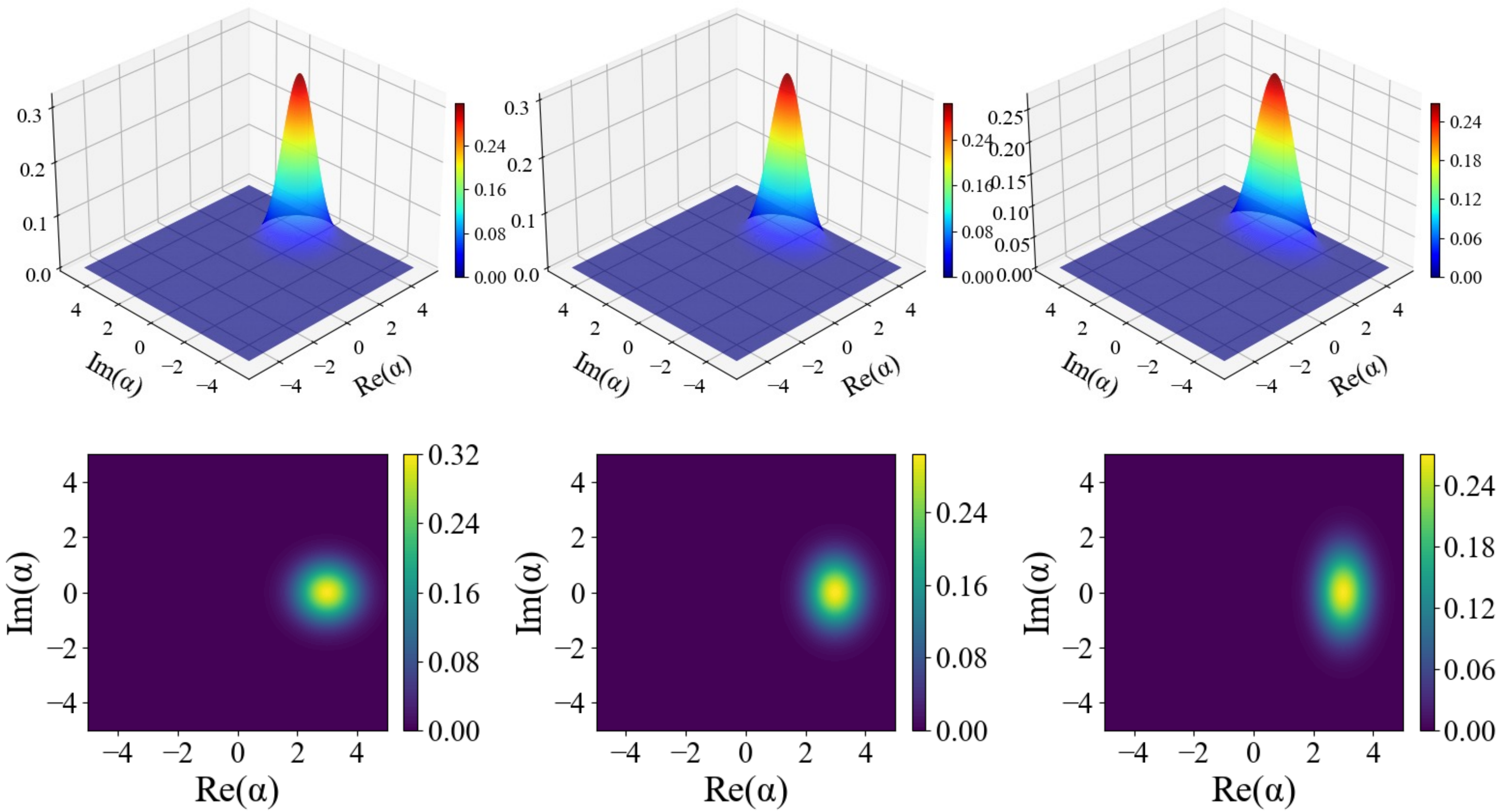
- **压缩态$|\beta, \xi\rangle$的Q-rep**

$$|\beta, \xi\rangle = \mathcal{D}(\beta)\mathcal{S}(\xi)|0\rangle$$

$$\beta = 3, \theta = 0$$

$$\xi = re^{i\theta}$$

$r = 0.0$ $\quad$ $r = 0.3$ $\quad$ $r = 0.6$



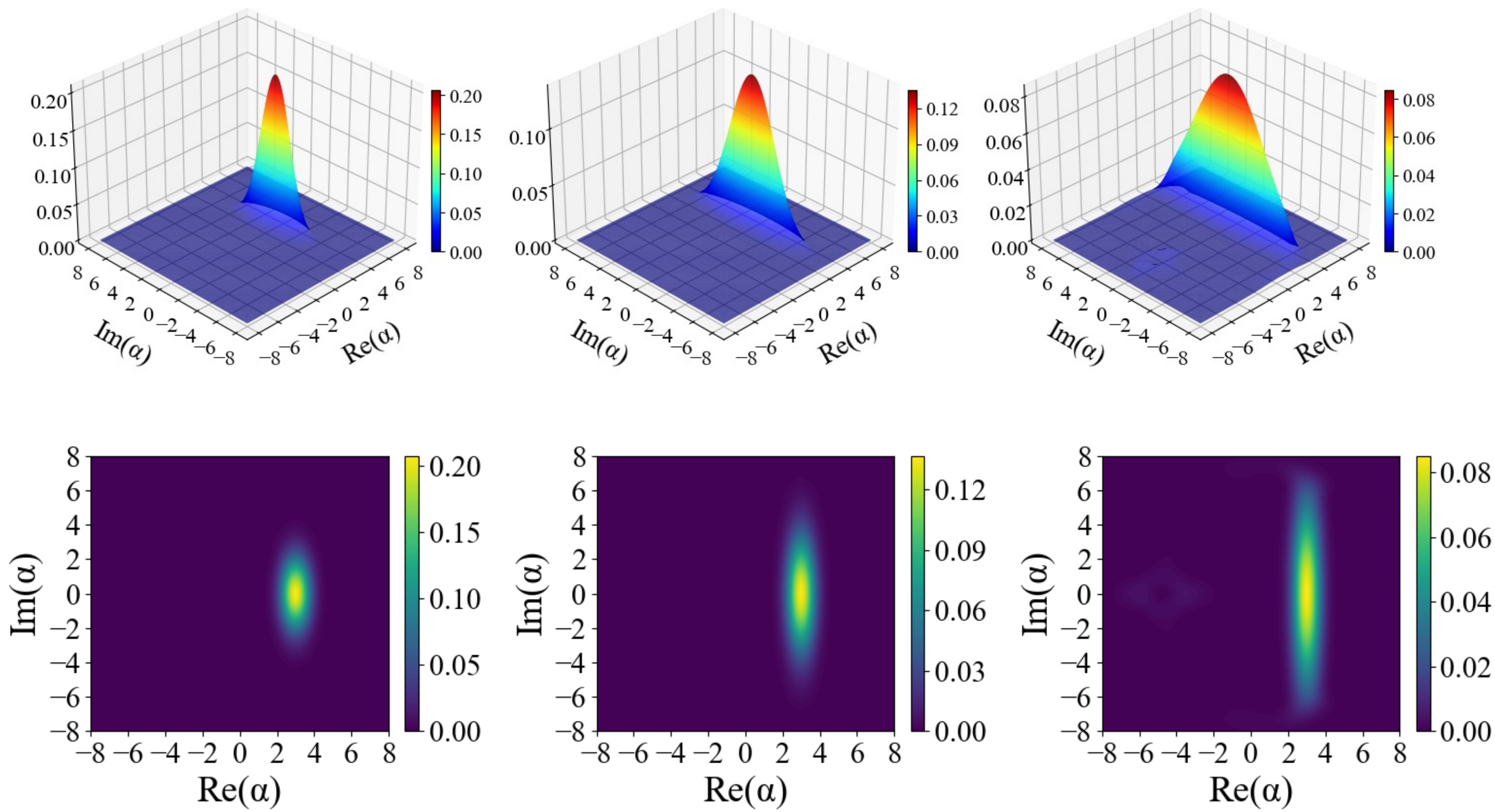
压缩态$|\beta,\xi\rangle$的Q-rep

$$|\beta,\xi\rangle = \mathcal{D}(\beta)\mathcal{S}(\xi)|0\rangle$$

$$\beta = 3, \theta = 0$$

$$\xi = re^{i\theta}$$

$r = 1.0$ $r = 1.5$ $r = 2.0$



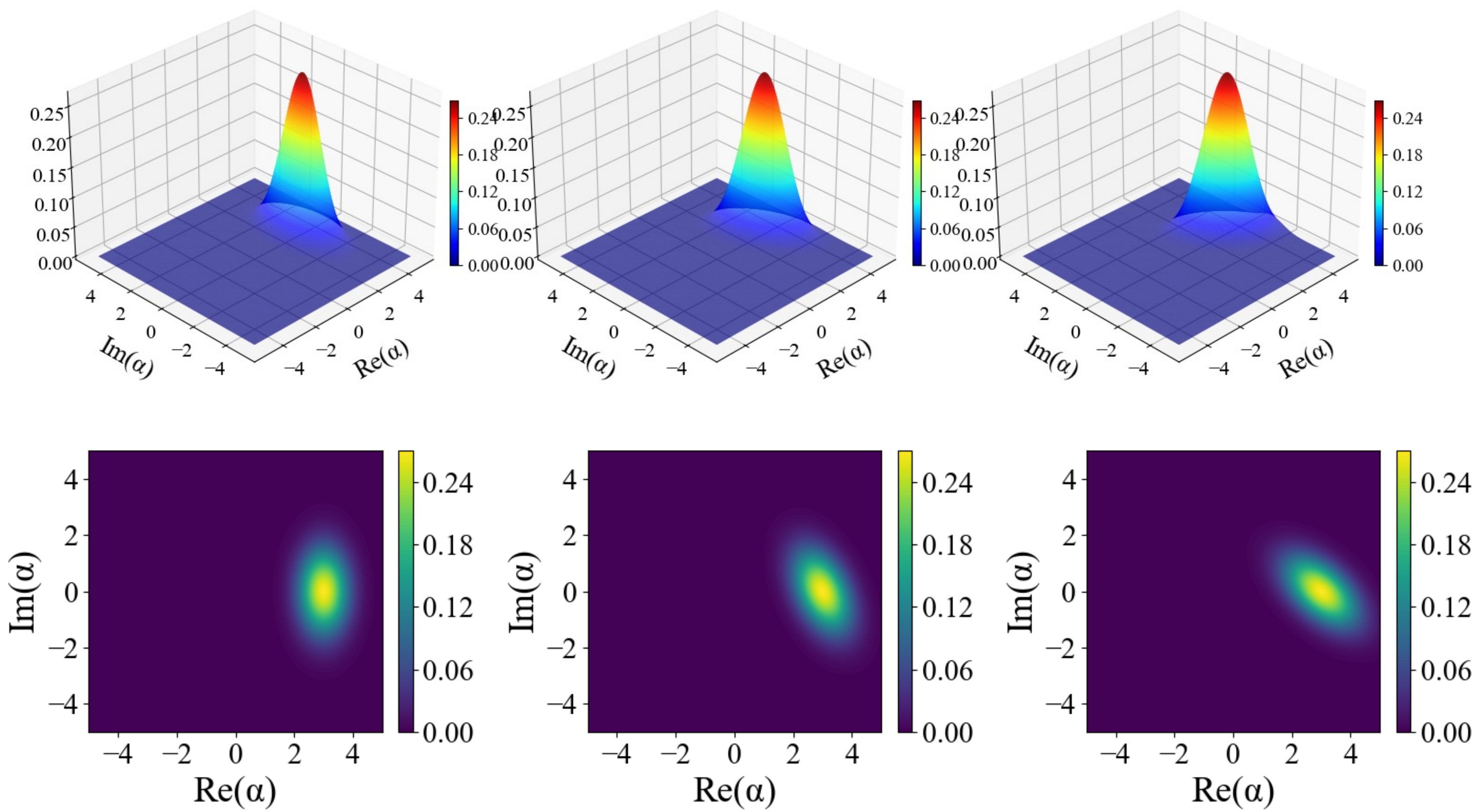
压缩态$|\beta,\xi\rangle$的Q-rep

$$|\beta,\xi\rangle = \mathcal{D}(\beta)\mathcal{S}(\xi)|0\rangle$$

$$\xi = re^{i\theta}$$

$\beta = 3, r = 0.6$

$\theta = 0$ $\quad$ $\theta = \pi/4$ $\quad$ $\theta = \pi/2$

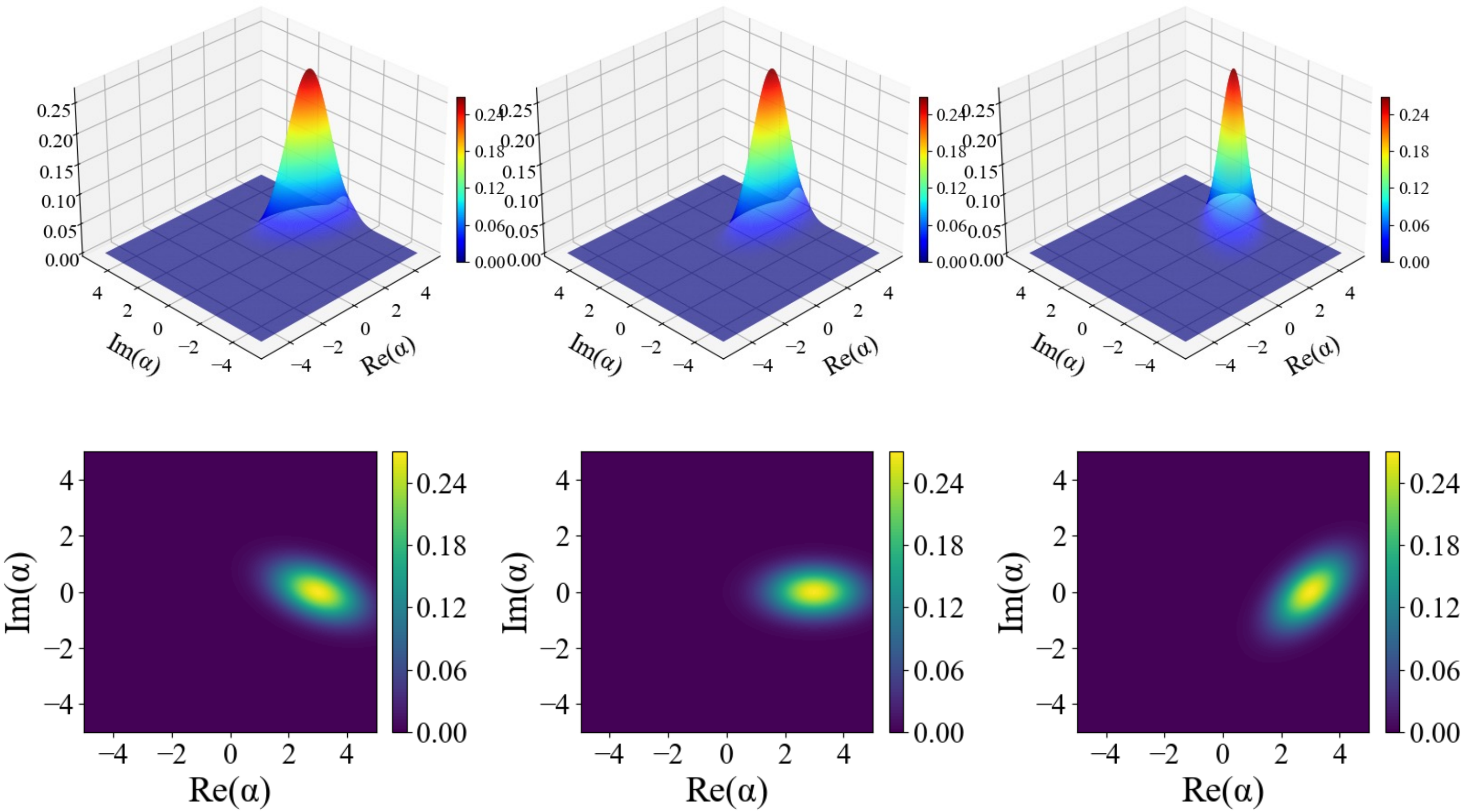


压缩态$|\beta,\xi\rangle$的Q-rep

$$|\beta,\xi\rangle = \mathcal{D}(\beta)\mathcal{S}(\xi)|0\rangle$$

$$\beta = 3, r = 0.6 \qquad \xi = re^{i\theta}$$

$\theta = 3\pi/4$ $\qquad$ $\theta = \pi$ $\qquad$ $\theta = 3\pi/2$



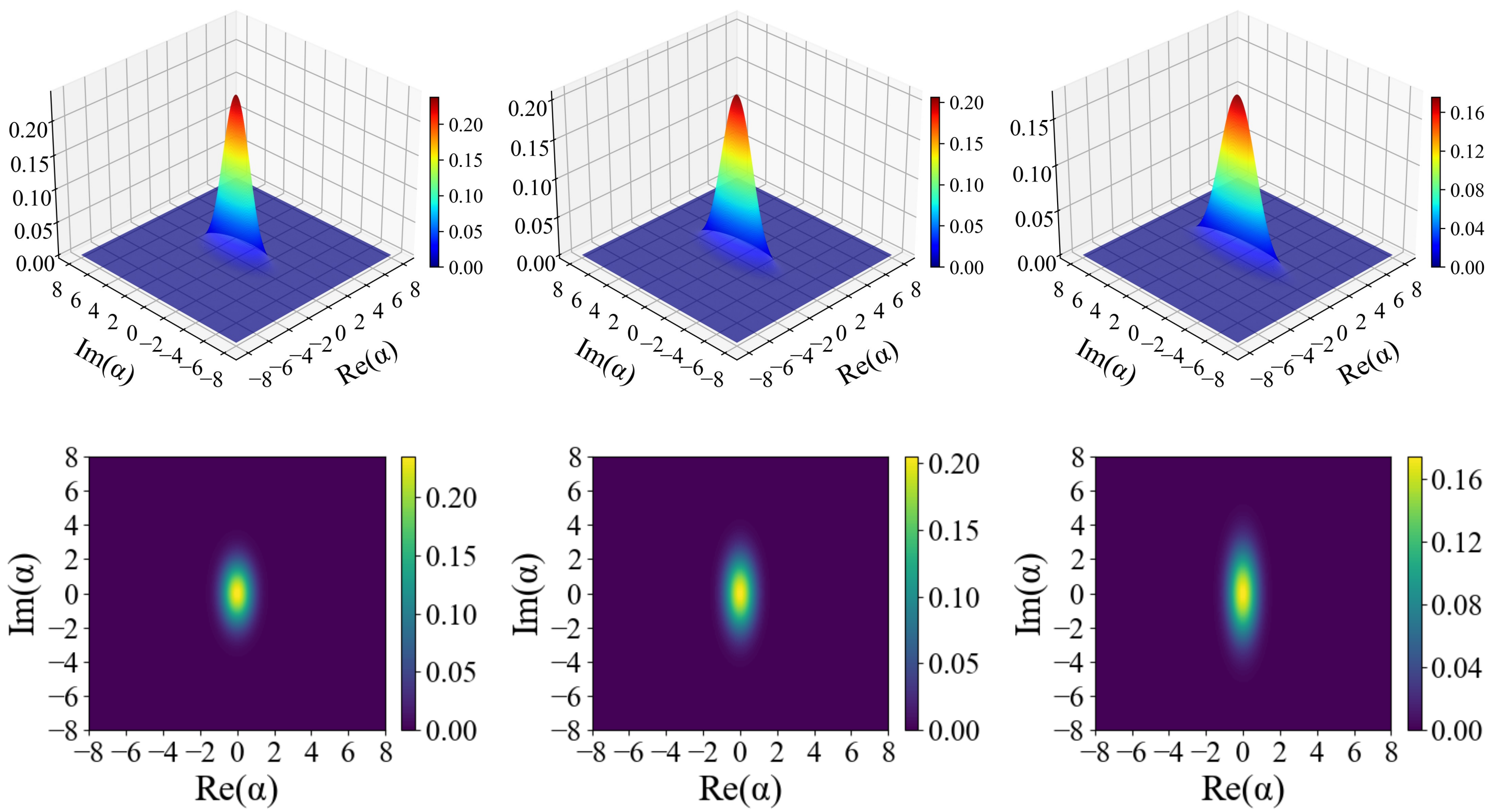
压缩态真空态$|0,\xi\rangle$的Q-rep

$$\beta = 0, \theta = 0$$

$$|\beta,\xi\rangle = \mathcal{D}(\beta)\mathcal{S}(\xi)|0\rangle$$

$$\xi = re^{i\theta}$$

$r = 0.8$ $\quad$ $r = 1$ $\quad$ $r = 1.2$



胡晓东 拍摄

三、 Wigner 函数(W-表示)

1. 定义： Wigner函数是由Eugene P. Wigner 在1932年提出来的，对于给定的密度矩阵ρ，定义为

$$W(x,p)=\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}d\xi\, exp\left(-\frac{i}{\hbar}p\xi\right)\left\langle x+\frac{1}{2}\xi\middle|\rho\middle|x-\frac{1}{2}\xi\right\rangle$$

看成是：“*质心 x 处、相距为 ξ 的两点$x+\frac{1}{2}\xi$和$x-\frac{1}{2}\xi$间的跃迁矩阵元，然后对其做傅立叶变换*”、“*系统的密度矩阵在相空间中的准概率分布*”

略去若干推导步骤。。。。。。

一般情况下，计算使用的是Wigner函数的等价积分形式

$$\overline{W}(\alpha,\alpha^*)=\frac{1}{\pi^2}\int d^2\lambda\, Tr\left(\rho e^{\lambda a^\dagger-\lambda^* a}\right)exp(-\lambda\alpha^*+\lambda^*\alpha)$$

$$\overline{W}(\alpha,\alpha^*)=\frac{1}{\pi^2}\int d^2\lambda\, Tr\left(\rho e^{\lambda a^\dagger-\lambda^* a}\right) exp(-\lambda\alpha^*+\lambda^*\alpha)$$

2. 三种表示的统一形式

P-表示、Q-表示、W-表示是以下三种特征函数的傅里叶变换

P-表示： $\chi_N(\lambda)=Tr(\rho e^{\lambda a^\dagger}e^{-\lambda^* a})$ **正规算符排序**

Q-表示： $\chi_A(\lambda)=Tr(\rho e^{-\lambda^* a}e^{\lambda a^\dagger})$ **反正规算符排序**

W-表示： $\chi_W(\lambda)=Tr(\rho e^{\lambda a^\dagger-\lambda^* a})$ **对称算符排序**

3.W-表示和P-表示之间的关系， $D(\lambda)=e^{\lambda a^{\dagger}-\lambda^* a}=e^{-\frac{1}{2}|\lambda|^2}e^{\lambda a^{\dagger}}e^{-\lambda^* a}$

P-表示： $\rho=\int P(\beta)|\beta\rangle\langle\beta|d^2\beta$

W-表示： $\overline{W}(\alpha,\alpha^*)=\frac{1}{\pi^2}\int d^2\lambda\, Tr(\rho D(\lambda))exp(-\lambda\alpha^*+\lambda^*\alpha)$

$$=\frac{1}{\pi^2}\int d^2\lambda\, Tr\left(\int P(\beta)|\beta\rangle\langle\beta|d^2\beta\, D(\lambda)\right)exp(-\lambda\alpha^*+\lambda^*\alpha)$$

$$=\frac{1}{\pi^2}\int d^2\beta\int d^2\lambda\,\sum_n\langle n|P(\beta)|\beta\rangle\,\langle\beta|D(\lambda)|n\rangle\, exp(-\lambda\alpha^*+\lambda^*\alpha)$$

$$=\frac{1}{\pi^2}\int d^2\beta\int d^2\lambda\,\sum_n\left\langle\beta\left|e^{\lambda a^{\dagger}-\lambda^* a}\right|n\right\rangle\langle n|P(\beta)|\beta\rangle\, exp(-\lambda\alpha^*+\lambda^*\alpha)$$

$$=\frac{1}{\pi^2}\int d^2\beta\int d^2\lambda\left\langle\beta\left|P(\beta)e^{\lambda a^{\dagger}-\lambda^* a}\right|\beta\right\rangle\, exp(-\lambda\alpha^*+\lambda^*\alpha)$$

$$D(\lambda) = e^{\lambda a^\dagger - \lambda^* a} = e^{-\frac{1}{2}|\lambda|^2} e^{\lambda a^\dagger} e^{-\lambda^* a}$$

$$= \frac{1}{\pi^2} \int d^2\beta . P(\beta) \int d^2\lambda \left\langle \beta \middle| e^{-\frac{1}{2}|\lambda|^2} e^{\lambda a^\dagger} e^{-\lambda^* a} \middle| \beta \right\rangle exp(-\lambda\alpha^* + \lambda^*\alpha)$$

$$= \frac{1}{\pi^2} \int d^2\beta . P(\beta) \int d^2\lambda e^{-\frac{1}{2}|\lambda|^2} e^{\lambda\beta^*} e^{-\lambda^*\beta} exp(-\lambda\alpha^* + \lambda^*\alpha)$$

$$= \frac{1}{\pi^2} \int d^2\beta . P(\beta) \int d^2\lambda \ e^{-\frac{1}{2}|\lambda|^2} e^{\lambda(\beta^* - \alpha^*)} e^{-\lambda^*(\beta - \alpha)}$$

$$= \frac{2}{\pi} \int P(\beta)\, e^{-2|\alpha-\beta|^2} d^2\beta$$

在这里，用到了积分关系式 $\frac{1}{\pi}\int d^2\lambda\, e^{-h|\lambda|^2} e^{-\lambda\alpha^* + \lambda^*\alpha} = \frac{1}{h} e^{-|\alpha|^2/h}$

4. 量子态的Wigner函数解析形式

$$D(\lambda) = e^{\lambda a^\dagger - \lambda^* a} = e^{-\frac{1}{2}|\lambda|^2} e^{\lambda a^\dagger} e^{-\lambda^* a}$$

$$\frac{1}{\pi}\int d^2\lambda \, e^{-h|\lambda|^2} e^{-\lambda\alpha^* + \lambda^*\alpha} = \frac{1}{h} e^{-|\alpha|^2/h}$$

- **相干态** $\rho = |\alpha_0\rangle\langle\alpha_0|$

$$\bar{W}(\alpha,\alpha^*) = \frac{1}{\pi^2}\int d^2\lambda \, Tr(\rho e^{\lambda a^\dagger - \lambda^* a}) exp(-\lambda\alpha^* + \lambda^*\alpha)$$

$$= \frac{1}{\pi^2}\int d^2\lambda \left\langle \alpha_0 \middle| e^{\lambda a^\dagger - \lambda^* a} \middle| \alpha_0 \right\rangle exp(-\lambda\alpha^* + \lambda^*\alpha)$$

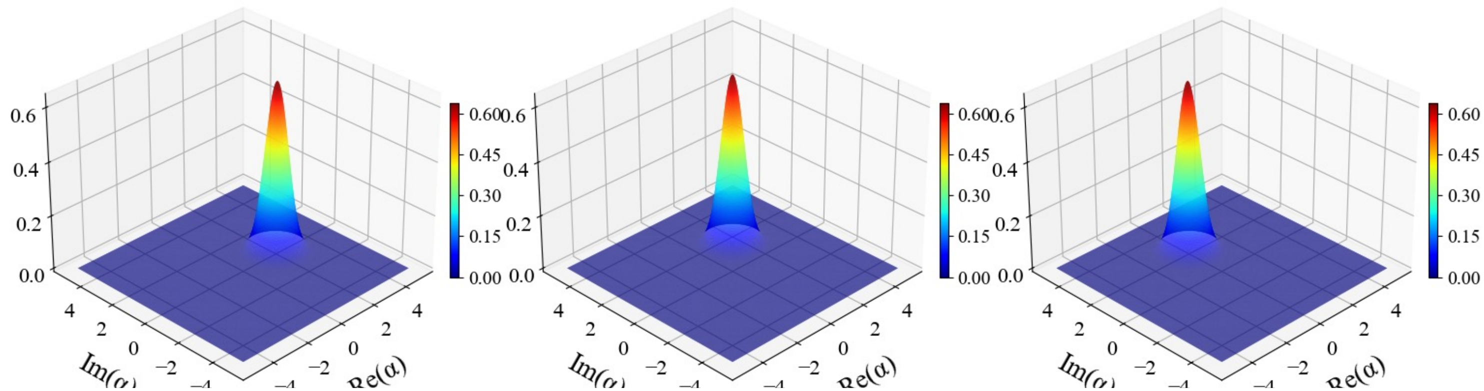
$$= \frac{1}{\pi^2}\int d^2\lambda \, e^{-|\lambda|^2/2} e^{-\lambda^*\alpha_0} e^{\lambda\alpha_0^*} \, exp(-\lambda\alpha^* + \lambda^*\alpha)$$

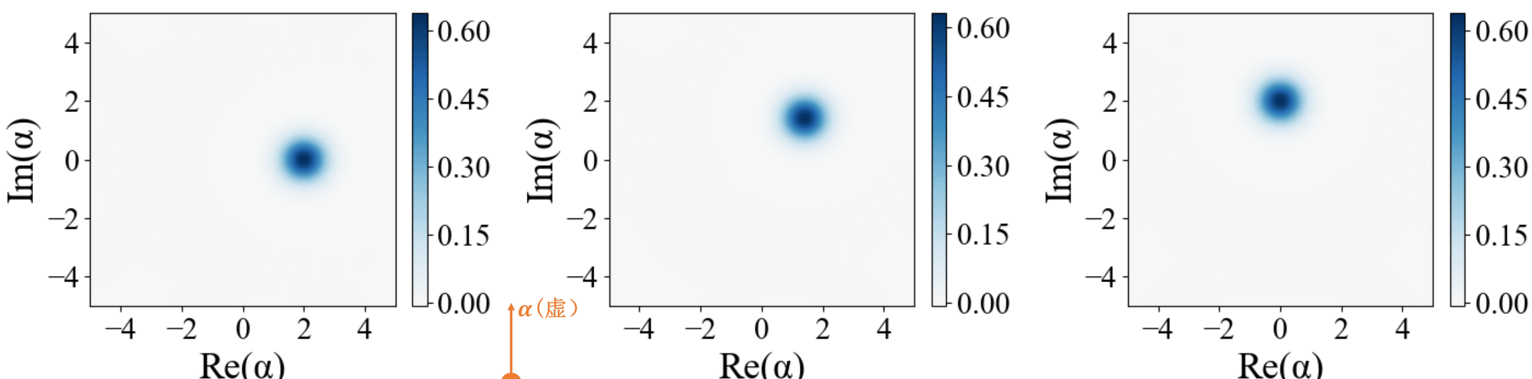
$$= \frac{2}{\pi} e^{-2|\alpha-\alpha_0|^2}$$

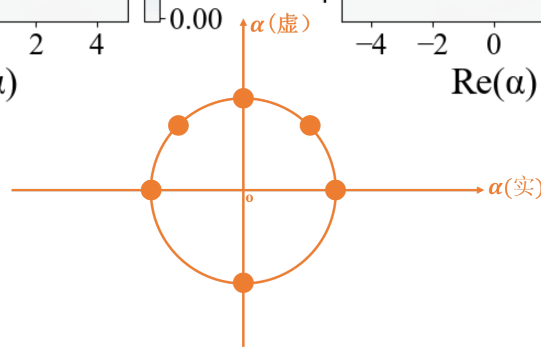
也可以直接利用P-表示和W-表示之间的关系得到（见上页）

● 相干态Wigner表示 $|\alpha_0\rangle$ $\alpha_0 = 2\, e^{i\theta}$ $W(\alpha, \alpha^*) = \frac{2}{\pi} e^{-2|\alpha-\alpha_0|^2}$

$\theta = 0$ $\theta = \pi/4$ $\theta = \pi/2$







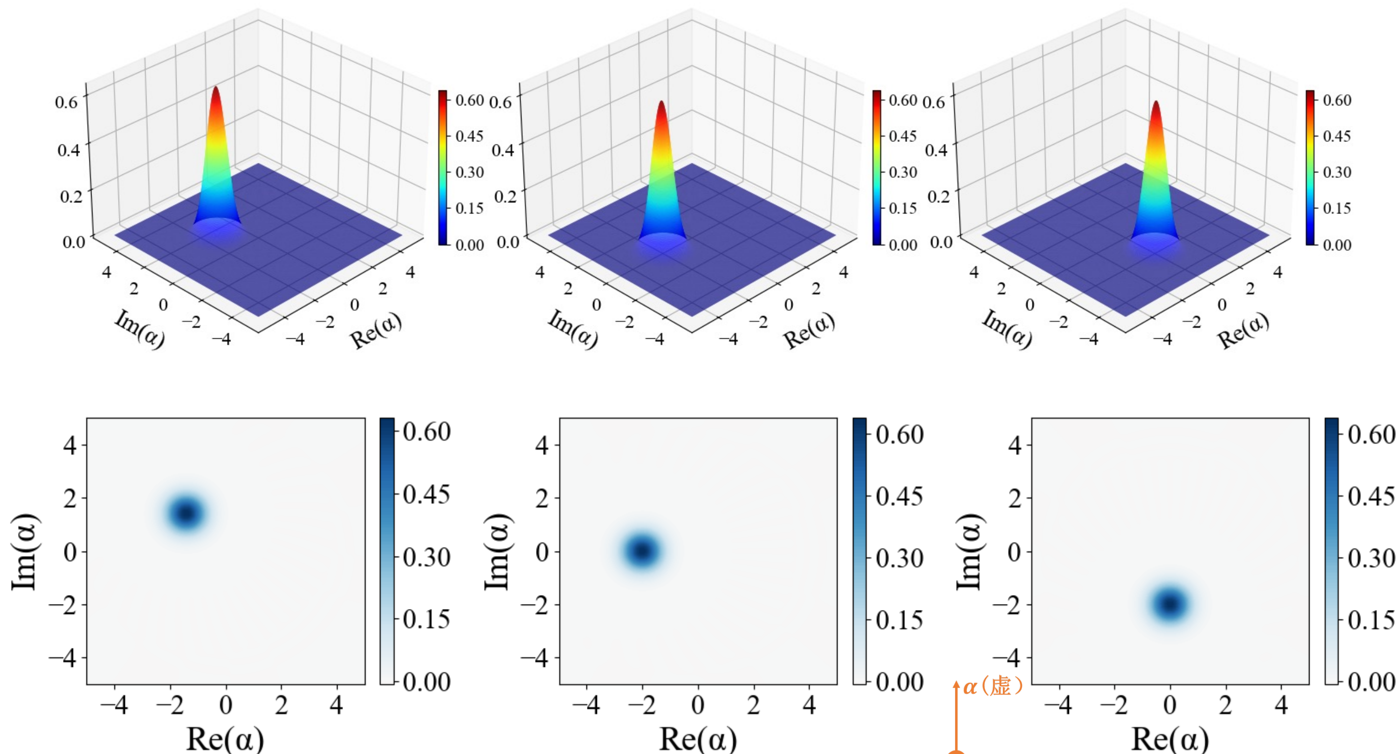
● 相干态Wigner表示

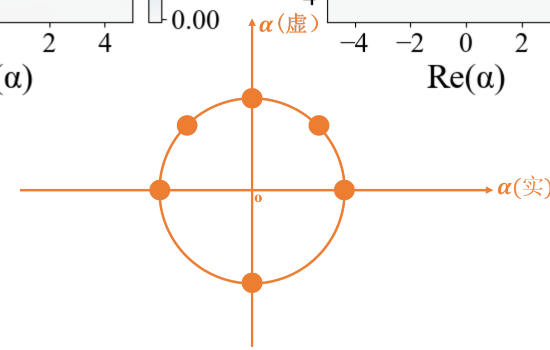
$|\alpha_0\rangle$

$\alpha_0 = 2\, e^{i\theta}$

$$W(\alpha, \alpha^*) = \frac{2}{\pi} e^{-2|\alpha - \alpha_0|^2}$$

$\theta = 3\pi/4$　　　　$\theta = \pi$　　　　$\theta = 3\pi/2$





$$\bar{W}(\alpha,\alpha^*) = \frac{1}{\pi^2}\int d^2\lambda\, Tr\left(\rho e^{\lambda a^\dagger - \lambda^* a}\right) exp(-\lambda\alpha^* + \lambda^*\alpha)$$

● 猫态Wigner表示

$|\psi\rangle = N\left(|\alpha_0\rangle + e^{i\phi}|-\alpha_0\rangle\right)$ N为归一化常数

$$\rho = |\psi\rangle\langle\psi|$$

$$= N^2\left(|\alpha_0\rangle + e^{i\phi}|-\alpha_0\rangle\right)\left(\langle\alpha_0| + \langle-\alpha_0|e^{-i\phi}\right)$$

$$= N^2\left(|\alpha_0\rangle\langle\alpha_0| + e^{-i\phi}|\alpha_0\rangle\langle-\alpha_0| + e^{i\phi}|-\alpha_0\rangle\langle\alpha_0| + |-\alpha_0\rangle\langle-\alpha_0|\right)$$

相干态Wigner函数已知，只需要求解$e^{-i\phi}|\alpha_0\rangle\langle-\alpha_0|$项的Wigner函数

$$W(\alpha) = \frac{1}{\pi^2}\int d^2\lambda\, Tr[|\alpha_0\rangle\langle-\alpha_0|D(\lambda)]exp(-\lambda\alpha^* + \lambda^*\alpha)$$

$$= \frac{1}{\pi^2}\int d^2\lambda \left\langle-\alpha_0\middle|e^{\lambda a^\dagger - \lambda^* a}\middle|\alpha_0\right\rangle exp(-\lambda\alpha^* + \lambda^*\alpha)$$

$$= \frac{1}{\pi^2}\langle-\alpha_0|\alpha_0\rangle\int d^2\lambda e^{-\lambda\alpha_0^*}e^{-\lambda^*\alpha_0}e^{-\frac{|\lambda|^2}{2}} exp(-\lambda\alpha^* + \lambda^*\alpha)$$

$$\langle-\alpha_0|\alpha_0\rangle = e^{-\frac{1}{2}|\alpha_0|^2 - \frac{1}{2}|-\alpha_0|^2 + (-\alpha_0^*\alpha_0)} = e^{-2|\alpha_0|^2}$$

$$= \frac{1}{\pi^2}\int d^2\lambda\, e^{-2|\alpha_0|^2}e^{-\lambda\alpha_0^*}e^{-\lambda^*\alpha_0}e^{-\frac{|\lambda|^2}{2}} exp(-\lambda\alpha^* + \lambda^*\alpha)$$

- **猫态**

$$\frac{1}{\pi}\int d^2\lambda\, e^{-\gamma|\lambda|^2} e^{-\lambda\alpha^*+\lambda^*\alpha} = \frac{1}{\gamma} e^{-|\alpha|^2/\gamma}$$

$$= \frac{1}{\pi^2}\int d^2\lambda\, e^{-\frac{1}{2}(\lambda+2\alpha_0)^2-(\lambda+2\alpha_0)\alpha^*+(\lambda^*+2\alpha_0^*)\alpha} e^{-2\alpha_0\alpha^*+2\alpha_0^*\alpha}$$

积分公式中

$$h = \frac{1}{2}$$

$$\lambda = \lambda + 2\alpha_0$$

$$= \frac{2}{\pi} e^{-2|\alpha|^2} e^{-2\alpha_0\alpha^*+2\alpha_0^*\alpha}$$

同理可以得到$|-\alpha_0\rangle\langle\alpha_0|$项的Wigner函数为$\frac{2}{\pi} e^{-2|\alpha|^2} e^{2\alpha_0\alpha^*-2\alpha_0^*\alpha}$

因此，猫态的Wigner函数的解析形式为（自己推一下）

$W_{cat}(\alpha)$

$$= N^2[\frac{2}{\pi} e^{-2|\alpha-\alpha_0|^2} + \frac{2e^{-i\phi}}{\pi} e^{-2|\alpha|^2} e^{-2\alpha_0\alpha^*+2\alpha_0^*\alpha} + \frac{2e^{i\phi}}{\pi} e^{-2|\alpha|^2} e^{2\alpha_0\alpha^*-2\alpha_0^*\alpha} + \frac{2}{\pi} e^{-2|\alpha+\alpha_0|^2}]$$

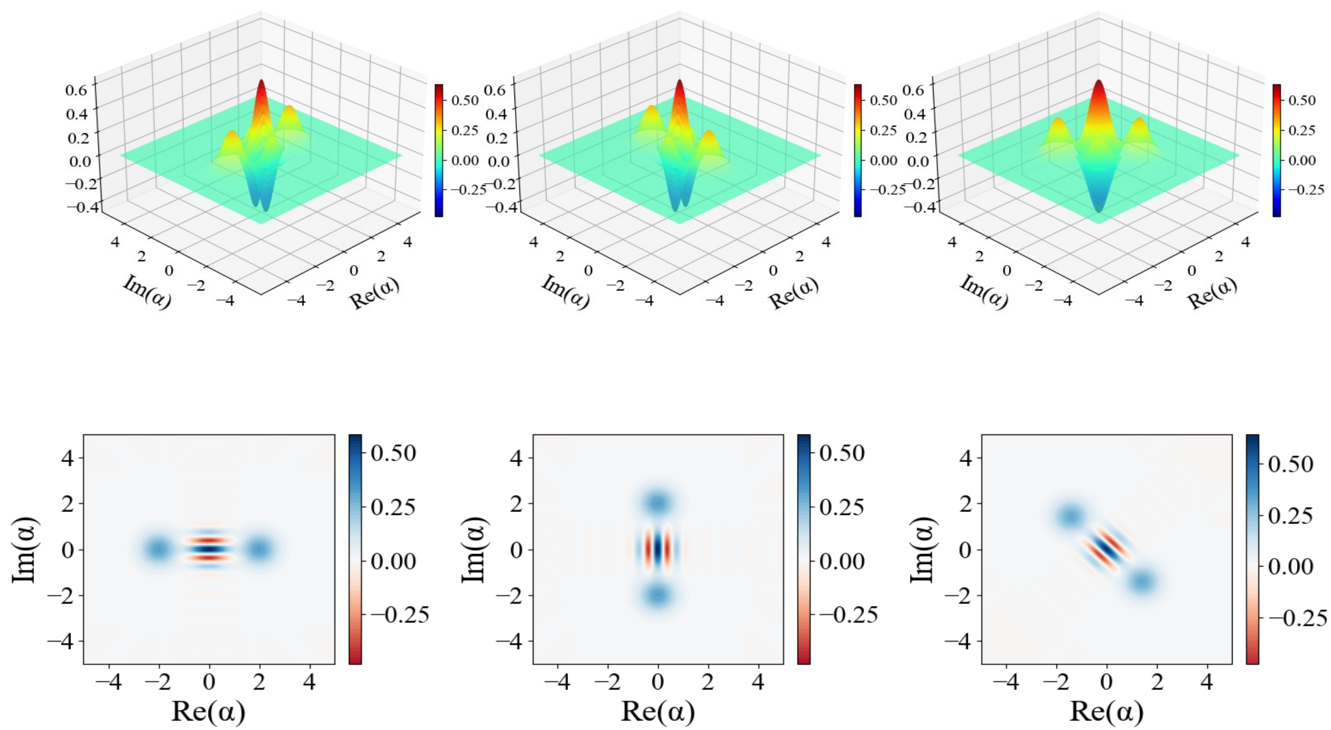
$$= \frac{2N^2}{\pi}[e^{-2|\alpha-\alpha_0|^2} + e^{-2|\alpha+\alpha_0|^2} + 2\, e^{-2|\alpha|^2}(e^{-i\phi} e^{-2\alpha_0\alpha^*+2\alpha_0^*\alpha} + e^{i\phi} e^{2\alpha_0\alpha^*-2\alpha_0^*\alpha}]$$

偶猫态的Wigner分布

$\alpha_0 = 2\, e^{i\theta}$ $\qquad$ $e^{i\phi} = 1$

$$Q_{cat}(\alpha) = \frac{2N^2}{\pi}\left[e^{-2|\alpha-\alpha_0|^2} + e^{-2|\alpha+\alpha_0|^2} + 2\, e^{-2|\alpha|^2}\left(e^{-2\alpha_0\alpha^* + 2\alpha_0^*\alpha} + e^{2\alpha_0\alpha^* - 2\alpha_0^*\alpha}\right)\right]$$

$\theta = 0$ $\qquad$ $\theta = \pi/2$ $\qquad$ $\theta = 3\pi/4$

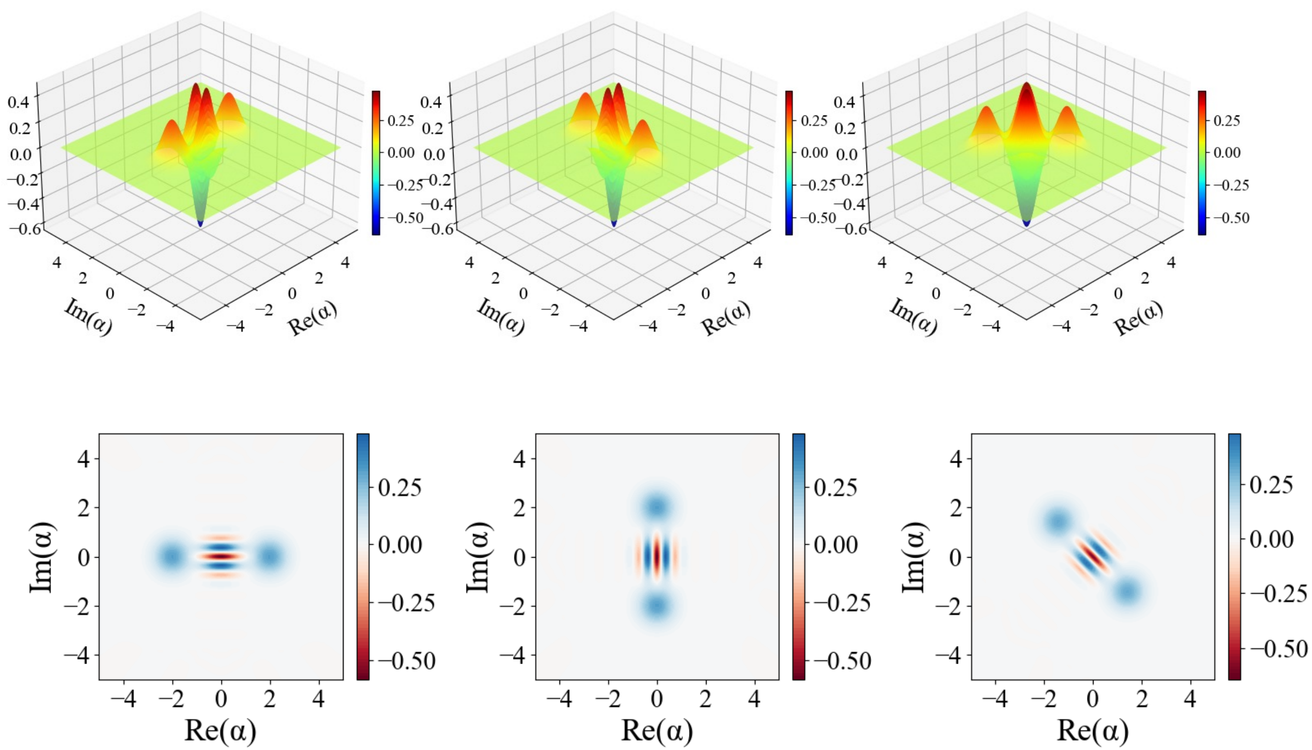


奇猫态的Wigner-分布

$\alpha_0 = 2\,e^{i\theta}$　　　$e^{i\phi} = -1$

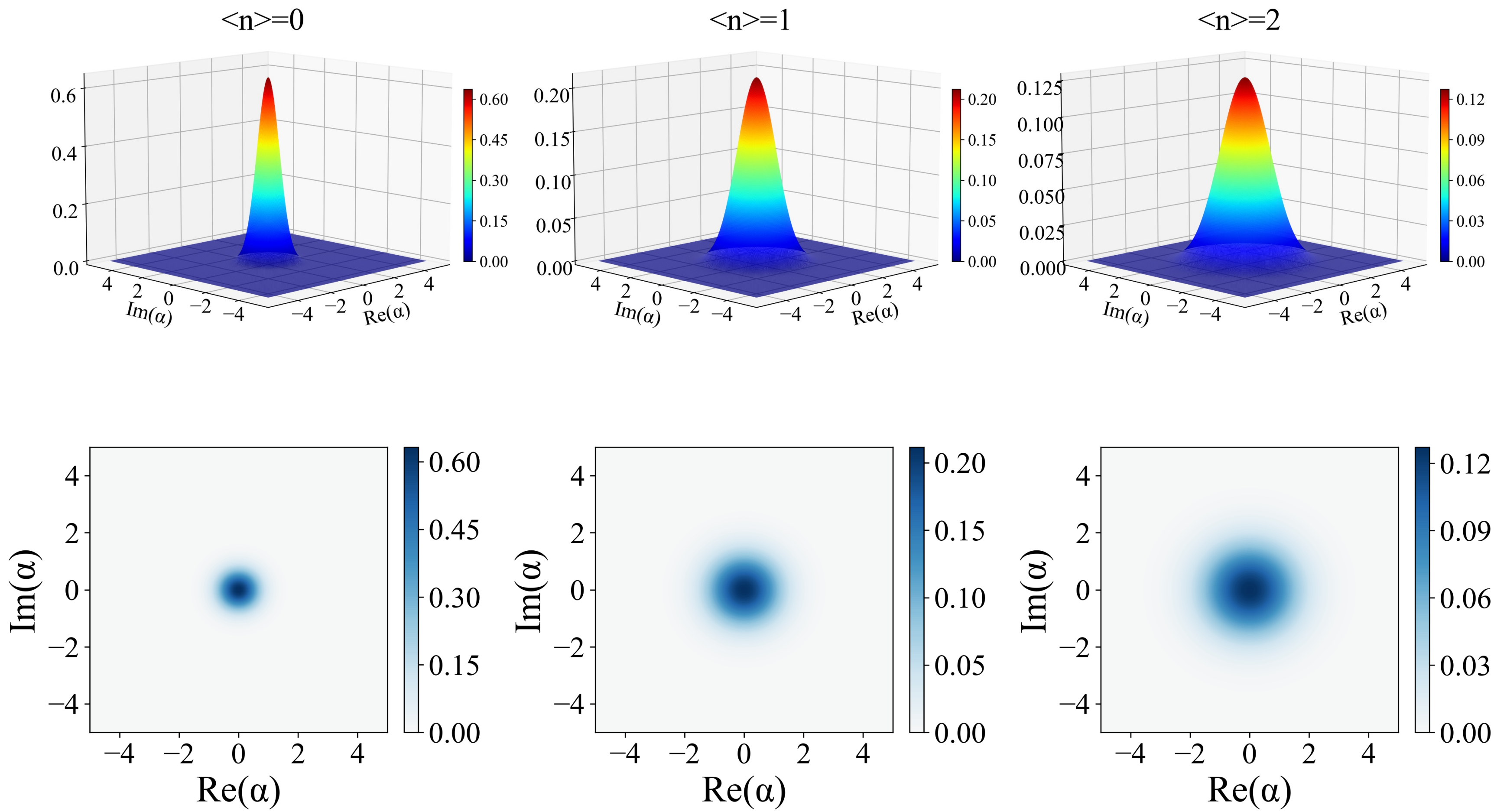
$$W_{cat}(\alpha) = \frac{2N^2}{\pi}\left[e^{-2|\alpha-\alpha_0|^2} + e^{-2|\alpha+\alpha_0|^2} - 2\,e^{-2|\alpha|^2}(e^{-2\alpha_0\alpha^*+2\alpha_0^*\alpha} + e^{2\alpha_0\alpha^*-2\alpha_0^*\alpha}\right]$$

$\theta = 0$　　　$\theta = \pi/2$　　　$\theta = 3\pi/4$



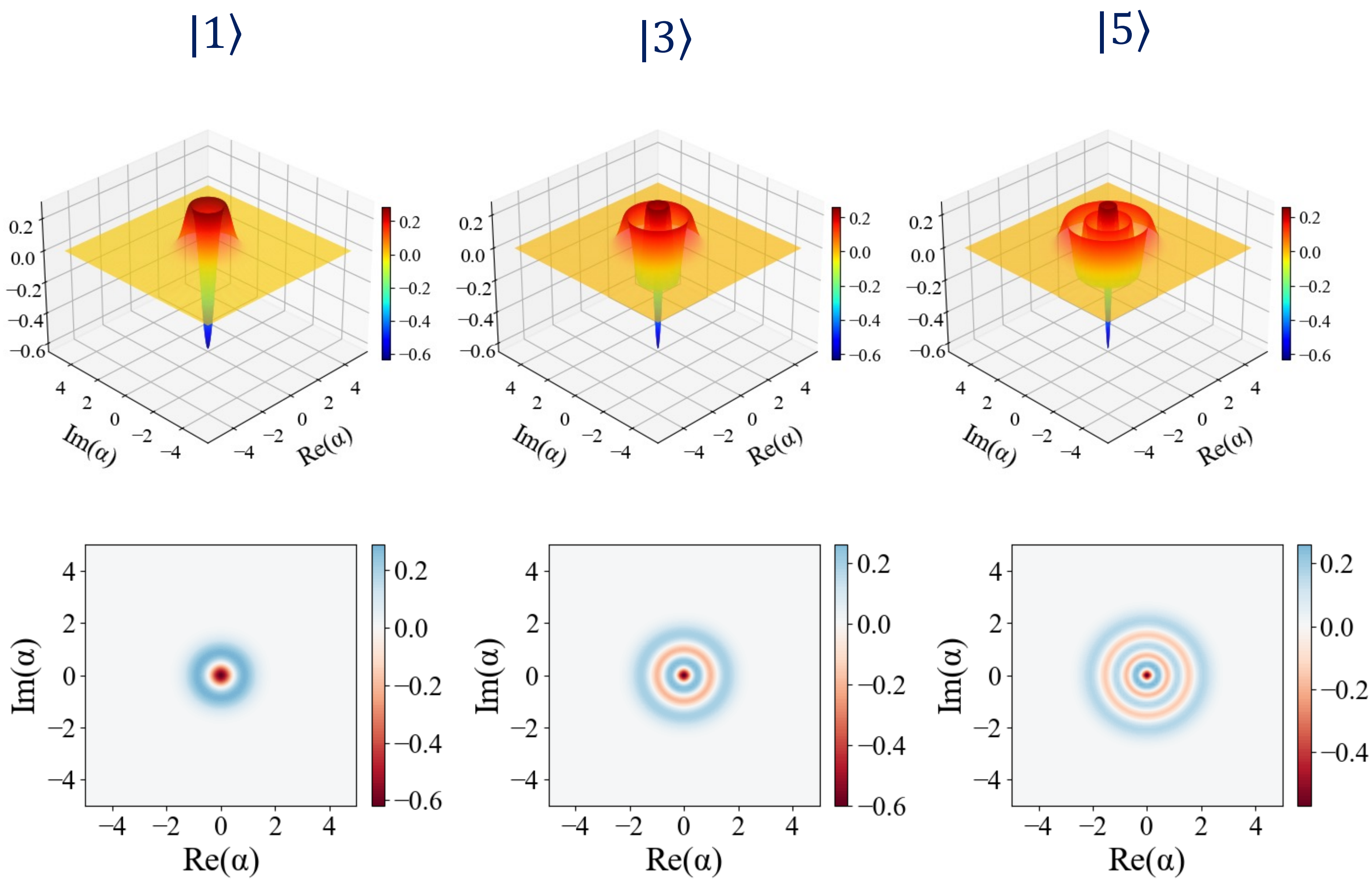
热光场态的Wigner分布

$$\bar{W}(\alpha,\alpha^*) = \frac{1}{\pi\left(\langle n\rangle+\frac{1}{2}\right)} exp[-\frac{|\alpha|^2}{\langle n\rangle+\frac{1}{2}}]$$



● Fock态 $|n\rangle$的Wigner表示

$$W_m(\alpha) = \frac{2(-1)^m}{\pi} exp(-2|\alpha|^2) L_m(4|\alpha|^2)$$



●压缩态的Wigner分布

$$\xi = re^{i\theta}$$

$$\rho = |\beta,\xi\rangle\langle\beta,\xi| \qquad |\beta,\xi\rangle = D(\beta)S(\xi)|0\rangle$$

$$\bar{W}(\alpha,\alpha^*) = \frac{1}{\pi^2}\int d^2\lambda\, Tr[\rho D(\lambda)]exp(-\lambda\alpha^* + \lambda^*\alpha)$$

$$= \frac{1}{\pi^2}\int d^2\lambda \left\langle 0\right| S^\dagger(\xi)D^\dagger(\beta)D(\lambda)D(\beta)S(\xi)\left|0\right\rangle exp(-\lambda\alpha^* + \lambda^*\alpha)$$

$$D^\dagger(\beta)D(\lambda)D(\beta) = D^\dagger(\beta)e^{\lambda a^\dagger - \lambda^* a}D(\beta) = e^{\lambda(a^\dagger+\beta^*)-\lambda^*(a+\beta)} = e^{\lambda a^\dagger-\lambda^* a}e^{\lambda\beta^*-\lambda^*\beta}$$

$$= D(\lambda)exp(\lambda\beta^* - \lambda^*\beta)$$

$$S^\dagger(\xi)D(\lambda)S(\xi) = S^\dagger(\xi)e^{\lambda a^\dagger-\lambda^* a}S(\xi) = e^{\lambda[S^\dagger(\xi)a^\dagger S(\xi)]-\lambda^*[S^\dagger(\xi)aS(\xi)]}$$

$$= e^{\lambda(a^\dagger \cosh r - ae^{-i\theta}\sinh r)-\lambda^*(a\cosh r - a^\dagger e^{i\theta}\sinh r)}$$

$$= e^{a^\dagger(\lambda\cosh r+\lambda^* e^{i\theta}\sinh r)-a(\lambda^*\cosh r+\lambda e^{-i\theta}\sinh r)}$$

$$= D\left(\lambda\cosh r + \lambda^* e^{i\theta}\sinh r\right)$$

$$\bar{W}(\alpha) = \frac{1}{\pi^2}\int d^2\lambda \left\langle 0\right|D(\lambda\cosh r + \lambda^* e^{i\theta}\sinh r)\left|0\right\rangle e^{-\lambda(\alpha^*-\beta^*)+\lambda^*(\alpha-\beta)}$$

令$\chi = \lambda \cosh r + \lambda^* e^{i\theta} \sinh r \qquad \lambda = \chi \cosh r - \chi^* e^{i\theta} \sinh r$

$$\bar{W}(\alpha) = \frac{1}{\pi^2}\int d^2\lambda \left\langle 0\middle|D(\lambda \cosh r + \lambda^* e^{i\theta} \sinh r)\middle|0\right\rangle e^{-\lambda(\alpha^*-\beta^*)+\lambda^*(\alpha-\beta)}$$

$$= \frac{1}{\pi^2}\int d^2\chi \langle 0|D(\chi)|0\rangle e^{-(\chi \cosh r-\chi^* e^{i\theta}\sinh r)(\alpha^*-\beta^*)+(\chi^* \cosh r-\chi e^{-i\theta}\sinh r)(\alpha-\beta)}$$

$$= \frac{1}{\pi^2}\int d^2\chi\langle 0|D(\chi)|0\rangle e^{-\chi[(\alpha-\beta)e^{-i\theta}\sinh r+(\alpha^*-\beta^*)\cosh r]+\chi^*[(\alpha^*-\beta^*)e^{i\theta}\sinh r+(\alpha-\beta)\cosh r]}$$

$$= \frac{1}{\pi^2}\int d^2\chi \langle 0|D(\chi)|0\rangle e^{-\chi A^*+\chi^* A}$$

$$= \frac{1}{\pi^2}\int d^2\chi \left\langle 0\middle|e^{-\frac{|\chi|^2}{2}} e^{\chi a^\dagger} e^{-\chi^* a}\middle|0\right\rangle e^{-\chi A^*+\chi^* A}$$

$$= \frac{1}{\pi^2}\int d^2\chi\, e^{-\frac{|\chi|^2}{2}} e^{-\chi A^*+\chi^* A} = \frac{2}{\pi} e^{-2|\chi|^2}$$

$$\frac{1}{\pi}\int d^2\lambda\, e^{-h|\lambda|^2} e^{-\lambda\alpha^*+\lambda^*\alpha} = \frac{1}{h} e^{-|\alpha|^2/h}$$

其中$A = (\alpha^*-\beta^*)e^{i\theta}\sinh r + (\alpha - \beta)\cosh r$

于是得到Wigner函数表达形式为

$$\bar{W}(\alpha) = \frac{2}{\pi} e^{-2|(\alpha^*-\beta^*)e^{i\theta}\sinh r+(\alpha-\beta)\cosh r\,|^2}$$

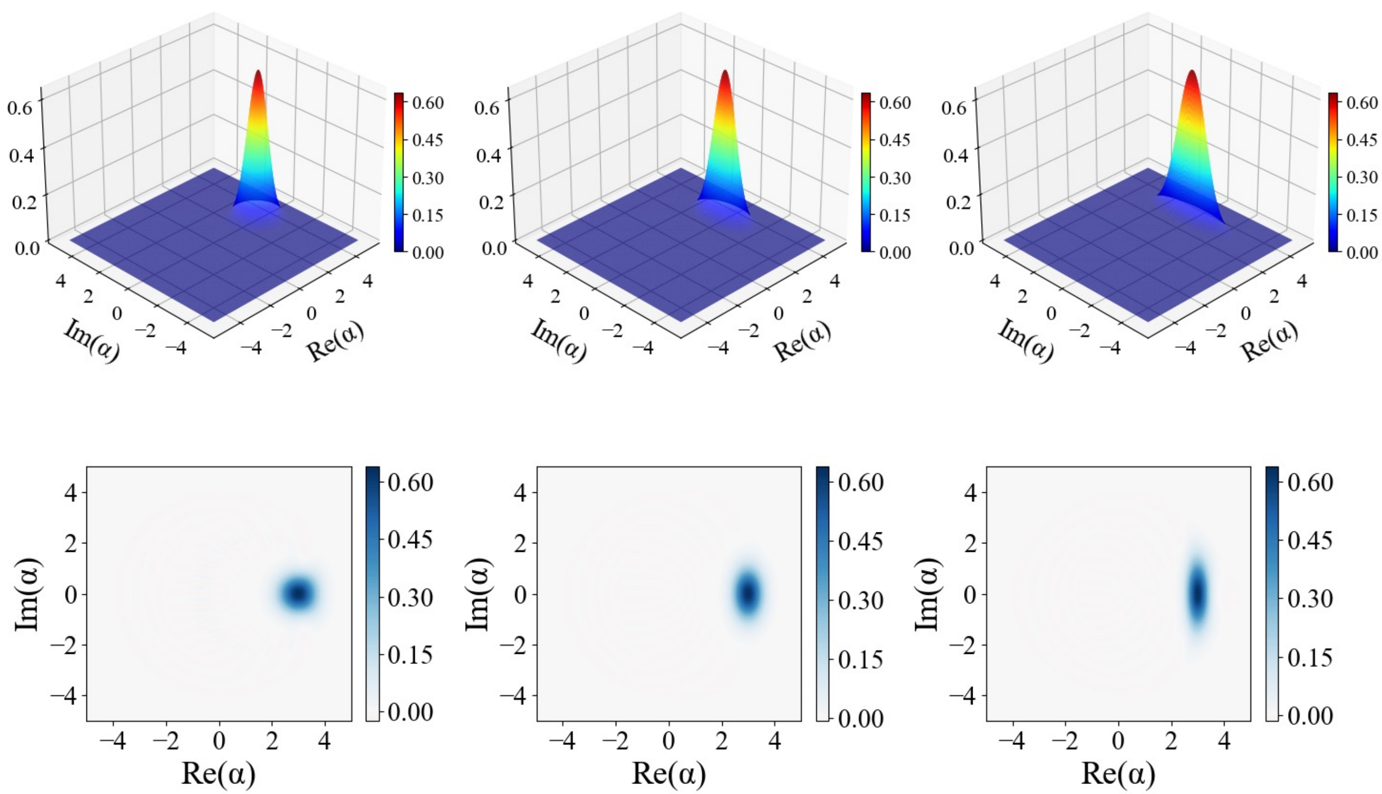
● 压缩态$|\beta,\xi\rangle$的Wigner分布

$$\bar{W}(\alpha) = \frac{2}{\pi} e^{-2|(\alpha^*-\beta^*)e^{i\theta}\sinh r + (\alpha-\beta)\cosh r|^2}$$

$$|\beta,\xi\rangle = \mathcal{D}(\beta)\mathcal{S}(\xi)|0\rangle, \quad \xi = re^{i\theta}$$

$$\beta = 3, \theta = 0$$

$r = 0.0$ $\quad$ $r = 0.3$ $\quad$ $r = 0.6$



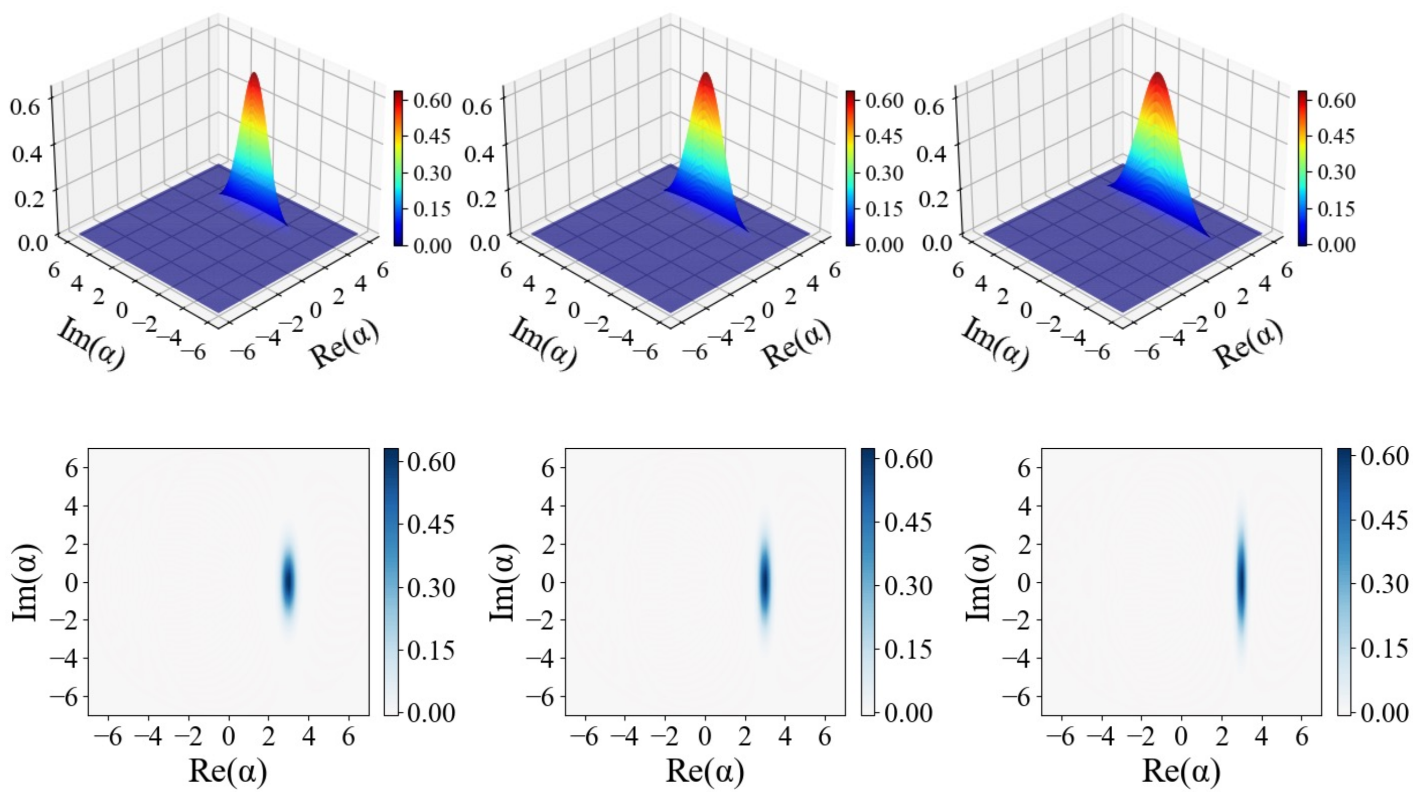
压缩态$|\beta,\xi\rangle$的Wigner分布

$$\beta = 3, \theta = 0$$

$$|\beta,\xi\rangle = \mathcal{D}(\beta)\mathcal{S}(\xi)|0\rangle$$

$$\xi = re^{i\theta}$$

$r = 0.8$ $\quad$ $r = 1$ $\quad$ $r = 1.2$

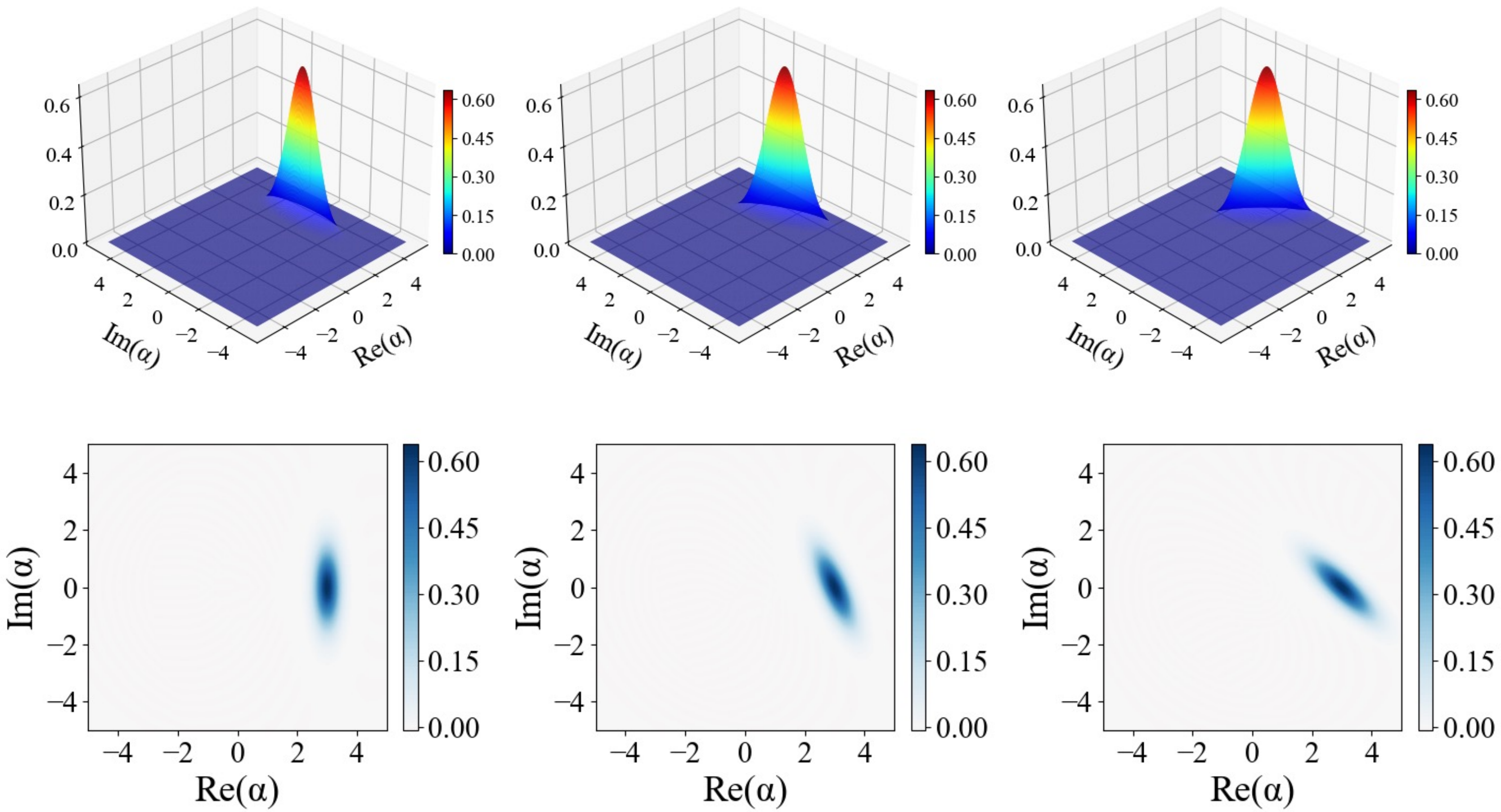


压缩态$|\beta,\xi\rangle$的Wigner分布

$$|\beta,\xi\rangle = \mathcal{D}(\beta)\mathcal{S}(\xi)|0\rangle$$

$$\beta = 3, r = 0.6 \qquad \xi = re^{i\theta}$$

$\theta = 0$ $\quad$ $\theta = \pi/4$ $\quad$ $\theta = \pi/2$



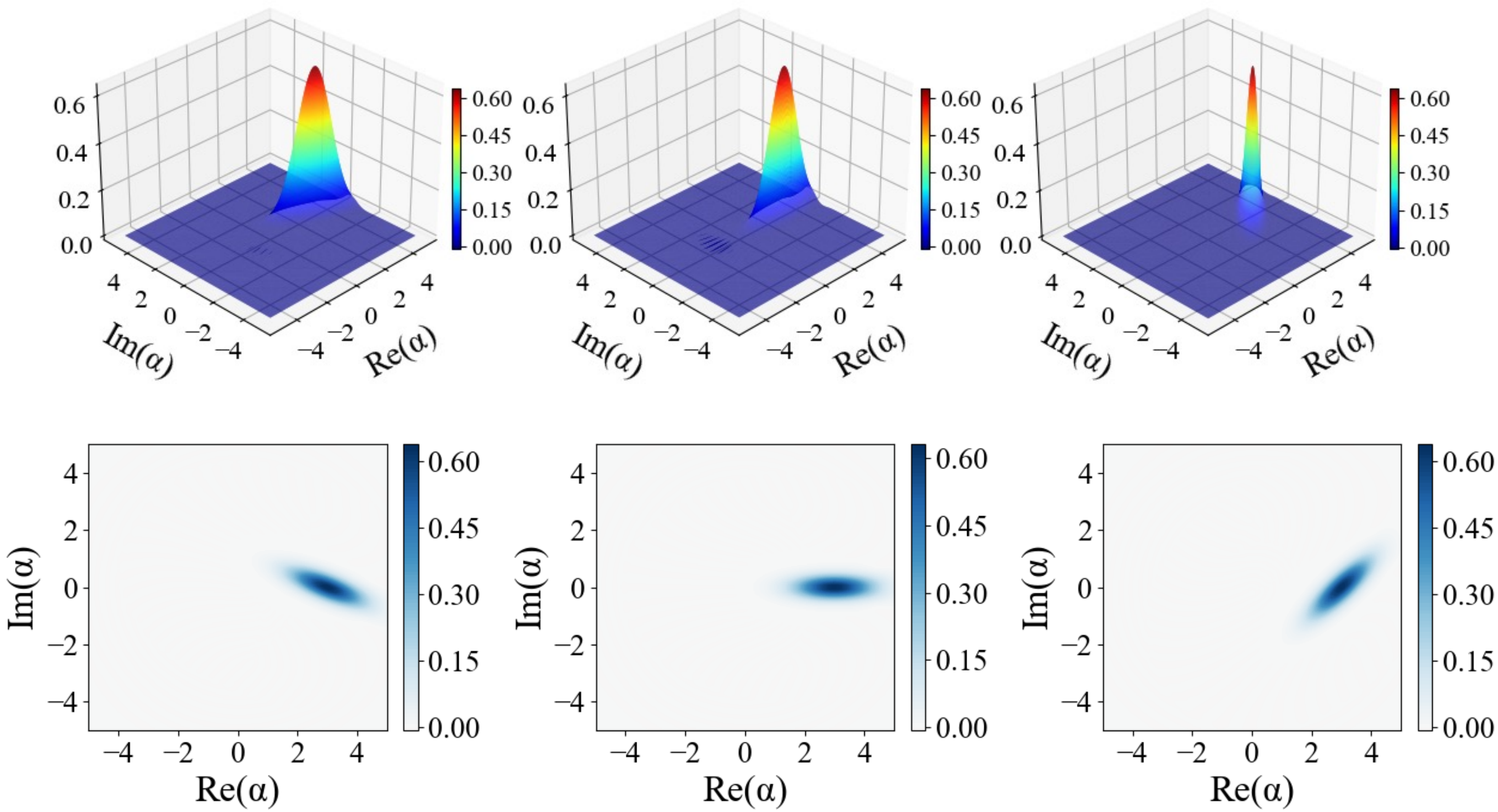
压缩态$|\beta, \xi\rangle$的Wigner分布

$\beta = 3, r = 0.6$

$$|\beta, \xi\rangle = \mathcal{D}(\beta)\mathcal{S}(\xi)|0\rangle$$

$$\xi = re^{i\theta}$$

$\theta = 3\pi/4$ $\quad$ $\theta = \pi$ $\quad$ $\theta = 3\pi/2$



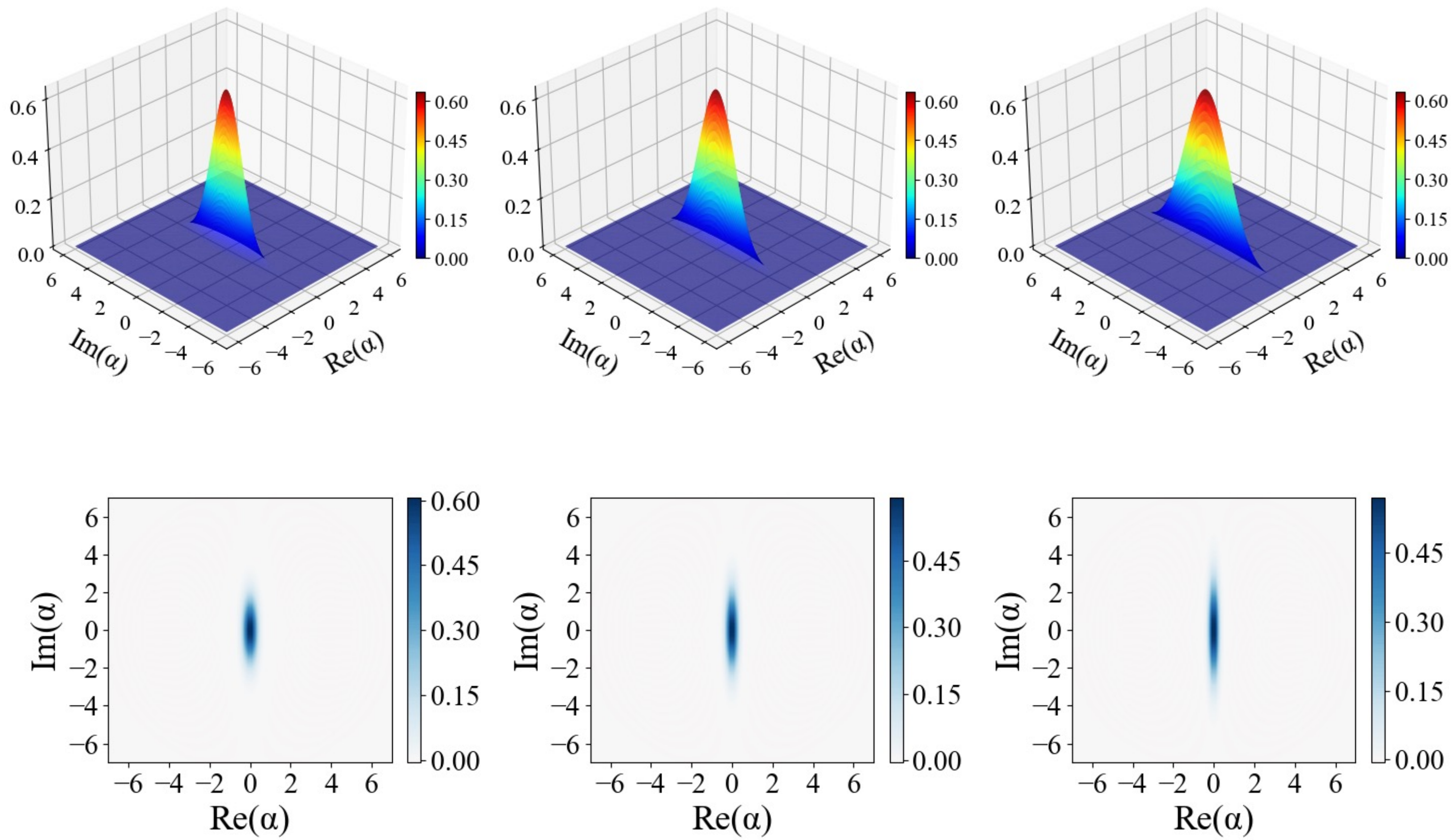
压缩态真空态$|0,\xi\rangle$的Wigner分布

$$|\beta,\xi\rangle = \mathcal{D}(\beta)\mathcal{S}(\xi)|0\rangle$$

$$\xi = re^{i\theta}$$

$$\beta = 0, \theta = \pi$$

$r = 0.8$ $r = 1$ $r = 1.2$



相关讨论：

连续变量和分立变量：

高斯态：用Wigner函数界定的

P表示----相干态

Q表示----压缩态

W表示----猫态

三种表示都是相空间的一种准概率分布

五、思考题

1. Fock态、相干态及一般情况下$\hat{\rho}$的表达式
2. 正规排列和反正规排列的定义
3. P表示的定义，$P-\rho$关系的适用范围
4. 证明：$P(\alpha,\alpha^*)=Tr[\rho\delta(\alpha^*-a^+)\delta(\alpha-a)]$与$\rho=\int P(\alpha,\alpha^*)\,|\alpha><\alpha|d^2\alpha$等价
5. P表示的应用实例
6. Q表示的定义，$Q-\rho$关系的适用范围
7. 多种量子态的Q表示实例
8. Wigner函数及多种量子态的Wigner函数实例

作业：无

报告（报告的具体要求与之前同）

了解几种分布函数的特点（课上内容）

读前沿文献：（按兴趣自己查找）

猫态、Wigner函数等相关、或相关其它

胡晓东 拍摄